THE

MECHANICS OF ENGINEERING

VOLUME I

KINEMATICS, STATICS, KINETICS, STATICS OF RIGID BODIES AND OF ELASTIC SOLIDS

BY

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Dedication

With the approval of the President and Fellows of Yale University, a series of volumes has been prepared by a number of the Professors and Instructors, to be issued in connection with the Bicentennial Anniversary, as a partial indication of the character of the studies in which the University teachers are engaged.

This series of volumes is respectfully dedicated to

The Graduates of the University

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PREFACE.

THIS work is presented as the first volume of a series dealing with the applications of Mechanics to engineering problems. It is issued uniform with "Stresses in Framed Structures" by the same author, which will be revised as soon as possible, so as to form the second volume of the series.

The present volume opens with a carefully considered presentation of the fundamental principles of Kinematics, Statics, and Kinetics. Then follow in proper order the practical applications. Throughout the work numerous problems and illustrative examples are given in direct connection with each important mechanical principle.

The method of presentation is the result of much thought and teaching experience. The author believes that this method will commend itself to teacher and student. Thus the first chapters are devoted to the preliminary discussions of mass and space only, without reference to time, motion, or force. Under this head are treated not only mass, density, and centre of mass, but also moment of inertia, considered simply as a mass and space quantity. The experience of the author has convinced him of the advantages of this method of presentation. The student in his preparatory study of Geometry has already become conversant with those space relations of which he must make future use. He should now become equally familiar with those mass and space relations of which he must also make future use, and it is both proper and advantageous that this subject should be treated, like Geometry, as a preparatory study. Then the student is in a position to take up not only the subject of Kinematics, which deals with relations of space and motion, but also the subject of Kinetics, which deals with mass, space, and motion.

In the chapters on Kinematics and Kinetics the method of presentation has been much abridged from the author's "Elementary Principles of Mechanics," and will be found simpler, more logical and direct. The numerous examples introduced here will be found of service both to teacher and student.

In the applications the author has included the results of his own work in this direction, and he believes that the professional reader will find here new and valuable discussions of engineering problems, especially in the chapters on Masonry Walls and Dams, the Strength of Long Columns, the Swing Bridge, the Metal Arch, the Suspension System, and the Stone Arch. In all these chapters the entire treatment and many results are different from those already given by the author in "Stresses in Framed Structures" and "Elementary Principles of Mechanics." By the application of the principle of least work many new and simple results and methods have been obtained. Especially is this the case as regards the stone arch. Treatises on this subject are prolix, overburdened with mathematical discussions and involved formulæ, limited at best in their application, diverse in their assumptions, and discouraging

alike to the student and engineer. The method of solution here given is simple, direct and practical, and adapts itself with equal ease to any form of arch and any surcharge.

The size of this volume is due to its scope and plan. It treats with thoroughness and with direct reference to practice, of several topics for which the student must otherwise have recourse to separate treatises. It thus offers in one volume and in proper sequence Courses on Elementary Mechanics, Kinematics, Statics, Kinetics, Framed Structures, Graphical Statics, Walls and Dams, Retaining Walls, Mechanics of Materials, Swing Bridge, Metal and Stone Arches, and Suspension System.

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GREEK ALPHABET.

In all mathematical works Greek letters are used. The reader should therefore be familiar with these letters, so that he can write them and recognize them at sight and call them by their names.

Letter.	Transliteration.	Name.
α	a	Alpha
β	b	Beta
Γ, γ	g	Gamma
Δ, δ	d	Delta
ϵ	e short	Epsilon
ζ	z	Zeta
η	e long	Eta
θ	th	Theta
ι	i	Iota
κ	k	Kappa
Λ, λ	l	Lambda
μ	m	Mu
ν	n	Nu
ξ	x	Xi
o	o short	Omicron
Π, π	p	Pi
ρ	r	Rho
Σ, σ	s	Sigma
τ	t	Tau
Υ, υ	u	Upsilon
ϕ	ph	Phi
χ	ch	Chi
ψ	ps	Psi
Ω, ω	o long	Omega

MECHANICS.

INTRODUCTION.

CHAPTER I.

MECHANICS. KINEMATICS. DYNAMICS. STATICS. KINETICS. MEASUREMENT. UNITS.

Physical Science.—We live in a world of matter, space and time. We do not know what these are in themselves, and we cannot explain or define any one of them in terms of the others.

Thus we recognize matter in certain *states* which we call SOLID, LIQUID, or GASEOUS. We distinguish also different *kinds* of matter, such as iron, wood, glass, water, air, etc., which we call SUBSTANCES. We also recognize limited portions of matter of definite shape and volume, such as a pebble, a rain-drop, a planet, etc., which we call BODIES. But what matter is in itself we do not know.

We also recognize matter as occupying space, and we note successive events as occupying time. But what space and time are in themselves we do not know.

We also recognize FORCE as causing change of motion of matter, or change of the volume or shape of bodies. But what force is in itself we do not know.

Yet, although we thus know nothing of matter, space, time and force in themselves, we can and do investigate them in their *measurable relations*, and such investigation is the object of all PHYSICAL SCIENCE.

Mechanics—Kinematics and Dynamics—Statics and Kinetics.—That branch of physical science which treats of the measurable relations of *space* alone is called geometry.

That which deals with the measurable relations of *space and time* only—that is, with pure motion—is called KINEMATICS (*κινήμα*, motion). To the ideas of geometry it adds the idea of motion.

That which deals with the measurable relations of *force*, and of those measurable relations of *space, time and matter* involved in the study of the change of motion of material bodies under the action of *force*, is called DYNAMICS (*δύναμις*, force). To the ideas of kinematics it adds the idea of force.

We may divide dynamics into two parts. That portion which treats of material bodies at rest, under the action of balanced forces, is called STATICS. That portion which treats of the change of motion of material bodies under the action of unbalanced forces is called KINETICS.

The term MECHANICS is used to include the general principles of both kinematics and dynamics.

Measurement.—Since, then, we have to do in all that follows with the measurable relations of force, matter, space and time, the subject of the measurement of these quantities should first engage our attention.

Unit.—In order to measure any quantity whatever, we must always compare its magnitude with the magnitude of another quantity *of the same kind*. The quantity thus taken as a standard of comparison is called the UNIT of measurement.

Thus the unit of length must itself be some specified length, as, for instance, one foot, one yard, one centimeter or one meter. The unit of time must be a specified time, as one second. The unit of mass must be a specified mass, as one pound or one gram or one kilogram.

The units of mass, length and time are called FUNDAMENTAL UNITS, because not derived from any others.

Statement of a Quantity.—The complete statement of a quantity requires, therefore, a statement of the unit adopted and also a statement of the result of comparison of the magnitude of the quantity with the magnitude of the unit.

The result of this comparison is always a ratio between the magnitudes of two quantities of the same kind and is, therefore, always an abstract number.

This ratio or abstract number is called the NUMERIC.

Thus we say 3 *feet*, 4 *seconds*, 5 *pounds*. In each of these cases we state both the unit and the numeric, or ratio of the magnitude of the quantity to that of the unit. Thus 3 feet denotes a quantity whose magnitude is three *times* the magnitnde of one foot.

So for any quantity. In general, if L stands for any length and $[L]$ stands for the unit of length, we have $L = l[L]$, or the length equals l times the unit of length. Here l is the numeric and is an abstract number.

Again, if T is a certain interval of time, and $[T]$ stands for the unit of time, we have $T = t[T]$, or the time equals t times the unit of time. Here t is the numeric and is an abstract number.

So also if M is a certain mass and $[M]$ stands for the unit of mass, we have $M = m[M]$, or the mass equals m times the unit of mass. Here m is the numeric and is an abstract number.

Derived Unit.—A unit of one kind which is derived by reference to a unit of another kind is called a DERIVED UNIT.

Thus the unit of area may be taken as a square whose side is one unit of length, or one square foot. The unit of volume may be taken as a cube whose edge is the unit of length, or one cubic foot. The unit of velocity may be taken as one unit of length *per unit of time*, or one foot *per* second.

Such units are derived units, while the units of mass, space and time, not being thus derived from any others, are FUNDAMENTAL units.

Dimensions of a Derived Unit.—A statement of the mode in which the magnitude of a derived unit varies with the magnitudes of the fundamental units which compose it, is a statement of the DIMENSIONS of the derived unit.

Thus let $[A]$ denote the unit of area, and $[L]$ the unit of length. Then if $A = a[A]$ is the area of a square whose side is $L = l[L]$, where a and l are abstract numbers, we shall have $a[A] = l^2[L]^2$.

Now we shall have the numeric equation $a = l^2$, or the number of units of area equals the square of the number of units of length, provided we have $[A] = [L]^2$, or the unit of area equal to the square of the unit of length.

The statement $[A] = [L]^2$ is a statement of the dimensions of the unit of area.

Again, let $[L]$ denote the unit of length, $[T]$ denote the unit of time, and $[V]$ denote the unit of velocity. Then if $D = d[L]$ is any displacement and $T = t[T]$ is the time and $V = v[V]$ is the mean velocity, we have

$$v[V] = \frac{d[L]}{t[T]}.$$

We shall then have the numeric equation $v = \frac{d}{t}$, or the number of units of velocity is equal to the number of units of displacement divided by the number of units of time, provided we have

$$[V] = \frac{[L]}{[T]};$$

that is, provided the unit of velocity is the unit of length *per* unit of time.

This, then, is a statement of the dimensions of the unit of velocity.

Meaning of "Per."—It will be observed that the statement

$$[V] = \frac{[L]}{[T]}$$

is read, "the unit of velocity is equal to the *unit of length per unit of time.*" The word *per* is indicated by the line of division between $[L]$ and $[T]$.

Now we can divide the numeric d by the numeric t and write $v = \frac{d}{t}$, because these are abstract numbers. But it would be nonsense to speak of *dividing* length by time, or a unit of length by a unit of time. We therefore avoid such a statement by the use of the word *per*. If, then, we give to the symbol of division this new meaning, we can treat it by the rules which apply to the old meaning, and thus avoid the invention of a new symbol by using an old one in a new sense. The sign of division stands then for actual division so far as numerics are concerned, but so far as units are concerned it stands for the word *per*.

Whenever, then, the word PER *is used it can be replaced by the sign of division.*

Homogeneous Equations.—The letters in all equations or statements of the relations of quantities always stand for the numerics, and the units are always understood but not written in.

Thus such an equation as $v = \frac{d}{t}$ or $d = vt$ is a numeric equation, where the units are understood and must be supplied when interpreting the equation. When the units are thus supplied, all terms on both sides of any equation, which are combined by addition or subtraction, must always denote quantities of the same kind. Such an equation is called HOMOGENEOUS.

If any numeric equation is not thus homogeneous, it is incorrectly stated.

It is also evident that all algebraic combinations of such homogeneous equations must always produce homogeneous equations. If not, some error must have been made in the algebraic work. Thus in the equation $d = vt$, if we supply the omitted units, we have

$$d[L] = v\frac{[L]}{[T]} \times t[T] = vt[L].$$

The equation is therefore homogeneous, since the unit of length is to be understood in both terms, and we have a distance equal to a distance.

Again, suppose the result of some investigation is expressed by

$$3d + 2t = 10v.$$

Then, without any reference whatever to the various steps by which this result may have been obtained, we can say at once that it is incorrect.

For if we insert the units we have

$$3d[L] + 2t[T] = 10v\frac{[L]}{[T]},$$

or

$$3d \text{ feet} + 2\,t \text{ seconds} = 10v \text{ feet per sec.}$$

We see at once that the quantities in each term are not of the same kind. The equation is not homogeneous and is absurd.

If, however, we had

$$3d + 2tv = 10vt,$$

this equation is homogeneous, because when we insert the units we have

$$3d[L] + 2t[T]v\frac{[L]}{[T]} = 10v\frac{[L]}{[T]}t[T], \quad \text{or} \quad 3d[L] + 2tv[L] = 10vt[L].$$

Here all the terms are quantities of the same kind and the equation is homogeneous. It reads now,

$$3d \text{ feet} + 2tv \text{ feet} = 10vt \text{ feet}.$$

The relations expressed by it are not absurd. It does not follow that it is correct. It may still have been incorrectly deduced. But *on its face* it is not absurd, while $3d + 2t = 10v$ is manifestly so.

The student should make it a rule to test in this manner any equation the truth of which is suspected, as it may often save the trouble of examining in detail the entire investigation by which it has been deduced. If, however, it stands this test, then the derivation must be examined also, if error is still suspected.

Unit of Time.—The unit of time ordinarily adopted in dynamics is the SECOND or some multiple of the second.

It is the time of vibration of an isochronous pendulum which vibrates or beats 86400 times in a mean solar day of 24 hours, each hour containing 60 minutes and each minute 60 seconds ($24 \times 60 \times 60 = 86400$).

The sidereal day contains 86164.09 of these mean solar seconds.

Unit of Length.—The unit of length ordinarily adopted in dynamics is the FOOT or the METER or some multiple of these.

Unit of Mass.—The unit of matter or mass ordinarily adopted in dynamics is the POUND or the KILOGRAM.

Standard Unit.—All units adopted are defined by reference to certain STANDARD units. A standard unit, in general, should possess, so far as possible, a permanent magnitude unchanged by lapse of time and unaffected by the action of the elements or by change of place or temperature. It should be capable of exact duplication and should admit of direct and accurate comparison with other quantities of the same kind.

Standard Unit of Time.—The *standard* unit of time is the period of the earth's rotation, or the *sidereal day*. This has been proved by Laplace, from the records of celes-

tial phenomena, not to have changed by so much as one eight-millionth part of its length in the course of the last two thousand years.

The length of the solar day is variable, but the mean solar day, which is the exact mean of all its different lengths, is the period already mentioned, which furnishes the second of time. It is 1.00273791 of a sidereal day.

The second can therefore be defined, with reference to the standard unit of time, as the time of one swing of a pendulum so adjusted as to make 86400 oscillations in 1.00273791 of a sidereal day.

Standard Units of Length.—The English standard unit of length is the length of a standard bronze bar, deposited in the Standards Department of the Board of Trade in London.

Since such a bar changes in length with its temperature, the length is taken at the specified temperature of 62° Fah.

The length of this bar at this temperature is the English *standard unit of length*, and is called the STANDARD YARD. Accurate copies of this standard are distributed in various places, and from these all local standards of length are derived.*

The foot is defined as one third the length of the standard yard at 62° Fah.

The French standard of length is the *meter*, and is the length of a bar of platinum at the temperature of melting ice, or 0° C. This bar is preserved at Paris. Its length was intended to be the ten-millionth part of a quadrant of the earth's meridian through Paris.

The quadrant of the meridian through Paris is 10001472 standard meters, according to *Colonel Clarke's* determinations of the size and figure of the earth, which are at present the most authoritative, and thus the standard Paris meter is slightly less than the length upon which it was founded. The *material bar* is therefore the standard, just as is the case with the English standard.

The relation of the meter to the meridian was intended as a means of reproduction in case of destruction of the standard, but in such case the standard would probably be reproduced from the best existing copies.

This was actually the case with the original English standard, which was destroyed by fire in 1834. It had been originally defined as having at 62° Fah. a length of $\frac{36}{39.1393}$ of the length of a pendulum vibrating seconds in the latitude of London at the sea-level. But this provision for its restoration was repealed and a new standard bar was constructed from authentic copies of the old one.

The English inch, or the 36th part of the length of the standard yard, is very nearly equal to the five-hundred-millionth part of the length of the earth's polar axis $\left(\frac{1}{500482296}\right)$.

The utility of the standard, however, does not depend upon any such earth relations, the only value of which is for reproduction in case of destruction—a value which, as we have seen, is practically disregarded.

The ultimate standards are therefore the *actual bars*.

Standard Units of Mass.—The English standard unit of mass is a piece of platinum deposited in the Office of the Exchequer at London and called the "Imperial Standard Pound Avoirdupois."

The French standard unit of mass is a piece of platinum preserved at Paris and called the kilogram.

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* The English standard yard is 1 part in 17230 shorter than the U. S. copy.

Unit of Angle.—There are two units of angle in use, the DEGREE and the RADIAN.

The degree is that angle subtended at the centre of any circle by an arc equal in length to $\frac{1}{360}$ part of the circumference of that circle. It is subdivided sexagesimally into degrees (°), minutes (′), and seconds (″). The seconds are subdivided decimally. Minutes and seconds of *time* are distinguished by being written *min.*, *sec.*

The radian is that angle subtended at the centre of any circle by an arc equal in length to the radius. It is subdivided decimally.

If then the length of any arc is $s[L]$, or s units of length, and the length of the radius is $r[L]$, or r units of length, and if the angle subtended at the centre is α radians, we have

$$\alpha = \frac{s[L]}{r[L]} = \frac{s}{r}, \quad \text{or} \quad r\alpha = s.$$

The number of radians in any angle is then found by dividing the number of units of length in the subtending arc by the number of units of length in the radius, and this number is *independent of the particular unit of length adopted*, whether feet or centimeters.

If the subtending arc is the entire circumference, the number of radians is $\frac{2\pi r}{r} = 2\pi$. Hence 2π radians correspond to 360 degrees, or 1 radian corresponds to $\frac{360°}{2\pi} = \frac{180°}{\pi}$ $= 57.29578$ degrees $= 57° \ 17' \ 44''.8$.

Any angle expressed in radians may then be converted into degrees by multiplying the number of radians by $\frac{180°}{\pi} = 57.29578$ degrees $= 1$ radian.

Any angle expressed in degrees may be converted into radians by multiplying the number of degrees by $\frac{\pi}{180} = 0.0174533$ radians $= 1$ degree.

Examples.—(1) *Express 12° 34′ 56″ in terms of radians; and 3 radians in terms of degrees.*

ANS. 0.2196 radians; 171° 53′ 14″.424.

(2) *The radius of a circle is 10 feet; what is the angle subtended at the centre by an arc of 3 feet?*

ANS. $\frac{3}{10}$ radian, or 17° 11′ 19″.44.

(3) *How much must a rail 30 feet long be bent in order to fit into a curve of half a mile radius?*

ANS. $\frac{1}{88}$ radian, or 0° 39′ 3″.92.

(4) *Express 45 degrees in terms of radians, and 45 radians in terms of degrees.*

ANS. $\frac{\pi}{4}$ radians $= 0.7854$ radians; 257° 49′ 51″.636.

(5) *The angle subtended at the centre of a circle by an arc whose length is 1.57 feet is 15°; what is the radius?*

ANS. $\frac{s}{r} = \frac{15\pi}{180}$, or $r = \frac{1.57 \times 180}{15\pi} = 6$ feet.

(6) *What is the sin $\frac{\pi}{6}$ radians; cos $\frac{\pi}{6}$ radians; cos $\frac{\pi}{3}$ radians; tan $\frac{\pi}{4}$ radians?*

ANS. 0.5; $\frac{1}{2}\sqrt{3}$; 0.5; 1.

(7) *Express in degrees and in radians the angle made by the hands of a clock at 35 minues past 3 o'clock.*

ANS. 102.5 degrees; 1.79 radians.

Unit of Conical Angle.—Let the area of any portion of the surface of a sphere be $A[A]$, or A units of area, and let the square of the radius be $r^2[A]$, or r^2 units of area.

If lines are drawn from the centre C of the sphere to every point of the area, they form a cone, and the angle subtended at the centre C by the area we call a CONICAL ANGLE.

The conical angle subtended at the centre of a sphere by a portion of its surface whose area is equal to the square of its radius we call a SQUARE RADIAN. If we denote the conical angle subtended by the area A by α square radians, we have

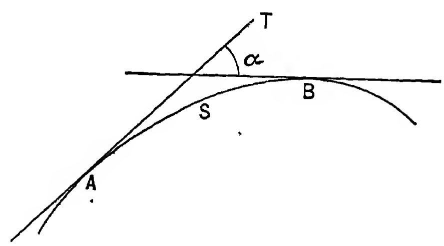
$$\alpha = \frac{A[A]}{r^2[A]} = \frac{A}{r^2}, \quad \text{or} \quad r^2\alpha = A.$$

The number of square radians in any conical angle is thus found by dividing the number of units of area in the subtending area by the number of units of area in the square of the radius, and this number *is independent of the unit of area adopted.*

If the subtending area is the entire surface of the sphere, the number of square radians is $\frac{4\pi r^2}{r^2} = 4\pi$. Hence the surface of a sphere subtends a conical angle of 4π square radians.

[The terms *solid angle* and *solid radian* are usually employed in place of conical angle and square radian as defined, but as they seem in no way descriptive, we have employed the latter terms as more expressive.]

Curvature.—The direction of a plane curve at any point is that of the tangent to the curve at this point.



Thus the direction of the curve AB at the point A is that of the tangent AT.

The *change* of direction between any two points of a plane curve is the angle between the tangents at these two points, and is called the INTEGRAL CURVATURE.

Thus the angle α, or change of direction between the tangents at A and B, is the integral curvature for the curve between A and B.

The integral curvature for any portion of a plane curve divided by the length of that portion is the *mean curvature.*

Thus if the length from A to B of the curve is $s[L]$, or s units of length, the mean curvature is $\frac{\alpha[\alpha]}{s[L]}$. Since α is given in radians, the unit of curvature is 57 *one radian per unit of length of arc.* When we say, therefore, that the mean curvature is $\frac{\alpha}{s}$, we mean $\frac{\alpha}{s}$ radians per unit of length of arc.

The limiting value of the mean curvature when the two points are indefinitely near is called the *curvature.*

The curvature, therefore, is the *limiting rate of change of direction per unit of length of arc.* Its unit is one radian per unit of length of arc.

Curvature of a Circle.—If the curve is a circle, the angle at the centre between the radii at A and B will be equal to the angle α between the tangents at A and B.

We have then $\alpha = \frac{s}{r}$ radians. The mean curvature is then $\frac{\alpha}{s} = \frac{1}{r}$ radians per unit of length of arc.

Since this is independent of s, the curvature at every point of a circle is constant and equal to the mean curvature for any two points, viz., $\frac{1}{r}$ radians per unit of length of arc.

Curvature of any Curve.—For any curve whatever a circle can always be described whose curvature is the same as that of the given curve at the given point. This is the *circle of curvature* of the curve at that point. Its radius is the *radius of curvature* of the curve at that point.

If then ρ is the radius of curvature of a curve at any given point, the curvature of the curve at that point is $\frac{1}{\rho}$ radians per unit of length of arc.

Since curvature then depends only upon the radius of curvature, the circle is the only curve whose curvature is constant.

(1) *A circle has a radius of 10 feet. What is its curvature?*

Ans. $\frac{1}{10}$ radian per foot of arc, or $5°.73$ per foot of arc.

(2) *If the radius is 10 yards, what is the curvature?*

Ans. $\frac{1}{10}$ radian per yard of arc, or $5°.73$ per yard of arc, or $\frac{1}{30}$ radian per foot of arc, or $1°\ 91$ per foot of arc.

Dimensions of Unit of Curvature.—If C is the curvature and c the number of units of curvature, we have by definition $c[C] = \frac{\alpha[\alpha]}{s[L]}$, where $[C]$ is the unit of curvature, and $[\alpha]$ is the unit of angle, $[L]$ the unit of length, and α, s the number of units of angle and length.

We shall always have $c = \frac{\alpha}{s}$, provided we take $[C] = \frac{[\alpha]}{[L]}$, that is, provided the unit of curvature is equal to the unit of angle divided by the unit of length.

This is a statement of the dimensions of the unit of curvature.

The unit of curvature is then one unit of angle per unit of length of arc, as, for instance, one radian per foot of arc, or one degree per foot of arc.

A railway curve has a length of one mile, the curvature is uniform, and the integral curvature is 30 degrees. What is the curve, the curvature, and the radius of curvature?

Ans. A circle; 0.5236 radian per mile arc; 1.9 miles radius.

Tables of Measures.—We shall deal in the course of this work with many other derived units, which will be explained as they occur. It will be useful to collect here for convenience of reference a number of such units.

I. MEASURES OF SPACE.

A. LENGTH.

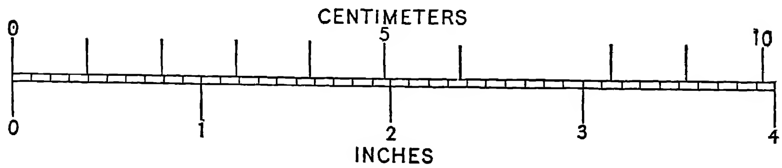


TABLE 1.

1 centimeter	= 0.3937079 inch
1 meter	= 39.37079 inches
	= 3.2809 feet
1 kilometer	= 0.62137 mile
	= 0.535987 nautical mile

TABLE 2.

1 inch	= 2.539954 centimeters
1 foot	= 30.479449 "
1 yard	= 0.91438347 meter
1 mile	= 1.60935 kilometers
1 nautical mile, or 6080.26 ft.	= 1.85327 kilometers

The following are approximate:

The centimeter is about $\frac{3}{8}$ inch. The meter is about 3 ft. $3\frac{3}{8}$ inches.
The decimeter is about 4 inches. One kilometer is about $\frac{5}{8}$ of a mile.
Distance from pole to equator is about 10000000 meters.
Earth's polar radius is about 500000000 inches.

B. AREA.

TABLE 3.

1 sq. centimeter	= 0.155006 sq. inch
1 sq. meter	= 10.7643 sq. feet
1 sq. hectometer, or 1 hectare	= 2.47114 acres
1 sq. kilometer	= 0.38611 sq. mile

TABLE 4.

1 sq. inch	= 6.45137 centimeters
1 sq. foot	= 928.997 "
1 sq. yard	= 0.836097 meter
1 acre	= 0.404672 hectare
1 sq. mile	= 2.58989 sq. kilometers

C. VOLUME.

TABLE 5.

1 cubic centimeter	= 0 0610271 cu. inch
1 liter, or 1 cubic decimeter	= 61.0271 cu. inches
	= 1.76172 pints
1 cubic meter	= 35.3166 cubic feet

TABLE 6.

1 cubic inch	= 16.3866 cubic centimeters
1 cubic foot	= 28.3153 liters
1 cubic yard	= 0.764513 cubic meter
1 pint	= 0.567627 liter
1 gallon	= 4.54102 liters

II. MEASURES OF MASS.

TABLE 7.

1 centigram	= 0.154323 grain
1 gram	= 15.4323 grains
	= 0.0353739 oz.
1 kilogram	= 2.20462 lbs.

TABLE 8.

1 grain	= 0.064799 gram
1 oz.	= 28.3496 grams
1 lb., or 16 oz.	= 0.453593 kilogram
1 ton, or 2240 lbs.	= 1016.05 kilos

1 gram = mass of 1 cubic centimeter of pure water at 4° C.

1 kilogram = 1 liter of pure water at 4° C.

1 gallon = 277.274 cubic inches. The gallon contains 10 lbs. of pure water at 62° F.

1 cubic foot of water contains about 1000 oz., or $62\frac{1}{2}$ lbs.

The pint contains 20 fluid oz.

Acceleration of gravity at London = 32.182 feet-per-second per second = 980.889 centimeters-per-second per second. Average value $32\frac{1}{6}$ feet-per-second per second or 980.3 centimeters-per-second per second.

1 dyne = force which will give a mass of 1 gram an acceleration of 1 centimeter-per-second per second = about $\frac{1}{981}$ weight of gram = weight of about 1 milligram.

1 poundal = force which will give a mass of 1 pound an acceleration of 1 foot-per-second per second = about weight of $\frac{1}{2}$ oz.

CHAPTER II.

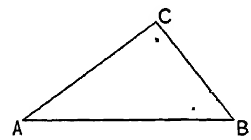
POSITION. TERMS AND DEFINITIONS.

Point.—A mathematical POINT has neither length, breadth, nor thickness. It is therefore without dimensions and indicates position only.

Point of Reference.—When we speak of a point as having position, some other point or points must always be assumed, by reference to which the position is given. Such a point is a POINT OF REFERENCE. It is also called a POLE, or ORIGIN.

Position, then, is always relative. We know nothing of "absolute" position.

Thus the position of the point C is known with respect to A when we know the length of the line AC and the angle BAC or the direction of the line AC. The points A and B are points of reference, by means of which C is located.

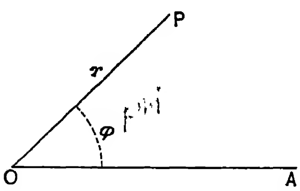


Position of a Point.—The position of a point with reference to other assumed points is then known when we have sufficient data to locate it. These data give rise to two methods of location:

1st, by polar co-ordinates;

2d, by Cartesian co-ordinates, so called because first employed by Descartes.

Plane Polar Co-ordinates.—The data necessary for locating a point by polar coordinates, when the point is situated in a given plane, consist of a distance and an angle; if the point is not in a known plane, of a distance and two angles.

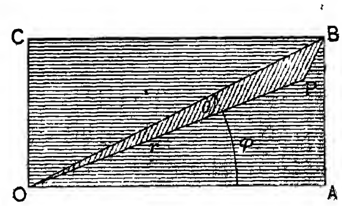


Thus, if the point P, in the plane of this page, is to be located, we first assume a line OA in the plane, as a line of reference. Then the position of P with reference to O is given by the angle AOP and by the distance OP.

The assumed point O is called the POLE; OA is the LINE OF REFERENCE; the distance OP is called the RADIUS VECTOR, and its magnitude is usually denoted by r; the angle AOP is the DIRECTION ANGLE; its magnitude is denoted by ϕ, and *it is measured around from OA to the left.*

The polar co-ordinates for a point in a given plane are therefore r and ϕ, or a distance and an angle. These are PLANE polar co-ordinates.

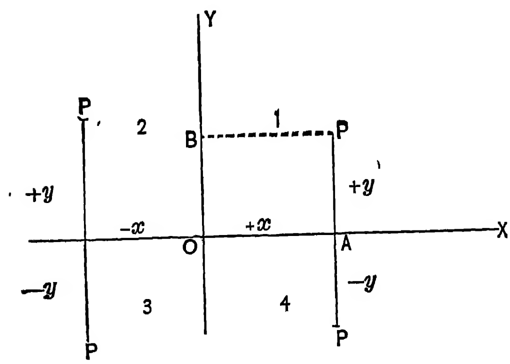
Space Polar Co-ordinates.—If the point P is not in a given plane, we assume as before a pole O, and a reference line OA in space. Through this line we assume any plane, as the plane of this page, $OABC$, and let OB be the intersection of this plane with a plane OPB, perpendicular to it and passing through OP. The location of P is then given by the length OP or the radius vector r, the angle AOB or ϕ, and the angle BOP or θ.



The polar co-ordinates for a point not in a given plane are therefore r, ϕ, and θ, or a distance and two angles. These are SPACE polar co-ordinates.

If O is a point on the earth's surface, and the reference line OA is a north and south line in the plane of the horizon, the angle ϕ would be the astronomical altitude of the point P.

Cartesian Co-ordinates—Plane.—The data necessary for locating a point by Cartesian co-ordinates, if the point is in a known plane, consists of two distances, parallel to two assumed lines of reference in that plane, passing through the point of reference, which is called the ORIGIN. The assumed lines of reference are usually taken at right angles.



Thus if the point P is known to be in the plane of this page, we assume any origin O and draw two reference lines OX and OY through O in this plane and at right angles. These two lines are called the AXES OF CO-ORDINATES, the horizontal one the *axis of* x, or the x axis, the other the *axis of* y, or the y axis. The distance BP or OA is denoted by x and called the ABSCISSA of the point P. The distance AP is denoted by y and called the ORDINATE of the point P.

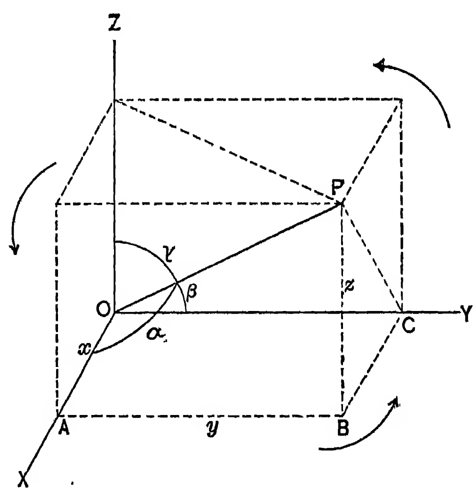
The abscissa x is positive to the right, negative to the left of the origin, while the ordinate y is positive when laid off above, and negative when below, the origin.

Any point in the plane is thus located with respect to O. If a point is in the first quadrant, its co-ordinates are $+x$, $+y$; if in the second quadrant, $-x$ and $+y$; if in the third quadrant, $-x$ and $-y$; if in the fourth quadrant, $+x$ and $-y$.

The axis OX is always taken so that $OA = +x$ is *towards the right* when we look along OY from O towards Y.

When the point is in a known plane, the co-ordinates are PLANE co-ordinates.

Cartesian Co-ordinates—Space.—If the point P is not in a known plane, we take three axes through the origin, all at right angles. Two of these we denote by X and Y as before; the third, at right angles to the plane of XY, we call the axis of z or the z axis.



Thus the position of the point P is given by the distance $OA = x$, the distance AB or $OC = y$, and the distance $BP = z$.

These are the SPACE co-ordinates of the point P.

The signs prefixed to the co-ordinates indicate the quadrant in which the point is located as before. Thus $+x$, $+y$ and $\pm z$ denote a point in the first quadrant either above or below the plane of XY; $-x$, $+y$, $\pm z$, a point in the second quadrant above or below the plane of XY; $-x$, $-y$, $\pm z$, and $+x$, $-y$, $\pm z$, points in the third and fourth quadrants above or below the plane of XY.

The plane of XY is usually taken as *horizontal*, so that OZ points to the *zenith*, and the axis OX is always taken so that $OA = +x$ is *towards the right* when we look along OY, from O towards Y. Thus if XOY is the plane of the horizon, Z is the zenith. If, then, OY is towards the north point of the horizon, OX is towards the east.

Angles about O measured in the plane XY are always measured round from OX towards OY; in the plane YZ from OY towards OZ; in the plane ZX from OZ towards OX, as indicated by the arrows in the figure.

Direction Cosines.—If we join the origin O and the point P by a line and denote the angle of OP with the X axis by α, with the Y axis by β, and with the Z axis by γ, we have the relations

$$x = OP \cos\alpha, \qquad y = OP\cos\beta, \qquad z = OP\cos\gamma.$$

These cosines are called the DIRECTION COSINES of OP, and they always have the same sign as the co-ordinates x, y and z respectively.

Since OP is the diagonal of a parallelopipedon, we have

$$\overline{OP}^2 = x^2 + y^2 + z^2 = \overline{OP}^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

Hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, therefore, any two of the direction cosines are given, the magnitude of the third can always be found.

Since, by trigonometry,

$$\cos 2\alpha = 2\cos^2\alpha - 1, \qquad \cos 2\beta = 2\cos^2\beta - 1, \qquad \cos 2\gamma = 2\cos^2\gamma - 1,$$

we have

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3,$$

or, from (1),

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2) if any two direction cosines are given in magnitude and sign, the third can be found in magnitude and sign.

Again, since, from trigonometry,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we have

$$\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta.$$

Since, by trigonometry,

$$\sin^2\alpha = 1 - \cos^2\alpha, \qquad \sin^2\beta = 1 - \cos^2\beta,$$

we have

$$\cos(\alpha + \beta)\cos(\alpha - \beta) = -1 + \cos^2\alpha + \cos^2\beta.$$

Hence, from (1),

$$\cos(\alpha + \beta)\cos(\alpha - \beta) + \cos^2\gamma = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We shall have occasion to refer to these equations hereafter.

MEASURABLE RELATIONS OF MASS AND SPACE.

CHAPTER I.

MASS AND DENSITY.

Mass and Space Relations.—As we have seen (page 1), we deal in mechanics with measurable relations of matter, space and time. Space relations alone constitute geometry, and a knowledge of geometry is therefore essential before taking up the subject of mechanics.

There are certain measurable relations of *matter and space alone* which are equally essential, and it will be found advantageous for the student to be familiar with these also, before taking up the subject of mechanics. We therefore give these relations first.

Mass—Unit of Mass.—The MASS of a body is proportional to the quantity of matter the body contains. We do not know what matter is, but if we take some one definite unchanging body as a standard, it is possible to compare other bodies with it. That is, we can say that, whatever matter may be, a given body has twice or three times, etc., as much matter as the standard.

Mass then, like position, rest and motion, is relative.

The unit of mass adopted is the standard pound or the standard kilogram. These units, as we have seen (page 6), are definite unchanging bodies and therefore contain definite unchanging quantities of matter.

Measurement of Mass.—We shall see later, when we consider the mutual relations of matter and force (page 212), that when two bodies exactly balance in an accurate equal-armed balance, then these two bodies must contain equal quantities of matter and therefore have equal mass.

By means of the balance, then, we can readily duplicate standard masses and determine the mass of any given body relatively to the mass of any assumed standard mass.

The student will at once recognize this as in accord with daily experience. Thus we say that the mass of a body is 2, 3 or 4 lbs. when it exactly balances 2, 3 or 4 standard lbs. In such case the quantity of matter contained by the body is 2, 3 or 4 times that contained by the standard.

Mass Independent of Gravity.—The mass of a body thus determined is in common language called the "weight of the body." Thus we speak of a "weight of four pounds."

In the language of mechanics this is incorrect. The "weight" of a body is, strictly speaking, the *force with which the earth attracts it*. This force varies with the locality and the height above sea-level. It is therefore a variable quantity for the same body.

But the mass or quantity of matter of course must remain the same whatever the locality. Two bodies of equal mass, which therefore balance on an equal-armed balance in any locality, would, as we shall see later (page 212), balance in any other locality. Gravity is thus made use of in determining mass, because weight is always *proportional to mass*. But the mass thus determined is *independent of gravity*.

When, therefore, we speak of a "mass of 4 lbs." we mean a definite invariable quantity of matter. When we speak of a "weight of 4 pounds" we mean the force of attraction of the earth for a mass of 4 lbs. at some specified locality.

The operation of finding the mass of a body by an equal-armed balance is then not, strictly speaking, "weighing" a body. It is "balancing" a body. The operation determines the mass and not the weight. It gives the same result for any locality, whereas the weight varies with the locality.

To determine the weight proper, we should use a spring or *dynamometer*. This would show a different weight in different localities, but the suspended mass would be always the same.

Notation for Mass.—When the mass of a body is 2, 3 or 4 pounds, it is customary to write it 2 lbs., 3 lbs., 4 lbs. Here the abbreviation "lb." stands for the Latin word "*libra*" (*balance*), and thus indicates that the mass *has been determined by balancing*.

To avoid confusion, it will be well for the student always to adhere to this notation. Thus "4 lbs." means a *mass*, while "4 pounds" means the *weight* of 4 lbs. at some specified locality. The expression "4 lbs." should really read "4 *libras*," but as this word "libra" has never come into common use, no one would understand what is meant, and "4 lbs." must therefore be read "four pounds." But as the abbreviation "lbs." is in common use, we can at least make the distinction to the eye, if not to the ear.

In the French system the distinction is complete, for we speak of "4 grams" when we mean mass and 4 *dynes* when we mean force, grams being measured by the balance, and dynes by the spring or "dynamometer."

We shall always denote the mass of a body by the letter *m* with a dash above it, thus: $\bar{m}$.

Density.—The number of units of mass of a body divided by its number of units of volume, or the *mass per unit of volume*, is the MEAN DENSITY of the body.

The mean density gives, then, the number of lbs. in a cubic foot, or the number of grams in a cubic centimeter.

The density *at a given point* of a body is the ratio of mass to volume of an indefinitely small volume of the body at that point. If this is the same at all points, the body is HOMOGENEOUS, or the density is UNIFORM. If it varies, the density is variable and the body is non-homogeneous.

The density of a body *in a given state* is the mass per unit of volume of any portion of the body in that state.

When the length of a body is great relatively to its other dimensions, the mass per unit of *length* is called its mean LINEAR DENSITY.

For a thin body the mass per unit of *area* is called its mean SURFACE DENSITY.

If $\bar{m}$ is the mass of a homogeneous body and V its volume and δ its density, we have then

Delta

$$\Delta\ \delta = \frac{\bar{m}}{V}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

or density equals mass per unit of volume.

Unit of Density.—If $[M]$ is the unit of mass and $\overline{\mathrm{m}}$ the number of units of mass, $[V]$ the unit of volume and V the number of units of volume, $[D]$ the unit of density and δ the number of units of density, we have

$$\delta[D] = \frac{\overline{\mathrm{m}}[M]}{V[V]}.$$

We shall have

$$\delta = \frac{\overline{\mathrm{m}}}{V}$$

provided we take

$$[D] = \frac{[M]}{[V]}.$$

The unit of density, then, is one unit of mass per unit of volume, as one lb. per cubic foot, or one gram per cubic centimeter.

Specific Mass.—The density-ratio of a body relatively to that of some *specified* standard substance is properly called its SPECIFIC MASS. It is often called "specific gravity," as a consequence of not distinguishing between weight and mass. The ideas are different, but the numerical values the same, since the weight of a body is proportional to its mass.

The standard substance taken is water. If γ is the density or mass of a unit of volume of water, and δ the density or mass of a unit of volume of any other body, then the specific mass ϵ is given by

$$\epsilon = \frac{\delta}{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $\delta = \frac{\overline{\mathrm{m}}}{V}$, where $\overline{\mathrm{m}}$ is the mass and V the volume of the body, we have

$$\epsilon = \frac{\overline{\mathrm{m}}}{\gamma V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since γ is the mass of a unit of volume of water, γV is the mass of a volume of water equal in volume to the body. Hence *the specific mass of any body is equal to the ratio of its mass to the mass of an equal volume of water.*

In the English system the mass of one cubic foot of pure water at 4° C., or the point of maximum density, is nearly 1000 ounces, or 62.5 lbs. (more exactly 998.6 ounces). The density of water is then about 62.5 lbs. per cubic foot, or

$$\gamma = \frac{62.5 \text{ lbs.}}{1 \text{ cub. ft.}}.$$

If, then, V is one cubic foot, we have, from (3),

$$\epsilon = \frac{\overline{\mathrm{m}} \text{ lbs.}}{62.5 \text{ lbs.}},$$

where $\overline{\mathrm{m}}$ is the mass in lbs. of one cubic foot of any body.

In the C.G.S. system, the mass of one cubic centimeter of pure water at 4° C. is very nearly one gram, and was intended to be so exactly. The density of water by this system is then

$$\gamma = \frac{1 \text{ gram}}{1 \text{ cub. c.}}.$$

If, then, V is one cubic centimeter, we have, from (3),

$$\epsilon = \frac{\overline{\mathrm{m}} \text{ grams}}{1 \text{ gram}} = \overline{\mathrm{m}},$$

where $\overline{\mathrm{m}}$ is the mass in grams of one cubic centimeter. That is, the mass in grams of one cubic centimeter gives at once the specific mass, while in the English system the mass in lbs. of one cubic foot mnst be divided by 62.5. Or inversely the specific mass of any body gives at once the mass in grams of one cubic centimeter, while it must be multiplied by 62.5 to obtain the mass in lbs. of one cubic foot.

Determination of Specific Mass.—A body totally immersed in water displaces its own volume of water. It is a well-known physical fact that a body so immersed is buoyed up by a force equal to the weight of the volume of water displaced.

If, then, a body is weighed in air and then weighed again while wholly immersed in water, the weight in air is proportional to the mass of the body, and the loss of weight is proportional to the mass of the displaced water. When very great accuracy is required the body should be weighed in a vacuum, or else allowance must be made for the buoyant force of the air. But in all cases of practical mechanics this is an unnecessary refinement.

To determine the specific mass, then, we have only to divide the weight of the body by its loss of weight in water, or, since weight is proportional to mass, we divide the number of units of mass of the body by the number of units of mass of an equal volume of water.

Table of Specific Mass.—In the following table the density-ratios, or specific mass, or so-called "specific gravity" relative to water, of a few substances are given.

The exact value in many cases will depend on the temperature and the mechanical process, such as hammering, etc., to which the bodies have been subjected in their manufacture.

Air at 0° C.	0.0012759	Tin	7.4
Alcohol at 0° C.	0.791	Iron	7.7
Turpentine at 0° C.	0.870	Copper	8.8
Ice	0.92	Silver	10.5
Sea-water at 0° C.	1.026	Lead	11.4
Crown glass	2.5	Mercury at 0° C.	13.596
Flint glass	3.0	Gold	19.3
Aluminum	2.6	Platinum	21.5
Zinc	7.0		

Examples.—(1) *The mass of a piece of limestone is 310 grams. When immersed in water it balances a mass of 188.5 grams. Find the specific mass.*

ANS. The mass of an equal volume of water is $310 - 188.5 = 121.5$ grams. Hence the specific mass is $\frac{310}{121.5} = 2.55$.

(2) *In order to find the specific mass of a piece of oak, a piece of wire which lost 10.5 grams when weighed in water was wrapped around the wood, the mass of which was 426.5 grams. The compound mass was 484.5 grams lighter in the water than in the air. Find the specific mass.*

ANS. The loss of the wood alone was $484.5 - 10.5 = 474$ grams. Hence the specific mass is $\frac{426.5}{474} = 0.9$.

(3) *An iron vessel filled with mercury has a mass of 500 lbs. and when weighed in water shows a loss of 40 lbs. If the specific mass of the iron is 7.7 and of the mercury 13.6, find the mass of the vessel and of the mercury.*

ANS. Since specific mass $\epsilon = \frac{\delta}{\gamma}$, where δ is density and γ is density of water, and since $\delta = \frac{\bar{m}}{v}$, where $\bar{m}$ is mass and v is volume, we have $\epsilon = \frac{\bar{m}}{v\gamma}$, or $v = \frac{\bar{m}}{\epsilon\gamma}$.

Let $\bar{m}_1$ be the mass of the iron and $\bar{m}_2$ of the mercury, and $\bar{m}$ the combined mass.

Then for the volume of the iron we can write $v_1 = \frac{\bar{m}_1}{\epsilon_1\gamma}$, for the volume of the mercury $v_2 = \frac{\bar{m}_2}{\epsilon_2\gamma}$, and for the combined volume $v = \frac{\bar{m}}{\epsilon\gamma}$.

Since the combined volume is equal to the sum of the volumes, we have

$$\frac{\bar{m}_1}{\epsilon_1\gamma} + \frac{\bar{m}_2}{\epsilon_2\gamma} = \frac{\bar{m}}{\epsilon\gamma}, \quad \text{or} \quad \frac{\bar{m}_1}{\epsilon_1} + \frac{\bar{m}_2}{\epsilon_2} = \frac{\bar{m}}{\epsilon}. \qquad (1)$$

Since the combined mass is equal to the sum of the masses, we have

$$\bar{m}_1 + \bar{m}_2 = \bar{m}. \qquad (2)$$

Combining (1) and (2) we obtain

$$\bar{m}_1 = \bar{m}\frac{\frac{1}{\epsilon} - \frac{1}{\epsilon_2}}{\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}}, \qquad \bar{m}_2 = \bar{m}\frac{\frac{1}{\epsilon} - \frac{1}{\epsilon_1}}{\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}}.$$

In the present case we have $\epsilon = \frac{500}{40}$, $\epsilon_1 = 7.7$, $\epsilon_2 = 13.6$ and $\bar{m} = 500$ lbs. Hence

$$\bar{m}_1 = 57\tfrac{25}{59} \text{ lbs.}, \quad \bar{m}_2 = 442\tfrac{34}{59} \text{ lbs.}$$

NOTE.—This is called the *problem of Archimedes*, because first solved by him for an alloy of gold and silver.

Its application to alloys or chemical compositions is, however, limited, as in general in such cases there is a change of volume, so that the combined volume is not equal to the sum of the volumes of the components, and equation (1) does not hold.

(4) *To find the specific mass of a mixture, given the volume or mass and specific mass of each constituent.*

ANS. We must assume the volume of the mixture equal to the sum of the volumes of the constituents. This is not invariably the case, especially where there is chemical union.

Let $\bar{m}_1, \bar{m}_2, \bar{m}_3$, etc., be the masses of the constituents;
$\epsilon_1, \epsilon_2, \epsilon_3$, " " " specific masses of the constituents;
v_1, v_2, v_3, " " " volumes of the constituents.

Let $\bar{m}$, v and ϵ be the mass, volume and specific mass of the mixture. Let γ be the density or mass of a unit of volume of water.

Then
$$\bar{m}_1 + \bar{m}_2 + \bar{m}_3 + \text{etc.} = \bar{m}.$$

But $\bar{m}_1 = \epsilon_1\gamma v_1$, $\bar{m}_2 = \epsilon_2\gamma v_2$, etc. Hence.

$$\epsilon_1 v_1\gamma + \epsilon_2 v_2\gamma + \text{etc.} = \epsilon v\gamma, \quad \text{or} \quad \epsilon = \frac{\epsilon_1 v_1 + \epsilon_2 v_2 + \text{etc.}}{v}.$$

We have also $v = v_1 + v_2 + v_3 + \text{etc}$ Therefore

$$\epsilon = \frac{\epsilon_1 v_1 + \epsilon_2 v_2 + \text{etc.}}{v_1 + v_2 + \text{etc.}} \qquad (1)$$

Again we have $v_1 = \frac{\bar{m}_1}{\epsilon_1 \gamma}$, $v_2 = \frac{\bar{m}_2}{\epsilon_2 \gamma}$, etc. Hence

$$\frac{\bar{m}}{\epsilon\gamma} = \frac{\bar{m}_1}{\epsilon_1\gamma} + \frac{\bar{m}_2}{\epsilon_2\gamma} + \text{etc.}\quad \text{Therefore}$$

$$\epsilon = \frac{\bar{m}_1 + \bar{m}_2 + \text{etc.}}{\frac{\bar{m}_1}{\epsilon_1} + \frac{\bar{m}_2}{\epsilon_2} + \text{etc.}} \qquad (2)$$

(5) *A flat bar of iron $4\frac{3}{4}$ inches wide and $\frac{5}{8}$ inch thick has a linear density of 9.91 lbs. per foot. Find the mass of a bar of iron 1 inch square and 1 yard long.*

ANS. 10 lbs.

(6) *From the preceding example state a rule for finding the mass per foot of a bar of iron of constant area of cross-section, also for finding the section if the mass per foot is given.*

ANS. To find the mass per foot in lbs., multiply the area of cross-section in square inches by 10 and divide by 3.

To find the area of cross-section in square inches, multiply the mass per foot in lbs. by 3 and divide by 10.

(7) *If a railroad rail weighs 50 lbs. per foot of length, what is its area of cross-section?*

ANS. 15 sq. inches.

CHAPTER II.

CENTRE OF MASS.

Elementary Mass or Particle.—We can consider the entire volume V of any body as subdivided into an indefinitely large number of indefinitely small ELEMENTS, each of *equal volume* v, so that $V = \Sigma v$. We can carry this subdivision as far as we please, until each equal elementary volume can be treated mathematically as a point.

If, then, δ is the density of such an elementary volume v, its mass is $m = \delta v$, and the entire mass $\bar{\mathrm{m}}$ of the body is then $\bar{\mathrm{m}} = \Sigma\delta v = \Sigma m$.

We thus consider the body as composed of elementary masses or PARTICLES of masses m_1, m_2, m_3, etc., if the body itself is not homogeneous, or of elementary masses or particles each of mass m if the body is homogeneous.

This conception lies at the basis of all our mathematical treatment, and the student should especially note *that it involves no theory whatever* as to the actual constitution of matter or as to what matter really is, i.e., as to whether a body is really composed of "atoms" or "molecules" or "particles," or as to whether matter is "continuous" or "discontinuous."

We commit ourselves, then, to no theory when we say that, whatever matter is, any body can be divided into equal portions so small that each portion can be considered as homogeneous and treated as a point. Whenever we speak of a "particle," it is to such a portion that we have reference.

Whatever, then, the constitution of matter may really be, we can consider and treat a body as composed of an indefinitely large number of indefinitely small homogeneous elements or material particles of equal volume, so small that each may be treated as a point. We denote the mass of such an element by m.

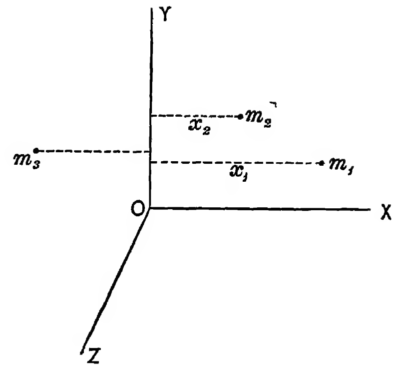
Material Surface.—A MATERIAL SURFACE, then, is a body whose uniform thickness is small compared to its other two dimensions. We can consider the entire area A of such a surface as made up of indefinitely small elements of area a, so that $A = \Sigma a$. Let δ be the *surface* density (page 15) of such an element. Then its mass is $m = \delta a$, and the entire mass $\bar{\mathrm{m}}$ of the material surface is $\bar{\mathrm{m}} = \Sigma\delta a$.

Material Line.—A MATERIAL LINE is a body whose length is large compared to its other dimensions, and whose sectional area at right angles to the length is constant. We can consider the length L of such a line as made up of indefinitely small elements of length l, so that $L = \Sigma l$. Let δ be the *linear* density (page 15) of such an element. Then its mass is $m = \delta l$, and the entire mass $\bar{\mathrm{m}}$ of the material line is $\bar{\mathrm{m}} = \Sigma\delta l$.

Centre of Mass.—We consider, then, a material body, surface or line as composed of an indefinitely large number of indefinitely small particles.

The centre of mass of such a body, surface or line is that point whose distance from any plane is equal to the average distance of all the particles from that plane.

Thus suppose a body composed of particles whose masses are m_1, m_2, m_3, etc. Suppose the first to contain a number n_1 of smaller particles of equal mass m, the second a number n_2 of the same mass m, and so on. Then



$$m_1 = n_1 m, \qquad m_2 = n_2 m, \qquad m_3 = n_3 m, \qquad \text{etc.}$$

Let x_1, x_2, x_3, etc., be the distances of m_1, m_2, m_3, etc., from the plane YZ, each x being taken with its proper sign. Let the entire number of equal particles of mass m be N, so that the total mass of the body is

$$\overline{\mathfrak{m}} = Nm.$$

Then we have for the average distance of all the particles from the plane YZ, that is for the distance of the centre of mass $\bar{x}$ from the plane YZ,

$$\bar{x} = \frac{n_1x_1 + n_2x_2 + n_3x_3 + \text{etc.}}{N}.$$

If we multiply numerator and denominator by m, we have

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \text{etc.}}{\overline{\mathfrak{m}}} = \frac{\Sigma mx}{\overline{\mathfrak{m}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the same way we obtain for the distance $\bar{y}$ of the centre of mass from the plane ZX

$$\bar{y} = \frac{\Sigma my}{\overline{\mathfrak{m}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for the distance $\bar{z}$ of the centre of mass from the plane XY

$$\bar{z} = \frac{\Sigma mz}{\overline{\mathfrak{m}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Position of Centre of Mass in General.—If, then, v is the volume of an element of a body, δ the density, and m the mass, we have $m = \delta v$. The entire mass $\overline{\mathfrak{m}}$ is $\overline{\mathfrak{m}} = \Sigma \delta v$, and we have, from the preceding equations,

$$\left.\begin{aligned} \bar{x} &= \frac{\Sigma mx}{\overline{\mathfrak{m}}} = \frac{\Sigma \delta vx}{\Sigma \delta v}, \\ \bar{y} &= \frac{\Sigma my}{\overline{\mathfrak{m}}} = \frac{\Sigma \delta vy}{\Sigma \delta v}, \\ \bar{z} &= \frac{\Sigma mz}{\overline{\mathfrak{m}}} = \frac{\Sigma \delta vz}{\Sigma \delta v}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

If the body is homogeneous, δ is constant for every element. It therefore cancels out, and we have *for a homogeneous body*

$$\bar{x} = \frac{\Sigma vx}{V}, \qquad \bar{y} = \frac{\Sigma vy}{V}, \qquad \bar{z} = \frac{\Sigma vz}{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where V is the entire volume.

In the same way if a is the area of an element of a material surface, δ the surface density, and m the mass, we have $m = \delta a$. The entire mass $\overline{\mathrm{m}}$ is $\overline{\mathrm{m}} = \Sigma\delta a$, and we have for a material surface

$$\bar{x} = \frac{\Sigma\delta ax}{\Sigma\delta a}, \qquad \bar{y} = \frac{\Sigma\delta ay}{\Sigma\delta a}, \qquad \bar{z} = \frac{\Sigma\delta az}{\Sigma\delta a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

If the *surface is homogeneous*, δ cancels out and we have

$$\bar{x} = \frac{\Sigma ax}{A}, \qquad \bar{y} = \frac{\Sigma ay}{A}, \qquad \bar{z} = \frac{\Sigma az}{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A is the entire area.

So also, if ds is the length of an element of a material line, and δ the linear density, we have $m = \delta.ds$. The entire mass is $\overline{\mathrm{m}} = \Sigma\delta.ds$, and we have for a material line

$$\bar{x} = \frac{\Sigma\delta.ds.x}{\Sigma\delta.ds}, \qquad \bar{y} = \frac{\Sigma\delta.ds.y}{\Sigma\delta.ds}, \qquad \bar{z} = \frac{\Sigma\delta.ds.z}{\Sigma\delta.ds}. \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

If the line is homogeneous, δ cancels out and we have

$$\bar{x} = \frac{\Sigma ds.x}{s}, \qquad \bar{y} = \frac{\Sigma ds.y}{s}, \qquad \bar{z} = \frac{\Sigma ds.z}{s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where s is the entire length.

Moment of Mass, Volume, Area, Line.—We call the product of any mass $\overline{\mathrm{m}}$, volume V, area A or line S by the distance of its *centre of mass* from any plane, the MOMENT of that mass, volume, area or line, relative to that plane.

We can then express all the preceding equations by the simple statement that *the moment of the total mass, volume, area, or length relative to any plane is equal to the algebraic sum of the moments of all the elementary masses, volumes, areas or lines relative to the same plane.*

In taking the algebraic sum, distances are to be taken with their proper signs, positive in the directions OX, OY, OZ, negative in the opposite directions.

Centre of Gravity.—We shall see hereafter (page 190) that the centre of mass of a body coincides with the point of application of the resultant of a system of parallel forces.

The attraction of the earth for a body is the resultant of a system of forces acting upon the particles of a body, each directed towards the centre of the earth.

We have thus a system of forces not strictly parallel. But practically the deviation from parallelism is insignificant, since the greatest dimension of any body on the earth with which we have to do is insignificant in comparison with the diameter of the earth. Hence the resultant force of gravity passes *practically* through the centre of mass. This resultant is the weight of the body. *The weight of the body acts practically, therefore, at the centre of mass.*

To determine the centre of mass by experiment, we have then only to suspend it successively from two different points of its surface. The two lines of suspension, if produced, must intersect practically at the centre of mass.

The centre of mass is therefore often called the "centre of gravity." This term is, however, strictly speaking, incorrect. Centre of mass is not dependent at all upon gravity any more than mass itself. We can make use of gravity in finding it, just as we do in measuring mass (page 14).

The term "centre of gravity" can only be properly applied to that point at which if the entire mass of the body were concentrated, it would attract and be attracted, for all positions of the body, just the same as the body itself. In this sense, as we shall see hereafter (page 207), only a few bodies possess a centre of gravity. But *all* bodies possess a centre of mass. The two conceptions are therefore entirely distinct.

Origin at the Centre of Mass.—In taking the algebraic sums of the elementary moments Σmx, Σmy, Σmz, we must, as already noted, take x, y, z for each element with their proper signs. If, then, we take the origin of co-ordinates at the centre of mass, we must have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$. Hence in this case

$$\Sigma mx = 0, \qquad \Sigma my = 0, \qquad \Sigma mz = 0. \qquad (1)$$

If we take polar co-ordinates and take the pole at the centre of mass, we have

$$\Sigma mr = 0, \qquad (2)$$

where r is the distance of any particle from the pole.

That is, *the algebraic sum of the moments of all the elements relative to the centre of mass is zero.*

Plane and Axis of Symmetry.—A body is symmetrical with respect to a plane when for each element on one side of the plane there is an equal, similarly situated element on the other side. In such case the centre of mass is in this plane.

A body is symmetrical with respect to an axis when it is symmetrical with respect to two planes passing through that axis. In such case the centre of mass is in that axis.

If a body is symmetrical with respect to two axes, the centre of mass is at their intersection.

Material Line, Area, Volume.—There is of course a certain inconsistency in speaking of the centre of mass of geometrical lines, areas and volumes. The expression is, however, allowable, since we are understood to mean a *material* line, area or volume, δ being linear, surface or body density, as defined on page 15. In the case of a homogeneous line, area or volume, δ cancels out, as we have seen in the equations of page 22, and we can then allowably speak of the centre of mass of the line, area or volume itself.

Determination of Centre of Mass.—We have already seen how to determine the position of the centre of mass by experiment (page 22).

The position of the centre of mass for many homogeneous bodies may often be told at once, by applying the principle of symmetry just given.

Thus the centre of mass of a homogeneous straight line is at its middle point. For a homogeneous circle, or circular area, or sphere, or spherical shell it is at the centre of figure. For a homogeneous parallelogram it is at the intersection of the two lines through the middle points of opposite sides, or at the intersection of the two diagonals, since these are lines of symmetry. So also for a homogeneous parallelopipedon the centre of mass is at the intersection of its diagonals, or any two lines through the centre of mass of opposite sides. For a homogeneous circular cylinder with parallel ends, at the middle point of the axis.

For a homogeneous triangle the centre of mass is at the intersection of any two lines from an apex to the middle point of the opposite side, since these are lines of symmetry.

For bodies in general we may find the centre of mass by applying the equations of page 22, or more generally by the application of the Integral Calculus to effect the summations indicated.

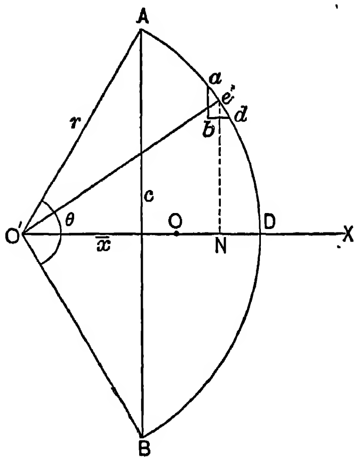
It is not necessary or desirable to occupy space here with such applications. It is of

far more importance that the student should understand clearly what the centre of mass is, as defined on page 20, and what use is made of it, and how it enters into mechanical problems, as will be explained later (pages 175 and 191), than that he should spend much time in finding its position.

In the following examples we show how to apply the equations of page 22 to a few cases.

Examples.—(1) *Find the centre of mass of a homogeneous circular arc.*

ANS. Let ADB be a homogeneous circular arc with centre at O', and let the axis of X pass through O' and the centre D of the arc.



Then $O'D$ is an axis of symmetry, and the centre of mass O is on this axis.

Let the chord $AB = c$, and the length of arc $ADB = s$, and r be the radius. Take an indefinitely small element ad of length ds whose centre of mass is at e, and let ab be the vertical projection of $ad = ds$. Drop eN perpendicular to $O'X$. Then we have, by similar triangles,

$$ds : ab :: r : O'N;$$

or, since $O'N = x$, the abscissa of the point e, we have

$$x \cdot ds = r \times ab.$$

By equation (3), page 22, we have then for the distance $O'O = \bar{x}$ of the centre of mass

$$\bar{x} = \frac{\Sigma x \cdot ds}{s} = \frac{r\Sigma ab}{s}.$$

But Σab is the chord $AB = c$. Hence

$$\bar{x} = \frac{rc}{s}.$$

Hence the centre of mass O of a circular arc ADB is in the axis of symmetry $O'D$ at a distance $\bar{x} = O'O$ from the centre O', which is a fourth proportional to the arc, the radius and the chord, or

$$s : r :: c : x.$$

If θ is the central angle $AO'B$ in radians, $c = 2r \sin \frac{\theta}{2}$, $s = r\theta$ and

$$\bar{x} = \frac{2r \sin\frac{\theta}{2}}{\theta}.$$

For a semicircle $s = \pi r$, $c = 2r$ or $\theta = \pi$, $\sin \frac{\theta}{2} = 1$ and

$$\bar{x} = \frac{2r}{\pi}.$$

For a circle $s = 2\pi r$, $c = 0$, or $\theta = 2\pi$, $\sin \frac{\theta}{2} = 0$ and $\bar{x} = 0$, or the centre of mass is at the centre of the circle.

BY CALCULUS.—The distance $ab = dy$, we have then

$$ds : dy :: r : x, \quad \text{or} \quad xds = rdy.$$

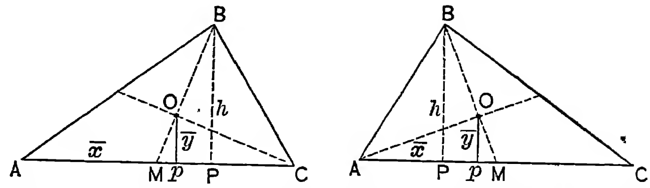
By equation (3), page 22,

$$\bar{x} = \frac{1}{s}\int_{-\frac{c}{2}}^{+\frac{c}{2}} xds = \frac{r}{s}\int_{-\frac{c}{2}}^{+\frac{c}{2}} dy = \frac{rc}{s}.$$

(2) *Find the centre of mass for a homogeneous triangle.*

ANS. Let ABC be the triangle, the base $AC = b$, and the altitude $BP = h$.

By the principle of symmetry the centre of mass O is at the intersection of any two lines from an apex to the middle point of the opposite side.



By geometry, then, the distance $\bar{y}$ of the centre of mass above the base is

$$\bar{y} = \tfrac{1}{3} h.$$

To find the distance $\bar{x} = Ap$, let M be the middle point of the base, and draw BM and drop the perpendiculars Op and BP. Then we have

$$AP = h \cot A,$$

and for the distance MP

$$MP = h \cot A - \frac{b}{2}, \quad \text{or} \quad MP = \frac{b}{2} - h \cot A,$$

according as A is the smaller or larger base angle.

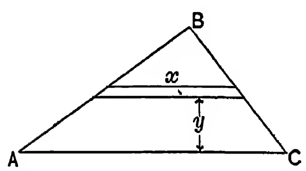
We have then $Mp = \frac{1}{3} MP$, or

$$Mp = \frac{h \cot A}{3} - \frac{b}{6}, \quad \text{or} \quad Mp = \frac{b}{6} - \frac{h \cot A}{3},$$

and hence in both cases, if A is acute,

$$\bar{x} = \frac{b}{2} - \frac{b}{6} + \frac{h \cot A}{3} = \frac{b + h \cot A}{3}. \qquad (2)$$

BY CALCULUS.—Take an elementary strip parallel to the base and at a distance y above it. Let the length of this strip be x. Its thickness will be dy, and its area xdy. We have then



$$\bar{y} = \frac{1}{A} \int_0^h xydy, \quad \bar{x} = \frac{1}{A} \int_0^h xdy\left(\frac{x}{2} + y \cot A\right).$$

But we have $A = \frac{bh}{2}$, and, by proportion,

$$h - y : x :: h : b, \quad \text{or} \quad x = \frac{b(h-y)}{h}.$$

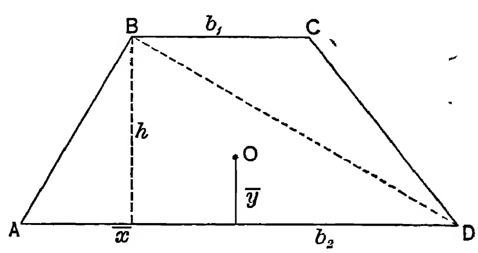
Substituting these values of A and x, we have

$$\bar{y} = \frac{2}{h^2} \int_0^h (h-y)ydy = \frac{h}{3}, \qquad \bar{x} = \frac{b}{h^3} \int_0^h (h-y)^2 dy + \frac{2 \cot A}{h^2} \int_0^h (h-y)ydy = \frac{b}{3} + \frac{h \cot A}{3}.$$

(3) *Find the centre of mass of a homogeneous trapezoid.*

ANS. Let $ABCD$ be the trapezoid. Denote the top or smaller base BC by b_1, and the bottom or larger base AD by b_2, and the altitude by h.

We have then for the area of the trapezoid



$$\text{area } ABCD = \frac{(b_1 + b_2)h}{2}.$$

We can divide the trapezoid into two triangles, ABD and BDC, by the diagonal BD.

The area of the triangle ABD is

$$\text{area } ABD = \frac{b_2 h}{2},$$

and, as just proved (example 2), the distance of its centre of mass above AD is $\frac{1}{3}h$, and on the right of A, $\frac{b_2 + h \cot A}{3}$.

The area of the triangle BDC is

$$\text{area } BDC = \frac{b_1 h}{2},$$

and the distance of its centre of mass above AD is $\frac{2}{3}h$, and on the right of A, from equation (2), example (2),

$$h \cot A + \frac{b_1 + b_2 - h \cot A}{3} = \frac{b_1 + b_2 + 2\,h \cot A}{3}.$$

We have then for the distance $\bar{y}$ of the centre of mass O above AB

$$\text{area } ABCD \times \bar{y} = \text{area } ABD \times \frac{h}{3} + \text{area } BDC \times \frac{2}{3}h,$$

and for the distance $\bar{x}$ of the centre of mass O on the right of A

$$\text{area } ABCD \times \bar{x} = \text{area } ABD \times \frac{b_2 + h \cot A}{3} + \text{area } BDC \times \frac{b_1 + b_2 + 2\,h \cot A}{3}.$$

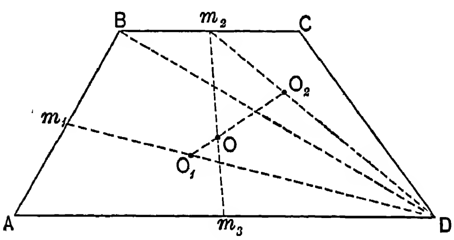
Inserting the values for the areas, we obtain, if A is acute,

$$\bar{y} = \frac{(2b_1 + b_2)h}{3(b_1 + b_2)},$$

$$\bar{x} = \frac{b_1^2 + b_1 b_2 + b_2^2 + h(2b_1 + b_2)\cot A}{3(b_1 + b_2)}.$$

For a parallelogram $b_1 = b_2$ and $\bar{y} = \frac{1}{2}h$, $\bar{x} = \frac{b}{2} + \frac{1}{2}h \cot A$. For a rectangle $A = 90°$ and $\bar{y} = \frac{1}{2}h$, $\bar{x} = \frac{1}{2}b$.

Graphical Construction.—1st. Draw lines Dm_1, Dm_2 from apex D to the middle points m_1 and m_2 of the opposite sides. Then the centre of mass O_1 of the triangle ABD is on the line Dm_1 at a distance DO_1 equal to $\frac{2}{3}$ of Dm_1. The centre of mass O_2 of the triangle BDC is on the line Dm_2 at a distance DO_2 equal to $\frac{2}{3}$ of Dm_2. If we join O_1 and O_2 thus found, the centre of mass must be on the line O_1O_2.



Draw m_2m_3 through the middle points of the parallel sides. This line is an axis of symmetry, since it passes through the centre of all elements parallel to BC and AD. The centre of mass must therefore lie in the line m_2m_3. It is therefore at the intersection C of m_2m_3 and O_1O_2.

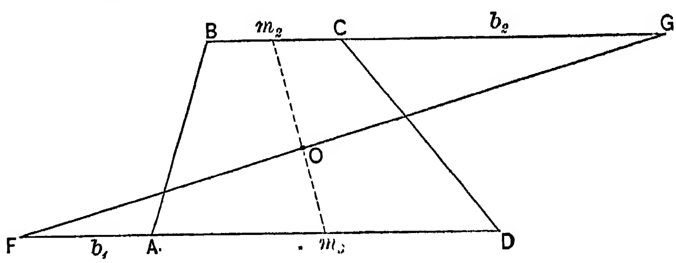
2d. We have, from the values of $\bar{x}$ and $\bar{y}$ just obtained,

$$\frac{b_1 + \bar{x}}{\bar{y}} = \frac{2b_1 + b_2 + h \cot A}{h}.$$

Now if we lay off CG parallel to AD and make it equal to b_2, and lay off AF parallel to BC and make it equal to b_1, and join F and G, the line FG must pass through O.

For we have, by proportion,

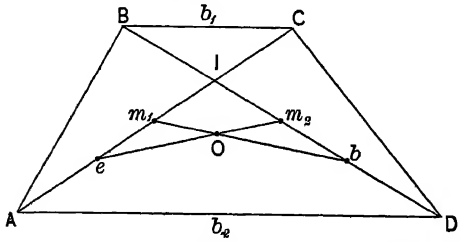
$$\frac{b_1 + \bar{x}}{\bar{y}} = \frac{b_1 + h \cot A + b_1 + b_2}{h}.$$



Hence the intersection of FG with the axis of symmetry m_2m_3 gives the centre of mass O.

3d. Another convenient construction is as follows:

Draw the diagonals AC and DB intersecting at I. Lay off along AC the distance $Ae = IC$, and along DB the distance $Db = IB$.



Bisect the diagonals at m_1 and m_2, and join m_1b and m_2e. The intersection O is the centre of mass.

Proof.—By construction the sum of the horizontal projections of Ae and Db is equal to b_1, and the points e and b are both at the same distance above AD.

Hence the line eb is horizontal and

$$eb = b_2 - b_1.$$

We have then, by construction, m_1m_2 also horizontal and

$$m_1m_2 = \frac{b_2 - b_1}{2}.$$

We have also, by proportion, if y_1 is the vertical from I on BC,

$$\frac{b_1}{y_1} = \frac{b_2}{h - y_1}, \quad \text{or} \quad y_1 = \frac{b_1 h}{b_2 + b_1}.$$

By similar triangles, if $\bar{y}$ is the distance of O above AD,

$$\frac{eb}{\bar{y} - y_1} = \frac{m_1 m_2}{\frac{h}{2} - \bar{y}}.$$

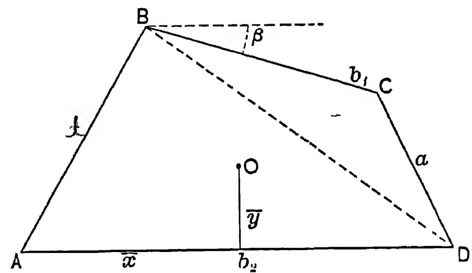
Inserting values,

$$\frac{b_2 - b_1}{\bar{y} - \frac{b_1 h}{b_2 - b_1}} = \frac{\frac{1}{2}(b_2 - b_1)}{\frac{h}{2} - \bar{y}}, \quad \text{or} \quad \bar{y} = \frac{(2b_1 + b_2)h}{3(b_2 + b_1)},$$

just as already found.

(4) *Find the centre of mass of a homogeneous trapezium.*

ANS Let $ABCD$ be the trapezium. Denote AD by b_2, BC by b_1, AB by d and DC by a. Also let β be the angle of b_1 with horizontal, and take AD horizontal.



We have then for the area of the trapezium

$$\text{area } ABCD = \frac{b_2 d \sin A + b_1 a \sin C}{2}.$$

We can divide the trapezium into two triangles, ADB and BDC, by the diagonal BD.

The area of the triangle ABD is

$$\text{area } ABD = \frac{b_2 d \sin A}{2};$$

the distance of its centre of mass above AD is $\frac{d \sin A}{3}$, and on the right of A (example (2)) $\frac{b_2 + d\cos A}{3}$.

The area of the triangle BDC is

$$\text{area } BDC = \frac{b_1 a \sin C}{2};$$

the distance of its centre of mass above AD is $\frac{d \sin A + a \sin D}{3}$, and on the right of A,

$$\frac{b_1 \cos\beta + b_2 + 2d\cos A}{3}.$$

We have then for the distance $\bar{y}$ of the centre of mass O above AB

$$\text{area } ABCD \times \bar{y} = \text{area } ABD \times \frac{d\sin A}{3} + \text{area } BDC \times \frac{d\sin A + a\sin D}{3},$$

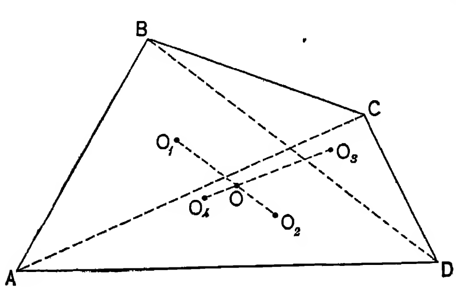
and for the distance $\bar{x}$ of the centre of mass O on the right of A

$$\text{area } ABCD \times x = \text{area } ABD \times \frac{b_2 + d\cos A}{3} + \text{area } BDC \times \frac{b_1\cos\beta + b_2 + 2d\cos A}{3}.$$

Inserting the values of the areas, we have

$$\bar{y} = \frac{b_1 a \sin C(d \sin A + a \sin D) + b_2 d^2 \sin^2 A}{3(b_2 d \sin A + b_1 a \sin C)},$$

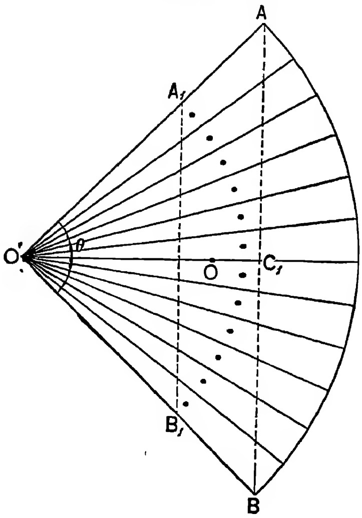
$$\bar{x} = \frac{b_1^2 a \sin C \cos\beta + b_1 b_2 a \sin C + b_2^2 d \sin A + d\cos A(b_2 d \sin A + 2b_1 a \sin C)}{3(b_2 d \sin A + b_1 a \sin C)}.$$



For a trapezoid, $\beta = 0$, $d \sin A = a \sin D = a \sin C = h$, and $\bar{y}$ and $\bar{x}$ reduce to the values already found.

Construction.—Draw the diagonals BD and CA, find the centre of mass O_1 of the triangle ABC and the centre of mass O_2 of the triangle ACD. Then the centre of mass O must be on the line O_1O_2. Again, find the centre of mass O_3 of the triangle DBC, and O_4 of DBA. The centre of mass O must be on the line O_3O_4. It is therefore at the intersection of O_1O_2 and O_3O_4.

(5) *Find the centre of mass of a homogeneous circular sector.*



ANS. Let $O'ACBO'$ be the sector with centre at O' and radius $O'A = r$.

The sector can be divided into an indefinitely large number of indefinitely small triangles. The centre of mass of each triangle is at a distance of $\frac{2}{3}r$ from O'. These centres form then a homogeneous circular arc $A_1C_1B_1$, and the centre of mass of the sector is the centre of mass of this arc.

From example (1), then, the centre of mass O lies upon the radius of symmetry $O'C$ and at a distance $O'O = \bar{x}$ from the centre given by

$$\bar{x} = \frac{\text{chord } A_1B_1}{\text{arc } A_1C_1B_1} \cdot \frac{2}{3}r.$$

Let c be the length of chord AB of the sector, and s the length of arc ACB of the sector, then

$$\frac{\text{chord } A_1B_1}{\text{arc } A_1C_1B_1} = \frac{c}{s},$$

and we have

$$\bar{x} = \frac{2}{3} \cdot \frac{rc}{s}.$$

If θ is the central angle $AO'B$ in radians, $c = 2r \sin\frac{\theta}{2}$, $s = r\theta$ and

$$\bar{x} = \frac{4r \sin\frac{\theta}{2}}{3\,\theta}.$$

For the semicircle, $c = 2r$, $s = \pi r$ or $\theta = \pi$, $\sin\frac{\theta}{2} = 1$; hence

$$\bar{x} = \frac{4r}{3\pi}.$$

For a quadrant

$$\bar{x} = \frac{4\sqrt{2}}{3\pi}r.$$

For a sextant

$$\bar{x} = \frac{2r}{\pi}.$$

Student should solve by Calculus.

(6) *Find the centre of mass for a homogeneous segment of a circle.*

ANS. Let ACB be the segment The centre of mass O is in the radius of symmetry $O'C$.

Let the radius $O'A = r$, and the length of arc $ACB = s$. Then the area of the sector $O'ACBO'$ is

$$A_1 = \frac{rs}{2}.$$

The distance $O'O_1$ of the centre of mass O_1 of the sector has just been found in example (5) to be

$$O'O_1 = \frac{2rc}{3s},$$

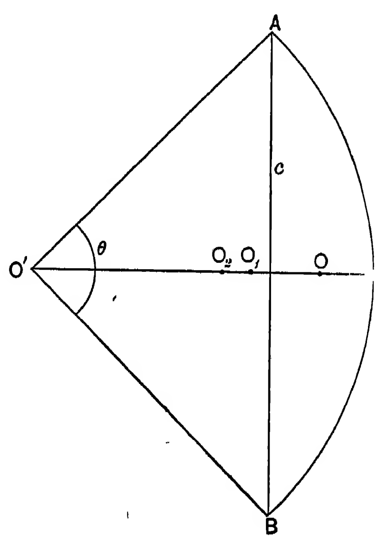
where c is the length of chord AB.

The height of the triangle $O'AB$ is $\sqrt{r^2 - \frac{c^2}{4}}$. Its area is then

$$A_2 = \frac{c}{2}\sqrt{r^2 - \frac{c^2}{4}}.$$

The distance $O'O_2$ of the centre of mass O_2 of the triangle is

$$O'O_2 = \frac{2}{3}\sqrt{r^2 - \frac{c^2}{4}}.$$



The area A of the segment is $A = (A_1 - A_2)$. Let $O'O = \bar{x}$ be the distance of its centre of mass. Then we have the moment of the sector equal to the sum of the moments of the triangle and segment, or

$$A_1 \,.\, O'O_1 = A_2 \,.\, O'O_2 + (A_1 - A_2)\bar{x}.$$

Substituting the values of A_1, A_2, $O'O_1$, and $O'O_2$, we obtain

$$\bar{x} = \frac{c^3}{12A} = \frac{c^3}{6\left(rs - c\sqrt{r^2 - \frac{c^2}{4}}\right)}.$$

If θ is the central angle $AO'B$ in radians, we have $s = r\theta$, $c = 2r \sin\frac{\theta}{2}$, and hence

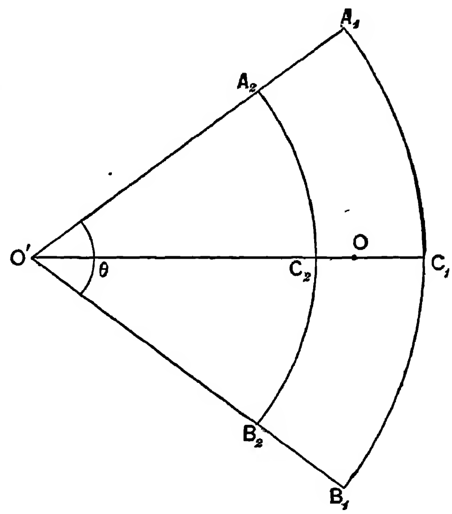
$$\bar{x} = \frac{c^3}{6r^2\left(\theta - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}.$$

For a semicircular segment, $\theta = \pi$, $\cos\frac{\theta}{2} = 0$, $c = 2r$, and hence $\bar{x} = \frac{4r}{3\pi}$, just as we have already found in example (5).

Student should solve by Calculus.

(7) *Find the centre of mass for a homogeneous circular ring.*

ANS. Let the outer radius $O'A_1$ be r_1, and the inner radius $O'A_2$ be r_2.



Let the length of the outer arc $A_1C_1B_1$ be s_1, and its chord A_1B_1 be c_1; the length of the inner arc $A_2C_2B_2$ be s_2, and its chord A_2B_2 be c_2.

Then, from example (5), we have for the distance of the centre of mass of the sector $O'A_1B_1O'$

$$\bar{x}_1 = \frac{2r_1c_1}{3s_1},$$

and its area

$$A_1 = \frac{r_1s_1}{2}.$$

The distance of the centre of mass of the sector $O'A_2B_2O'$ is

$$\bar{x}_2 = \frac{2r_2c_2}{3s_2} = \frac{2r_2^2c_1}{3r_1s_2},$$

and its area is

$$A_2 = \frac{r_2s_2}{2}.$$

The area of the ring is

$$A = A_1 - A_2 = \tfrac{1}{2}(r_1s_1 - r_2s_2) = \frac{s_1}{2r_1}(r_1^2 - r_2^2).$$

Let $O'O = \bar{x}$ be the distance of its centre of mass.

Then we have the moment of the outer sector equal to the sum of the moments of the inner sector and ring, or

$$A_1\bar{x}_1 = A_2\bar{x}_2 + (A_1 - A_2)\bar{x}.$$

Substituting the values of A_1, A_2, $\bar{x}_1$, $\bar{x}_2$, we obtain

$$\bar{x} = \frac{2c_1}{3s_1}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right),$$

or, since $s_1 = r_1\theta$, where θ is the central angle A_1OB_1 in radians, and $c_1 = 2r_1 \sin\frac{\theta}{2}$,

$$\bar{x} = \frac{4\sin\frac{\theta}{2}}{3\theta}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right).$$

(8) *Find the centre of mass for the homogeneous surface of a cylinder, or of any prism.*

ANS. The centre of mass for the homogeneous surface of a cylinder is at the middle point of the axis. For all the elements of the surface, obtained by taking slices parallel to the base, have their centres of mass upon the axis. The centre of mass of the surface is then the centre of mass of the axis.

For the same reason the centre of mass for the homogeneous surface of a prism is at the middle of the line which unites the centres of mass of its bases.

(9) *Find the centre of mass for the homogeneous surface of a right cone or pyramid.*

ANS. The centre of mass for the homogeneous surface of a right cone is in the axis at two-thirds its length from the apex. For the curved surface can be divided into an indefinite number of indefinitely small triangles. The centres of mass of all these triangles form a circle at a distance of two-thirds of the axis from the apex, whose centre of mass is in the axis.

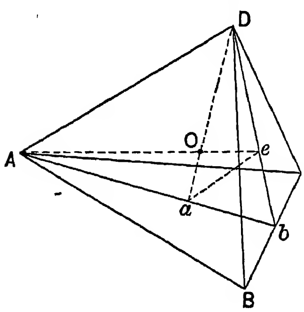
The same holds for any right pyramid.

(10) *Find the centre of mass of a solid homogeneous prism or cylinder with parallel bases.*

ANS. The centre of mass of a solid homogeneous prism is at the middle of the line joining the centres of mass of its bases. For by passing planes parallel to the bases we divide it into equal slices whose centres of mass lie in the axis.

The same holds for a cylinder.

(11) *Find the centre of mass for a homogeneous pyramid or cone.*



ANS. Let $ABCD$ be a homogeneous triangular pyramid. Take b at the middle point of BC, and draw Ab and Db. Take a point a on Ab, so that $ab = \frac{1}{3}Ab$, and a point e on Db, so that $eb = \frac{1}{3}Db$. Join ae, and draw Da and Ae. Then Da and Ae are axes of symmetry and the centre of mass O is at their intersection. But ae is parallel to AD and equal to $\frac{1}{3}AD$, and the triangle aOe is similar to AOD. Hence $aO = OD$, or $\frac{1}{3}OD = 3 . aO$, and therefore $aD = 4 . aO$, or $aO = \frac{1}{4} . aD$.

The centre of mass is then on the line joining a vertex with the centre of mass of the opposite base, and at a distance from the vertex of $\frac{3}{4}$ the length of this line.

Since any pyramid or cone is composed of triangular pyramids with a common vertex, the same holds true for the centre of mass.

We can therefore determine the centre of mass of a pyramid or cone by passing a plane parallel to the base at a distance from the base of $\frac{1}{4}$ the altitude, and finding the centre of mass of this section, or the point where it is pierced by the line from the apex to the centre of mass of the base.

CHAPTER III.

MOMENT OF INERTIA.*

Moment of Inertia.—The product of any indefinitely small elementary mass or area or volume by the *square* of its distance from any given point, line or plane is called the MOMENT OF INERTIA of the elementary mass or area or volume relative to that point, line or plane.

If, then, a is an elementary area and r its distance from any point, line or plane, ar^2 is the moment of inertia. If m is the mass of a particle, mr^2 is the moment of inertia. If v is the volume of an element, vr^2 is the moment of inertia.

The summation for any volume, area or mass of all these products is the moment of inertia of the entire volume, area or mass, relative to the assumed point, line or plane. Thus for any area the moment of inertia is Σar^2, for any volume Σvr^2, for any mass Σmr^2.

The point chosen is always the centre of mass of the entire body, mass, volume, or area, unless otherwise specified. The plane or line chosen is always a plane or line through this centre of mass, unless otherwise specified.

We use the term "centre of mass of an area" or of a volume in the sense already defined (page 23).

The line relative to which the moment of inertia is determined is the AXIS.

We always denote the moment of inertia relative to the centre of mass, or an axis or plane through the centre of mass, by I.

We have then for the moment of inertia of an area relative to the centre of mass, or any axis or plane through the centre of mass,

$$I = \Sigma ar^2.$$

For the moment of inertia of a volume we have

$$I = \Sigma vr^2.$$

For the moment of inertia of a mass,

$$I = \Sigma mr^2.$$

Significance of the Term.—The term "moment of inertia" is not well chosen according to the terminology of modern science. As we see from the definition, it has nothing to do with "inertia" (see page 169). It has been proposed to call it "second moment" of area or mass. The term "moment of inertia" is, however, of such general use that it is hardly worth while to try to introduce such a change. It is sufficient for the student to note that it is the arbitrary name for a certain quantity. This quantity occurs so often in mechanical problems that it is desirable to have a special name for it, and to discuss it thoroughly in advance of its use.

* This chapter may be omitted here if thought desirable, but should be taken before Kinetics of a Material System, page 297.

Radius of Gyration.—That distance at which, if the entire volume, area or mass were concentrated in a point, the moment of inertia would be the same as for the volume, area or body itself, is called the RADIUS OF GYRATION for the volume, area or body.

We denote the radius of gyration relative to the centre of mass, or a line or plane through the centre of mass, by k.

We have then, by our definition and notation, for a body of mass $\overline{\mathrm{m}}$,

$$I = \overline{\mathrm{m}}k^2, \quad \text{or} \quad k^2 = \frac{I}{\overline{\mathrm{m}}} = \frac{\Sigma mr^2}{\overline{\mathrm{m}}};$$

for an area A,

$$I = Ak^2, \quad \text{or} \quad k^2 = \frac{I}{A} = \frac{\Sigma ar^2}{A};$$

for a volume V,

$$I = Vk^2, \quad \text{or} \quad k^2 = \frac{I}{V} = \frac{\Sigma vr^2}{V}.$$

Moment of Inertia and Radius of Gyration in General.—Unless otherwise specified the term moment of inertia always signifies the moment of inertia for a body of mass $\overline{\mathrm{m}}$. The letter I always means this moment of inertia for the centre of mass, or an axis or plane through the centre of mass. The letter k denotes the corresponding radius of gyration.

Any other point, axis or plane is called an ECCENTRIC point, axis or plane. The moment of inertia is denoted in such case by I', and the corresponding radius of gyration by k'.

Everything in what follows which holds for I and k, I' and k', as thus defined, for a body of mass $\overline{\mathrm{m}}$, holds good also for a surface of area A or a volume V, unless otherwise specified.

Reduction of Moment of Inertia.—If, then, I is the moment of inertia for any axis through the centre of mass, and I' the moment of inertia for any parallel axis at the distance d, we can easily prove the relation

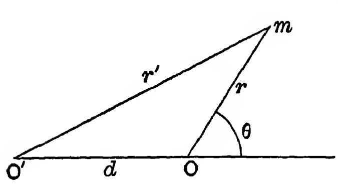
$$I' = I + \overline{\mathrm{m}}d^2.$$

That is, *the moment of inertia of a body relative to any eccentric axis is equal to the moment of inertia relative to a parallel axis through the centre of mass plus the product of the mass by the square of the distance between the two axes.*

The same holds for the moment of inertia of a surface if we replace $\overline{\mathrm{m}}$ by the area A of the surface, or for a volume if we replace $\overline{\mathrm{m}}$ by the volume V. We have then

$$I' = I + Ad^2, \qquad I' = I + Vd^2.$$

This is called the theorem of moment of inertia for parallel axes. By means of it we can find I' for any axis, if I for a *parallel axis* through the centre of mass and the distance d between these axes are given. Conversely, we can find I if I' and d are given.



The proof is simple. Let m be the mass of a particle of a body whose distance from an axis through the centre of mass O of the body is r. Let r' be its distance from a parallel axis at O', the distance $O'O$ between the axes being d. Let θ be the angle of r with $O'O$.

Then we have for any body

$$I' = \Sigma mr'^2 \quad \text{and} \quad I = \Sigma mr^2.$$

But we have

$$r'^2 = r^2 + d^2 \pm 2rd \cos \theta.$$

Hence

$$\Sigma mr'^2 = \Sigma mr^2 + d^2 \Sigma m \pm 2d\Sigma mr \cos \theta.$$

But $mr \cos \theta$ is the moment of m relative to the axis through the centre of mass O. Therefore, as we have seen (page 23),

$$\Sigma mr \cos \theta = 0.$$

We have then

$$\Sigma mr'^2 = \Sigma mr^2 + d^2 \Sigma m,$$

or, since $\Sigma m = \overline{\mathrm{m}}$, the entire mass of the body

$$I' = I + \overline{\mathrm{m}}d^2.$$

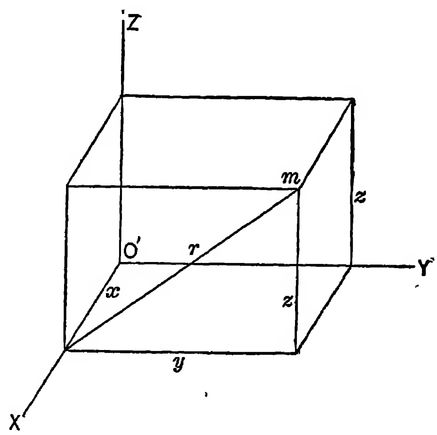
The same holds for an area if we replace $\overline{\mathrm{m}}$ by A and Σm by $\Sigma a = A$, or for a volume if we replace $\overline{\mathrm{m}}$ by V and Σm by $\Sigma v = V$.

We have then also

$$k'^2 = k^2 + d^2.$$

It is evident, then, that the moment of inertia relative to any axis through the centre of mass is *less than for any parallel axis*, and the radius of gyration relative to any axis through the centre of mass is *less than for any parallel axis*.

Moment of Inertia relative to an Axis.—Let $O'X$ be any axis and $XO'Y$, $XO'Z$ any two rectangular planes passing through that axis.



Then for any particle of a body of mass m whose co-ordinates are x, y, z, we have the moment of inertia relative to $O'X$

$$mr^2 = my^2 + mz^2 = m(y^2 + z^2).$$

Summing the moments of inertia for all the particles of the entire body, we have for the moment of inertia of the body relative to the axis $O'X$

$$\Sigma mr^2 = \Sigma my^2 + \Sigma mz^2 = \Sigma[m(y^2+z^2)].$$

But Σmr^2 is the moment of inertia of the body relative to the axis $O'X$. We denote it, therefore, by I'_x. Also, Σmy^2 is the moment of inertia of the body relative to the plane ZX, and Σmz^2 relative to the plane XY. We denote them, therefore, by I'_{zx} and I'_{xy}. We can therefore write

$$I'_x = I'_{xy} + I'_{zx} = \Sigma[m(y^2 + z^2)].$$

In the same way we have

$$I'_y = I'_{xy} + I'_{yz} = \Sigma[m(z^2 + x^2)],$$
$$I'_z = I'_{yz} + I'_{zx} = \Sigma[m(x^2 + y^2)];$$

or, *the moment of inertia of any body with reference to a line is equal to the sum of the moments of inertia for any two rectangular planes passing through that line.*

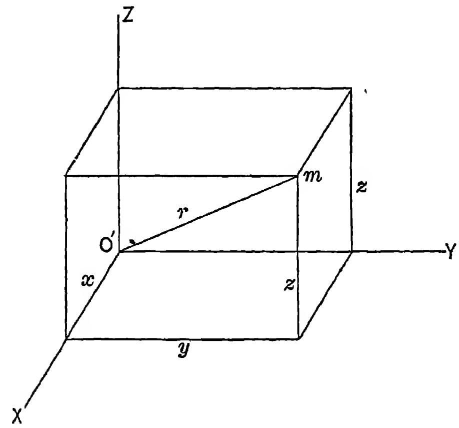
Polar Moment of Inertia for a Plane Area.—For any plane area, as $XO'Y$, we have $I_{xy} = 0$, and hence $I'_x = I'_{zx}$ and $I'_y = I'_{yz}$; hence

$$I'_z = I_y + I_x;$$

or, *the moment of inertia of any plane area relative to an axis perpendicular to the plane is equal to the sum of the moments of inertia for any two rectangular lines in the plane through the foot of the perpendicular.*

The moment of inertia for a plane area relative to an axes at right angles to the plane is called the POLAR moment of inertia relative to that line.

Moment of Inertia Relative to a Point.—Let O' be any point, $O'X$, $O'Z$ three co-ordinate planes.



Then for any particle of mass m whose co-ordinates are x, y, z, we have for the moment of inertia with reference to O'

$$mr^2 = mx^2 + my^2 + mz^2.$$

Summing the moments of inertia for all the particles, we have for the moment of inertia of the body with reference to O'

$$\Sigma mr^2 = \Sigma mx^2 + \Sigma my^2 + \Sigma mz^2.$$

But $\Sigma mr^2 = I'_0$ is the moment of inertia of the body with reference to the point O', and $\Sigma mx^2 = I'_{yz}$, $\Sigma my^2 = I'_{zx}$, $\Sigma mz^2 = I'_{xy}$, are the moments of inertia of the body with reference to the co-ordinate planes YZ, ZX, XY. Hence

$$I'_0 = I'_{xy} + I'_{yz} + I'_{zx}.$$

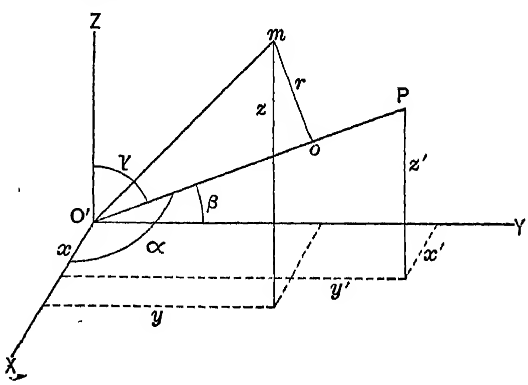
But we have just seen that $I'_{xy} + I'_{zx} = I'_x$. Hence

$$I'_0 = I'_x + I'_{yz}.$$

That is, *the moment of inertia with reference to any point is equal to the sum of the moments of inertia for any three rectangular planes through that point;*

or, *is equal to the sum of the moment of inertia for any line through the point and a plane through the point at right angles to this line.*

Moment of Inertia for an Inclined Axis, in General.—Let m be the mass of any particle of a body whose co-ordinates are x, y, z for any assumed origin O' and co-ordinate axes $O'X$, $O'Y$, $O'Z$, and let $O'o$ be any line through the origin O', making the angles α, β, γ with the co-ordinate axes. Let r be the perpendicular from m upon $O'o$.



Then we have the distance

$$O'o = x \cos \alpha + y \cos \beta + z \cos \gamma;$$

also, the square of the distance $O'm$,

$$\overline{O'm}^2 = x^2 + y^2 + z^2.$$

Hence $\qquad r^2 = \overline{O'm}^2 - \overline{O'o}^2$, or

$$r^2 = (x^2 + y^2 + z^2) - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2.$$

The moment of inertia of a particle of mass m relative to any axis $O'o$ is, then,

$$mr^2 = m(x^2 + y^2 + z^2) - m(x \cos \alpha + y \cos \beta + z \cos \gamma)^2.$$

Summing the moments of inertia for all the particles of the body we have, since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

$$\Sigma(mr^2) = \Sigma[m(x^2+y^2+z^2)(\cos^2\alpha + \cos^2\beta+\cos^2\gamma)] - \Sigma[m(x\cos\alpha + y\cos\beta + z\cos\gamma)^2].$$

Multiplying out and reducing, we can write this

$$\Sigma(mr^2) = \Sigma[m(y^2+z^2)]\cos^2\alpha + \Sigma[m(z^2+x^2)]\cos^2\beta + \Sigma[m(x^2+y^2)]\cos^2\gamma$$
$$- 2\cos\alpha\cos\beta\,\Sigma(mxy) - 2\cos\beta\cos\gamma\,\Sigma(myz) - 2\cos\gamma\cos\alpha\,\Sigma(mzx).$$

But, as we have seen, page 33,

$$\Sigma[m(y^2+z^2)] = I'_x, \qquad \Sigma[m(z^2+x^2)] = I'_y, \qquad \Sigma[m(x^2+y^2)] = I'_z,$$

are the moments of inertia of the body relative to the axis of X, Y, and Z. Also, $\Sigma(mr^2)$ is the moment of inertia I' relative to the axis $O'o$. Hence

$$I' = I'_x\cos^2\alpha + I_y\cos^2\beta + I'_z\cos^2\gamma - 2\cos\alpha\cos\beta\,\Sigma(mxy) - 2\cos\beta\cos\gamma\,\Sigma(myz)$$
$$- 2\cos\gamma\cos\alpha\,\Sigma(mzx). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) gives the moment of inertia of a body for any axis $O'o$ making any given angles α, β, γ with the co-ordinate axes.

Principal Axes.—If for any origin O' we take the rectangular axes X, Y, Z in such directions that for any body

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0,$$

these axes are called PRINCIPAL AXES *for the point* O', the moments of inertia I'_x, I'_y, I'_z relative to these principal axes are called PRINCIPAL MOMENTS OF INERTIA, and the corresponding radii of gyration k'_x, k'_y, k'_z are called PRINCIPAL RADII OF GYRATION.

If we take the origin at the centre of mass, the principal axes are called CENTRAL PRINCIPAL AXES, the corresponding moments of inertia I_x, I_y, I_z are central principal moments of inertia, and the corresponding radii of gyration k_x, k_y, k_z are central principal radii of gyration.

Properties of Principal Axes.—The three conditions for principal axes are then

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0.$$

Introducing these conditions in (1) we have

$$I' = I'_x\cos^2\alpha + I'_y\cos^2\beta + I'_z\cos^2\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, *the moment of inertia of a body relative to any line is equal to the sum of the products obtained by multiplying the moments of inertia for the principal axes for any point of the line, by the square of the cosines of the angles which the line makes* with these principal axes.

For any plane material area, taking the plane as that of ZX, we have for any point of the area given by x and z, $y = 0$. Hence, no matter where the origin nor what direction the axes of x and y may have, we have

$$\Sigma mxy = 0, \qquad \Sigma myz = 0.$$

Also, if a body has a plane of symmetry, then, taking this as the plane of ZX, we have for any origin in the plane, for any particle given by x, z and $+y$ above the plane, an equal particle given by x, z and $-y$ below. Hence, no matter where the origin nor what direction the axes of x and y may have, we have

$$\Sigma mxy = 0, \qquad \Sigma myz = 0.$$

Introducing these conditions in (1) we obtain

$$I' = I_x' \cos^2 \alpha + I_y' \cos^2 \beta + I_z' \cos^2 \gamma - 2 \cos \gamma \cos \alpha \Sigma\ mzx.$$

This reduces to (2) when $\beta = 0$, and hence $\alpha = 90$, $\gamma = 90$, and we have in such case $I' = I_y'$. That is, the perpendicular to the plane is a principal axis for the point of intersection.

It is not, however, in general a principal axis for any other point of the perpendicular. For if we take some other point on y at a distance d from the origin, then if the axis of y is a principal axis for this point also, we must have

$$\Sigma mx\,(y - d) = 0, \qquad \Sigma mz\,(y - d) = 0.$$

These conditions can only be satisfied when $\Sigma mx = 0$ and $\Sigma my = 0$, that is when the perpendicular passes through the centre of mass.

Hence, *if any two of the three conditions for principal axes are fulfilled, we have either a plane area or a plane of symmetry.*

Any perpendicular to a plane area or a plane of symmetry is a principal axis for the point of its intersection with the plane. If this point of intersection is the centre of mass, the perpendicular is a principal axis for any of its points.

A line cannot be a principal axis at more than one of its points, unless it passes through the centre of mass, in which case it is a principal axis at every one of its points.

Let a body have two planes of symmetry at right angles. Taking one plane as the plane of ZX, we have, as already proved,

$$\Sigma mxy = 0, \qquad \Sigma myz = 0,$$

and any perpendicular to this plane is a principal axis for its point of intersection. Taking the other plane as the plane of YZ, we have

$$\Sigma mxy = 0, \qquad \Sigma mzx = 0,$$

and any perpendicular to this plane is a principal axis for its point of intersection. We have then all three conditions

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0$$

fulfilled, and hence for two rectangular planes of symmetry *the principal axes for any point on the line of intersection of these two planes are this line of intersection and the two perpendiculars to it at the point in each plane.*

If a body has three planes of symmetry at right angles, their point of intersection is the centre of mass (page 23).

Hence for three rectangular planes of symmetry *any line of intersection is a principal axis for any of its points. The principal axes for any point on a line of intersection are this line and the two perpendiculars to it at the point in each plane. The three lines of intersection are principal axes for the centre of mass, or central principal axes.*

Ellipsoid of Inertia.—Since the mass $\overline{\mathfrak{m}}$ of a body multiplied by the square of its radius of gyration k for any axis gives the moment of inertia I for that axis, we have, by dividing equation (2) by $\overline{\mathfrak{m}}$,

$$k'^2 = k_x'^2 \cos^2 \alpha + k_y'^2 \cos^2 \beta + k_z'^2 \cos^2 \gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the preceding Fig., page 34, take a point P on the line $O'o$ at any convenient distance $O'P = l$ from O'. Denote the product lk' by s^2, so that

$$lk' = s^2, \quad \text{hence} \quad k'^2 = \frac{s^4}{l^2}.$$

Substituting in (3) we have

$$\frac{l^2 \cos^2 \alpha}{\frac{s^4}{k_x'^2}} + \frac{l^2 \cos^2 \beta}{\frac{s^4}{k_y'^2}} + \frac{l^2 \cos^2 \gamma}{\frac{s^4}{k_z'^2}} = 1.$$

Let the co-ordinates of the point P be x', y', z'. Then

$$l \cos \alpha = x', \qquad l \cos \beta = y', \qquad l \cos \gamma = z',$$

and we have

$$\frac{x'^2}{\frac{s^4}{k_x'^2}} + \frac{y'^2}{\frac{s^4}{k_y'^2}} + \frac{z'^2}{\frac{s^4}{k_z'^2}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equation of an ellipsoid referred to its major and minor axes is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1,$$

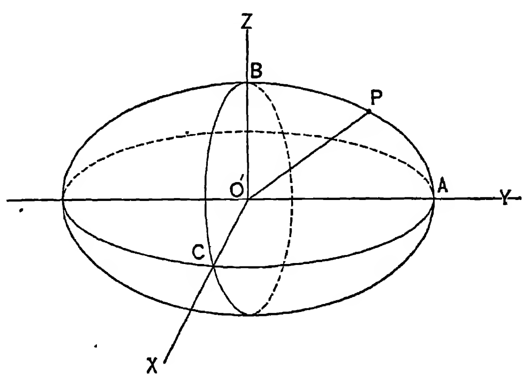
where A, B and C are the semi-axes.

We see, then, that (4) is the equation of an ellipsoid whose semi-axes are principal axes given by

$$A = \frac{s^2}{k_x'}, \qquad B = \frac{s^2}{k_y'}, \qquad C = \frac{s^2}{k_z'},$$

where s may have any convenient value.

Hence if we lay off on every line $O'o$ through the origin O' a distance $O'P = l = \frac{s^2}{k'}$, where the distance s may be taken any convenient distance, all the points P thus determined will lie on the surface of an ellipsoid whose equation is (4), whose axes are the principal axes for the point O', and whose semi-axes are given by the values of A, B and C just found.



The square of the reciprocal of any semi diameter l is, then,

$$\frac{1}{l^2} = \frac{k'^2}{s^4}.$$

This ellipsoid is called the ELLIPSOID OF INERTIA *for the point* O', because the square of the reciprocal of any semi-diameter $O'P = l$ multiplied by the mass $\overline{m}$ of the body, or $\frac{1}{s^4} \cdot \overline{m}k'^2$, is proportional to the moment of inertia $I' = mk'^2$ of the body relative to an axis coinciding with this semi-diameter, and if we take $s = 1$, will be equal to this moment of inertia.

The axes of this ellipsoid are principal axes for the point O'.

We see at once that the principal moments of inertia must include the greatest and least of all the moments of inertia for the point O', the least corresponding to the longest semi-diameter, $O'A$, and the greatest to the least semi-diameter, either $O'B$ or $O'C$.

For any point O', then, there must be at least one set of rectangular axes, OA, OB, OC, which are principal axes.

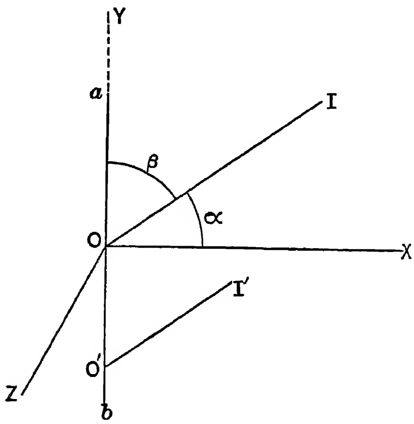
Determination of Moment of Inertia.—The moment of inertia of a body can be determined experimentally, as will be explained hereafter (page 340). For many bodies it is readily determined by a simple application of the Integral Calculus.

It is not necessary or desirable to occupy space here with such applications. It is of much greater importance that the student should understand clearly what the moment of inertia of a body is, as defined on page 31, and what use is made of it and how it enters into mechanical problems, as will be explained later (page 321).

In the following examples we show how the moment of inertia may be found for a number of simple cases, with and without the use of the Calculus.

Examples.—(1) *Find the moment of inertia for a homogeneous straight line.*

ANS. Let the line ab coincide with the axis of Y, and let O be its centre of mass. Then by the principle of symmetry OX, OY, OZ are central principal axes (page 35). Let the length of the line be L, and δ the linear density. Then the mass $\overline{m}$ is



$$\overline{m} = \delta L.$$

Divide the line above and below O into an indefinitely large number n of short elementary lines. The mass of each is $\delta\frac{L}{2n}$, and the distances of these elementary masses above and below OX or OZ are $\frac{L}{2n}$, $2\frac{L}{2n}$, $3\frac{L}{2n}$, etc. Multiply the mass of each element by the square of its distance, and summing the products, we have for the moment of inertia I_x or I_z relative to the central principal axes OX and OY

$$I_x = I_z = 2\,.\,\delta\frac{L}{2n}\left[\frac{L^2}{4n^2} + 4\frac{L^2}{4n^2} + 9\frac{L^2}{4n^2} + \dots\, n^2\frac{L^2}{4n^2}\right]$$

$$= \frac{\overline{m}L^2}{4n^3}(1 + 4 + 9 + \dots n^2).$$

The sum of the series

$$1 + 4 + 9 + \dots n^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n^3}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right).$$

As n increases, this sum approaches the limit $\frac{n^3}{3}$ Hence for n indefinitely large

$$I_x = I_z = \frac{\overline{m}L^2}{12}.$$

The moment of inertia I_y for the axis OY is $I_y = 0$

These are the moments of inertia for the central principal axes.

Let OI be any inclined axis in the plane XY making the angles α, β, γ with the central principal axes. Then $\cos\gamma = 0$ and $\cos\alpha = \sin\beta$. We have then from equation (2), page 35, the moment of inertia I for any axis through the centre of mass making the angle β with the line

$$I = I_x \sin^2\beta = \frac{\overline{m}L^2}{12}\sin^2\beta.$$

For a parallel axis $O'I'$ through any point O' of the line, that is for an axis through any point of the line, making the angle β with the line, if $\bar{y}$ is the distance $O'O$ from the point O' to the centre of mass, we have, from page 32,

$$I' = I + \bar{m}\bar{y}^2 \sin^2\beta = \bar{m}\left(\frac{L^2}{12} + \bar{y}^2\right)\sin^2\beta. \quad \text{(1)}$$

We can write this last in the form

$$I' = \frac{\bar{m}}{6}\left(\frac{L}{2} + \bar{y}\right)^2 \sin^2\beta + \frac{\bar{m}}{6}\left(\frac{L}{2} - \bar{y}\right)^2 \sin^2\beta + \frac{2}{3}\bar{m}\bar{y}^2 \sin^2\beta.$$

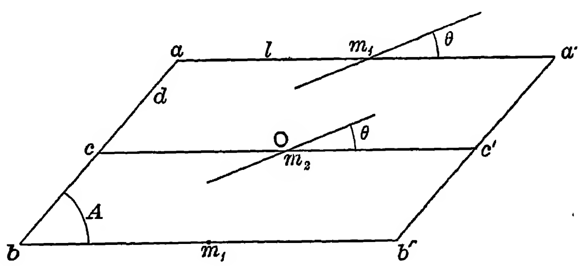
From this we see at once that the moment of inertia of a homogeneous line for any axis whatever *is the same as for a system of three material particles consisting of one sixth the mass of the line at each end and two thirds the mass of the line at the centre.*

BY CALCULUS.—The mass of any element of the line is δdy. The moment of inertia of the element relative to OI is then $\delta dy \times y^2 \sin^2\beta$. Hence, since $\bar{m} = \delta L$,

$$I = \int_{-\frac{L}{2}}^{+\frac{L}{2}} \delta y^2 dy \,.\, \sin^2\beta = \frac{\bar{m}L^2}{12}\sin^2\beta.$$

(2) *Find the moment of inertia for a system consisting of three parallele equidistant homogeneous straight lines of equal length l, the ends being in parallel straight lines, the mass of the centre line being m_2 and of each of the outer lines m_1.*

ANS. Let aa', bb', cc' be the lines of length l, of mass m_1, m_1 and m_2 respectively, and let d be the length of the line ab through the ends, and A be the angle abb'.



The centre of mass O of the system is evidently at the middle of the centre line.

The moment of inertia of aa' relative to any axis through its centre of mass in the plane of the lines making the angle θ with aa' is, from example (1),

$$\frac{m_1}{12}l^2 \sin^2\theta.$$

For a parallel axis through the centre of mass O, since the distance between the axes is $\frac{d}{2}\sin(A - \theta)$, we have (page 32)

$$\frac{m_1}{12}l^2 \sin^2\theta + \frac{m_1 d^2}{4}\sin^2(A - \theta).$$

We have the same result for the line bb'. For the line cc' we have

$$\frac{m_2}{12}l^2 \sin^2\theta.$$

The moment of inertia of the system relative to any axis in the plane of the lines through the centre of mass O, making the angle θ with the lines, is then

$$I = \frac{m_1}{6}l^2 \sin^2\theta + \frac{m_1 d^2}{2}\sin^2(A - \theta) + \frac{m_2}{12}l^2 \sin^2\theta.$$

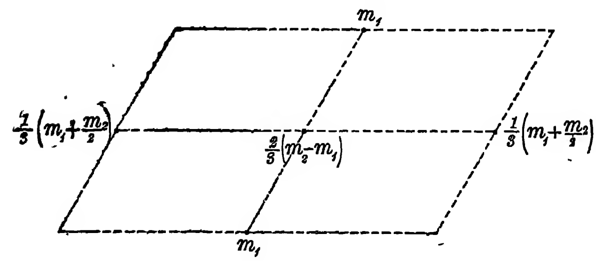
For any parallel axis in the plane at a distance p we have, then (page 32),

$$I' = \left(m_1 + \frac{m_2}{2}\right)\frac{l^2}{6}\sin^2\theta + \frac{m_1 d^2}{2}\sin^2(A - \theta) + (2m_1 + m_2)p^2. \quad \text{(2}$$

We can write this in the equivalent form

$$I' = \frac{1}{3}\left(m_1 + \frac{m_2}{2}\right)\left[\left(\frac{l}{2}\sin\theta + p\right)^2 + \left(\frac{l}{2}\sin\theta - p\right)^2\right] + m_1\left[\left(\frac{d}{2}\sin(A-\theta) + p\right)^2 + \left(\frac{d}{2}\sin(A-\theta) - p\right)^2\right] + \frac{2}{3}p^2(m_2 - m_1).$$

We see from this that the moment of inertia for the system *is the same as for a particle of mass m_1 at the middle of each outer line, a particle of mass $\frac{1}{3}\left(m_1 + \frac{m_2}{2}\right)$ at each end of the centre line, and a particle of mass $\frac{2}{3}(m_2 - m_1)$ at the centre of mass of the system.*



SPECIAL CASES.—If $m_2 = 4m_1$, we have m_1 at the middle of each outer line, m_1 at each end of the centre line, and $2m_1$ at the centre of mass of the system.

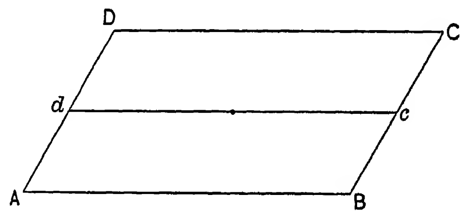
If $m_2 = 2m_1$, we have m_1 at the middle of each outer line, $\frac{2}{3}m_1$ at each end of the centre line and $\frac{2}{3}m_1$ at the centre of mass of the system.

If $m_2 = m_1$, we have m_1 at the middle of each outer line and $\frac{1}{2}m_1$ at each end of the middle line.

(3) *Find the moment of inertia for a homogeneous parallelogram and rectangle.*

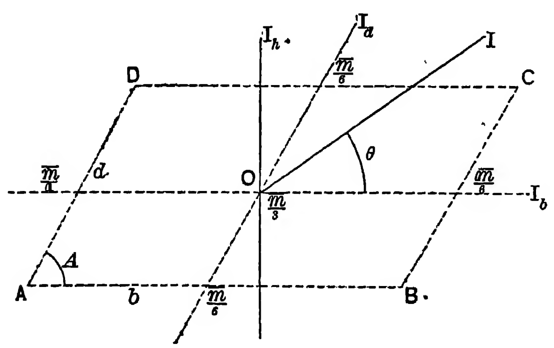
ANS. We have seen that the moment of inertia for a homogeneous straight line for any axis is the same as for a particle of one sixth its mass at each end and of four sixths its mass at the middle point.

Since the parallelogram $ABCD$ is generated by a straight line AD moving parallel to itself, the particles at A and D will generate lines AB and DC, the mass of each being one sixth the mass of the parallelogram, and the particle at d will generate the line dc, whose mass is four sixths the mass of the parallelogram.



We have then a system of three parallel lines, the mass m_1 of each outer line being $m_1 = \frac{1}{6}\overline{m}$ and the mass of the centre line being $m_2 = \frac{4}{6}\overline{m}$, where $\overline{m}$ is the mass of the parallelogram.

We have then $m_2 = 4m_1$, and, as just proved in the preceding example, *the moment of inertia of the parallelogram is the same as for a particle of mass $m_1 = \frac{1}{6}\overline{m}$ at the middle of each outer line and at the ends of the centre line, and a particle of $\frac{2}{6}\overline{m}$ at the centre of mass.*



For the axis I_b through the centre of mass parallel to the base $AB = b$ we have then

$$I_b = \frac{\overline{m}}{12} d^2 \sin^2 A.$$

For the axis I_d through the centre of mass parallel to the side $AD = d$ we have

$$I_d = \frac{\overline{m}}{12} b^2 \sin^2 A.$$

For the axis I_h through the centre of mass perpendicular to the base AB we have

$$I_h = \frac{\overline{m}}{12}(b^2 + d^2 \cos^2 A).$$

For the axis OZ through the centre of mass O at right angles to the plane we have (page 34)

$$I_z = I_b + I_h = \frac{\overline{m}}{12}(b^2 + d^2).$$

For any axis in the plane through the centre of mass O making the angle θ with the base AB we have

$$I = \frac{\overline{m}}{12} b^2 \sin^2\theta + \frac{\overline{m}}{12} d^2 \sin^2(A - \theta).$$

For any parallel axis in the plane at a distance p (page 33)

$$I' = \frac{\overline{m}}{12}b^2 \sin^2 \theta + \frac{\overline{m}}{12}d^2 \sin^2 (A - \theta) + \overline{m}p^2. \qquad (3)$$

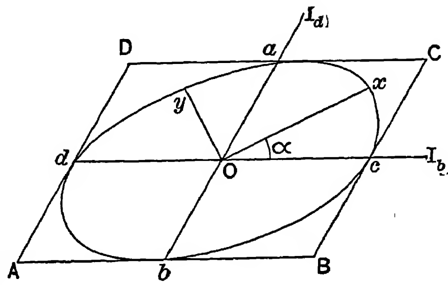
The axis OZ is a principal axis (page 36), but the axes for I_b and I_h are not principal axes.

PRINCIPAL AXES.—If δ is the surface density, the mass $\overline{m}$ of the parallelogram is

$$\overline{m} = \delta bd \sin A.$$

Hence we have

$$I_b = \frac{\overline{m}^3}{48\delta^2} \cdot \frac{1}{\left(\frac{b}{2}\right)^2}, \qquad I_d = \frac{\overline{m}^3}{48\delta^2} \cdot \frac{1}{\left(\frac{d}{2}\right)^2}.$$



If, then, we inscribe an ellipse in the parallelogram tangent to the sides at their middle points, the square of the reciprocal of any semi-diameter is proportional to the moment of inertia for the coincident axis. Hence this ellipse is the central ellipse of inertia (page 37), and if l is the length of any semi-diameter and I the moment of inertia for the coincident axis, we have

$$l^2 = \frac{\overline{m}^3}{48\delta^2 I}.$$

Let OX, OY be the central principal semi-axes denoted by l_x, l_y, and I_x, I_y the corresponding moments of inertia. Then

$$l_x^2 = \frac{\overline{m}^3}{48\delta^2 I_x}, \qquad l_y^2 = \frac{\overline{m}^3}{48\delta^2 I_y}.$$

Now in an ellipse the area of the circumscribed parallelogram is equal to four times the product of the principal semi-axes, or

$$4l_x l_y = bd \sin A = \frac{\overline{m}}{\delta}.$$

Inserting the values of l_x and l_y we have, since $I_x = \overline{m}k_x^2$ and $I_y = \overline{m}k_y^2$,

$$k_x k_y = \frac{\overline{m}}{12\delta},$$

where k_x and k_y are the central principal radii of gyration.

We have also (page 34)

$$I_z = I_x + I_y, \quad \text{or} \quad \frac{b^2 + d^2}{12} = k_x^2 + k_y^2.$$

Solving these two equations we obtain for the central principal axes

$$k_x^2 = \frac{1}{24}\left[b^2 + d^2 + \sqrt{(b^2 + d^2)^2 - \frac{4\overline{m}^2}{\delta^2}} \right],$$

$$k_y^2 = \frac{1}{24}\left[b^2 + d^2 - \sqrt{(b^2 + d^2)^2 - \frac{4\overline{m}^2}{\delta^2}} \right].$$

We thus know the central principal moments of inertia $I_x = \overline{m}k_x^2$, $I_y = \overline{m}k_y^2$.

If the principal axis OX makes the angle α with the base AB, we have

$$I_b = I_x \cos^2 \alpha + I_y \sin^2 \alpha.$$

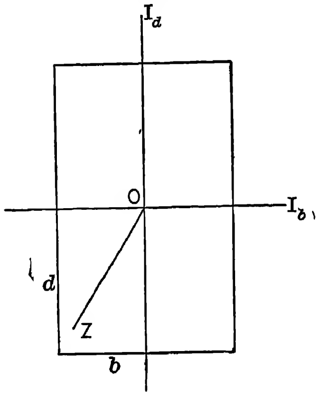
Hence

$$\cos^2 \alpha = \frac{I_b - I_y}{I_x - I_y} = \frac{\frac{d^2 \sin^2 A}{12} - k_y^2}{k_x^2 - k_y^2}.$$

We thus know the position of the principal axis of X and the principal moments of inertia I_x, I_y I_z. The moment of inertia for any axis through the centre of mass is then given by

$$I = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma.$$

For a rectangle we have $A = 90$, and hence for any axis equation (2) becomes



$$I' = \frac{\overline{m}}{12}[b^2 \sin^2 \theta + d^2 \cos^2 \theta] + \overline{m}p^2.$$

For the axis OI_b, $p = 0$, $\theta = 0$ and

$$I_b = \frac{\overline{m}d^2}{12}.$$

For the axis OI_d, $p = 0$, $\theta = 90$ and

$$I_d = \frac{\overline{m}b^2}{12}.$$

For the axis OZ

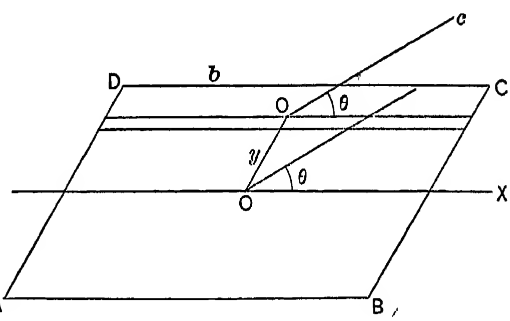
$$I_z = I_b + I_d = \frac{\overline{m}}{12}(b^2 + d^2).$$

In this case $\overline{m} = \delta bd$, and we have from our general formula $k_y^2 = \frac{d^2}{12}$, or $I_y = \frac{\overline{m}d^2}{12}$. Hence $\cos^2 \alpha = 0$, or $\alpha = 90°$. Hence the axes of I_b, I_d and I_z are central principal axes, as they should be, since the planes of I_bI_d and I_dOZ are planes of symmetry (page 36)

BY CALCULUS.—The mass of an element parallel to AB at a distance y from OX is $m = \delta b dy$. Its moment of inertia for an axis Oc making the angle θ with AB is, from (1),

$$\frac{mb^2 \sin^2 \theta}{12},$$

and transferring to a parallel axis through the centre of mass O at the distance $\frac{y \sin (A - \theta)}{\sin A}$, we have



$$\frac{mb^2 \sin^2 \theta}{12} + my^2 \frac{\sin^2 (A - \theta)}{\sin^2 A}.$$

Hence

$$I = \int_{-\frac{d \sin A}{2}}^{+\frac{d \sin A}{2}} \frac{\delta b^3 dy \sin^2 \theta}{12} + \frac{\delta b y^2 dy \sin^2 (A - \theta)}{\sin^2 A},$$

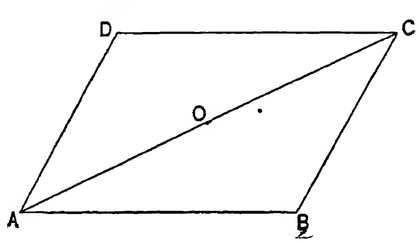
or, since $\delta bd \sin A = \overline{m}$,

$$I = \frac{\overline{m}}{12}[b^2 \sin^2 \theta + d^2 \sin^2 (A - \theta)],$$

as already found.

(4) *Find the moment of inertia for a homogeneous triangle.*

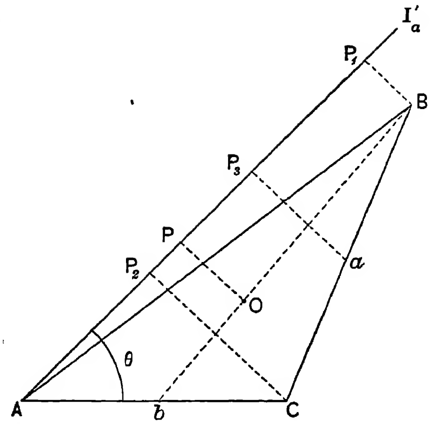
ANS We have just seen in example (3) that for a homogeneous parallelogram the moment of inertia for any axis is the same as for a particle of one sixth the mass of the parallelogram at the middle point of each side, and a particle of two sixths the mass of the parallelogram at the centre of mass.



If, then, we draw the diagonal AC, we divide the parallelogram into two equal triangles, and if $\overline{m}$ is the mass of each triangle, we have $\frac{1}{3}\overline{m}$ at the middle point of each side.

Hence for any axis the moment of inertia of a homogeneous triangle *is the same as for a particle of one third the mass of the triangle at the middle point of each side.*

Let ABC be a triangle whose angles are A, B, C and sides a, b, c. Take any axis AI'_a through the vertex A in the plane of the triangle, making the angle θ with the base b. Drop the perpendiculars p_1 and p_2 from B and C, the perpendicular p_3 from the middle point of the side a, and the perpendicular p from the centre of mass O.



Then we have

$$p_1 = c \sin(\theta - A), \qquad p_2 = b \sin\theta,$$

$$p_3 = \frac{p_1 + p_2}{2} = \frac{b \sin\theta + c \sin(\theta - A)}{2}.$$

If p_1 is less than $\frac{b}{2}\sin\theta$, we have

$$p = p_1 + \frac{2}{3}\left(\frac{b}{2}\sin\theta - p_1\right), \quad \text{or} \quad p = \frac{1}{3}(p_1 + b\sin\theta).$$

If p_1 is greater than $\frac{b}{2}\sin\theta$, we have

$$p = \frac{b}{2}\sin\theta + \frac{1}{3}\left(p_1 - \frac{b}{2}\sin\theta\right), \quad \text{or} \quad p = \frac{1}{3}(p_1 + b\sin\theta).$$

In all cases, then,

$$p = \frac{1}{3}[\, b\sin\theta + c\sin(\theta - A)\,].$$

If, then, we take one third of the mass $\overline{m}$ at the middle point of each side, we have for the moment of inertia for the axis I'_a

$$I'_a = \frac{\overline{m}}{3}\cdot\frac{p_1^2}{4} + \frac{\overline{m}}{3}\cdot\frac{p_2^2}{4} + \frac{\overline{m}}{3}p_3^2,$$

or

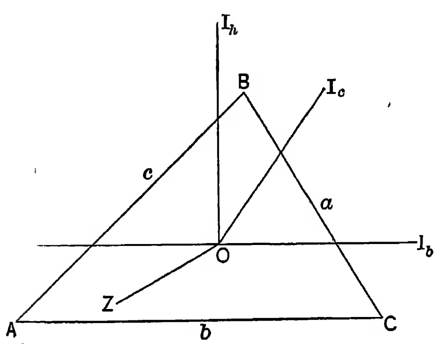
$$I'_a = \frac{\overline{m}}{6}[b^2\sin^2\theta + c^2\sin^2(\theta - A) + bc\sin\theta\sin(\theta - A)].$$

Reducing to a parallel axis through the centre of mass O by subtracting $\overline{m}p^2$ (page 33), we have

$$I = \frac{\overline{m}}{18}[b^2\sin^2\theta + c^2\sin^2(\theta - A) - bc\sin\theta\sin(\theta - A)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For any axis in the plane of the triangle, then, if p is the perpendicular distance from the centre of mass to the axis,

$$I' = \frac{\overline{m}}{18}[b^2\sin^2\theta + c^2\sin^2(\theta - A) - bc\sin\theta\sin(\theta - A)] + \overline{m}p^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$



For the axis I_b through the centre of mass O parallel to the base b we have $p = 0$, $\theta = 0$, and equation (3) becomes

$$I_b = \frac{\overline{m}}{18}c^2\sin^2 A.$$

For the axis I_h through the centre of mass perpendicular to the base b we have $p = 0$, $\theta = 90°$, and equation (3) becomes

$$I_h = \frac{\overline{m}}{18}[b^2 + c^2\cos^2 A - bc\cos A].$$

For the axis I_c through the centre of mass parallel to the side c we have $p = 0$ $\theta = A$, and equation (3) becomes

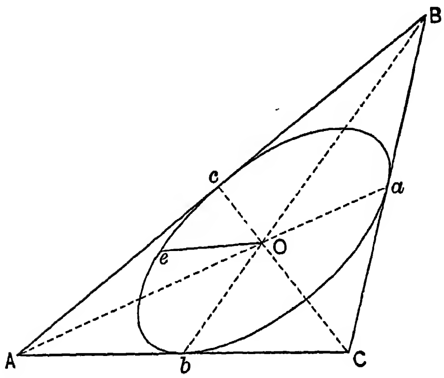
$$I_c = \frac{\overline{m}}{18}b^2\sin^2 A.$$

For the axis OZ through the centre of mass at right angles to the plane of the triangle we have (page 34)

$$I_z = I_b + I_h = \frac{\mathfrak{m}}{18}(b^2 + c^2 - bc \cos A),$$

or, since $2\,bc \cos A = b^2 + c^2 - a^2$, we can write

$$I_z = \frac{\mathfrak{m}}{36}(a^2 + b^2 + c^2).$$



The axis OZ is a principal axis, but the axes of I_b and I_h are not principal axes.

Principal Axes.—For the axis Bb through the vertex B and middle point of AC we have $p = 0$ and $\theta - A$ equal to the angle ABb, and hence

$$c \sin(\theta - A) = \frac{b}{2} \sin\theta,$$

and equation (3) becomes

$$I_B = \frac{\mathfrak{m}}{24} b^2 \sin^2\theta.$$

We also have $\overline{Bb} \sin\theta = c \sin A$, or

$$\sin^2\theta = \frac{c^2 \sin^2 A}{\overline{Bb}^2} = \frac{c^2 \sin^2 A}{9 \times \overline{Ob}^2}.$$

Hence, substituting this value of $\sin^2\theta$,

$$I_B = \frac{\mathfrak{m} b^2 c^2 \sin^2 A}{216} \cdot \frac{1}{\overline{Ob}^2}.$$

In the same way we have for the axis Cc

$$I_C = \frac{\mathfrak{m} b^2 c^2 \sin^2 A}{216} \cdot \frac{1}{\overline{Oc}^2},$$

and for the axis Aa

$$I_A = \frac{\mathfrak{m} b^2 c^2 \sin^2 A}{216} \cdot \frac{1}{\overline{Oa}^2}.$$

We see, then, that the moments of inertia relative to Bb, Cc and Aa are proportional to $\frac{1}{\overline{Ob}^2}$, $\frac{1}{\overline{Oc}^2}$, $\frac{1}{\overline{Oa}^2}$. Hence if we inscribe an ellipse in the triangle tangent at b, c and a, it will be the central ellipse of inertia, and the reciprocal of the square of any semi-diameter will be proportional to the moment of inertia for the coincident axis.

If, then, we draw the semi-diameter Oe parallel to AC, we have

$$\overline{Oe}^2 = \frac{\mathfrak{m} b^2 c^2 \sin^2 A}{216 I_b} = \frac{b^2}{12}.$$

Now Oe and Ob are conjugate semi-diameters, and since the parallelogram upon two conjugate semi-diameters is equal to the rectangle of the principal semi-axes,

$$\overline{Ob} \times \overline{Oe} \sin\theta = \frac{b^2 c^2 \sin^2 A}{216\, k_x k_y}, \quad \text{or} \quad \overline{Ob}^2 \times \overline{Oe}^2 \sin^2\theta = \frac{b^4 c^4 \sin^4 A}{216^2 k_x^2 k_y^2}.$$

Inserting the value of $\sin^2\theta$ and of $\overline{Oe}^2$ already found, we have

$$k_x^2 k_y^2 = \frac{b^2 c^2 \sin^2 A}{432}.$$

We have also

$$I_x + I_y = I_z, \quad \text{or} \quad k_x^2 + k_y^2 = \tfrac{1}{36}(a^2 + b^2 + c^2).$$

Solving these two equations we obtain

$$k_x^2 = \tfrac{1}{72}\left[(a^2 + b^2 + c^2) + \sqrt{(a^2 + b^2 + c^2)^2 - 12b^2c^2 \sin^2 A}\right],$$

$$k_y^2 = \tfrac{1}{72}\left[(a^2 + b^2 + c^2) - \sqrt{(a^2 + b^2 + c^2)^2 - 12b^2c^2 \sin^2 A}\right].$$

We thus know the central principal moments of inertia $I_x = \overline{\text{m}}k_x^2$ and $I_y = \overline{\text{m}}k_y^2$.
If the principal axis of X makes the angle α with the base $b = AC$, we have

$$I_b = I_x \cos^2 \alpha + I_y \sin^2 \alpha, \quad \text{or} \quad \frac{\overline{\text{m}}}{18} c^2 \sin^2 A = \overline{\text{m}}k_x^2 \cos^2 \alpha + \overline{\text{m}}k_y^2 \sin^2 \alpha.$$

Hence

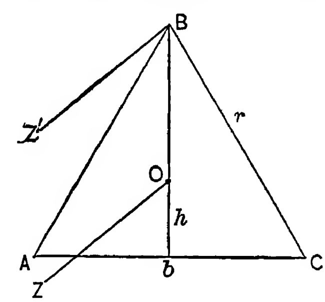
$$\cos^2 \alpha = \frac{I_b - I_y}{I_x - I_y} = \frac{\dfrac{c^2 \sin^2 A}{18} - k_y^2}{k_x^2 - k_y^2}.$$

We thus know the position of the principal axis of X and the principal moments of inertia I_x, I_y, I_z. The moment of inertia for any axis through the centre of mass is then given by

$$I = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma.$$

(5) *Find the moment of inertia for a homogeneous circular disc.*

ANS. Let b be the base AC of an isosceles triangle ABC, and r, r, the equal sides and h the altitude. Then $h^2 = r^2 - \frac{b^2}{4}$ and, from example (4), we have for the axis OZ through the centre of mass at right angles to the plane of the triangle

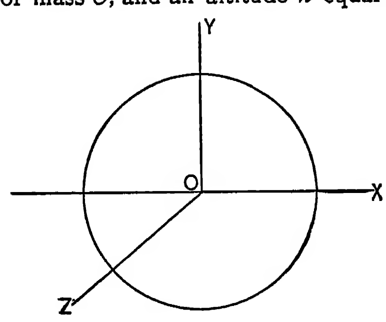


$$I_z = \frac{\overline{\text{m}}}{36}(b^2 + 2r^2) = \frac{\overline{\text{m}}}{18}(3r^2 - 2h^2).$$

For a parallel axis BZ' through the vertex B we have

$$I_z' = \frac{\overline{\text{m}}}{18}(3r^2 - 2h^2) + \overline{\text{m}}\left(\frac{2}{3}h\right)^2 = \frac{\overline{\text{m}}}{3}\left(h^2 + \frac{r^2}{2}\right).$$

Now the circular disc may be considered as composed of an indefinitely large number of isosceles triangles all having a vertex at the centre of mass O, and an altitude h equal to r. We have therefore, for a circular disc, for the axis OZ through the centre of mass O at right angles to the plane of the disc



$$I_z = \frac{\overline{\text{m}}}{3}\left(r^2 + \frac{r^2}{2}\right) = \frac{\overline{\text{m}}r^2}{2}.$$

Any three rectangular planes through the centre of mass are planes of symmetry. Hence (page 36) any three rectangular axes with origin at centre of mass O are central principal axes.

We have also $\quad I_z = I_x + I_y$;
or, since I_x and I_y are evidently equal,

$$I_x = I_y = \frac{I_z}{2} = \frac{\overline{\text{m}}r^2}{4}. \qquad (5)$$

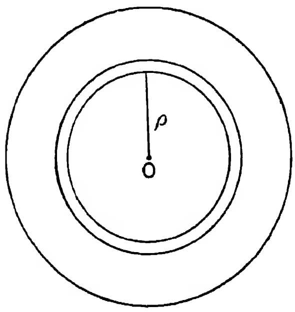
For any axis in the plane of the disc we have, then, if p is the distance from the centre of mass to the axis,

$$I' = \frac{\overline{\text{m}}r^2}{4} \cos^2 \alpha + \frac{\overline{\text{m}}r^2}{4} \sin^2 \alpha + \overline{\text{m}}p^2.$$

We can write this in the form

$$I' = \frac{\overline{\text{m}}}{8}(r \cos \alpha + p)^2 + \frac{\overline{\text{m}}}{8}(r \cos \alpha - p)^2 + \frac{\overline{\text{m}}}{8}(r \sin \alpha - p)^2 + \frac{\overline{\text{m}}}{8}(r \sin \alpha + p)^2 + \frac{\overline{\text{m}}}{2}p^2.$$

Hence the moment of inertia for a homogeneous circular disc *is the same as for a particle of one eighth the mass of the disc at the extremities of any two rectangular diameters, and a particle of one half the mass of the disc at the centre.*



BY CALCULUS.—Take an elementary circular strip of radius ρ and thickness $d\rho$. The area of this strip is $2\pi\rho d\rho$, and its mass is

$$m = 2\delta\pi\rho d\rho.$$

Its moment of inertia for the axis OZ is, then,

$$m\rho^2 = 2\delta\pi\rho^3 d\rho.$$

For the disc, then, we have

$$I_z = \int_0^r 2\delta\pi\rho^3 d\rho = \frac{\delta\pi r^4}{2}.$$

But $\delta\pi r^2$ is the mass $\overline{m}$ of the disc. Hence

$$I_z = \frac{\overline{m}r^2}{2},$$

as already found.

(6) *Find the moment of inertia for a homogeneous hollow circular disc.*

ANS. Let r_1 be the outer and r_2 the inner radius. Then, if δ is the surface density, we have for the mass of a disc of radius r_1

$$\overline{m}_1 = \delta\pi r_1^2,$$

and, from example (5),

$$I_z = \overline{m}_1 \frac{r_1^2}{2} = \delta\pi r_1^2 \cdot \frac{r_1^2}{2}.$$

For the mass of a disc of radius r_2 we have

$$\overline{m}_2 = \delta\pi r_2^2,$$

and

$$I_z = \overline{m}_2 \cdot \frac{r_2^2}{2} = \delta\pi r_2^2 \cdot \frac{r_2^2}{2}.$$

Hence the polar moment of inertia for the hollow disc is

$$I_z = \frac{\delta\pi}{2}(r_1^4 - r_2^4) = \frac{\delta\pi}{2}(r_1^2 - r_2^2)(r_1^2 + r_2^2).$$

But $\delta\pi(r_1^2 - r_2^2)$ is the mass $\overline{m}$ of the hollow disc. Hence

$$I_z = \frac{\overline{m}}{2}(r_1^2 + r_2^2).$$

Hence for any diameter

$$I_x = I_y = \frac{I_z}{2} = \frac{\overline{m}}{4}(r_1^2 + r_2^2). \quad \text{. . . . .} \quad (6)$$

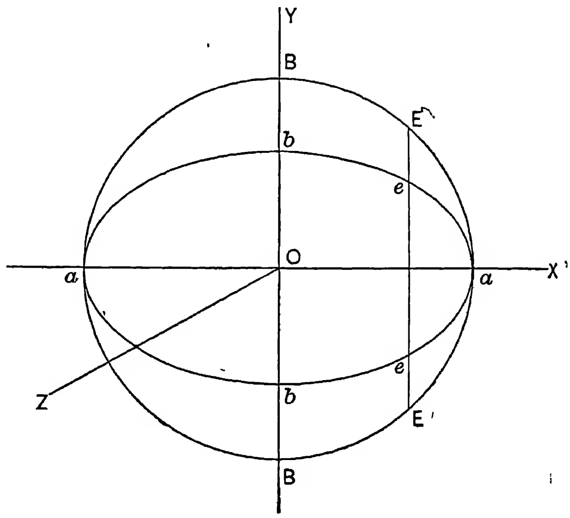
BY CALCULUS.—We have, as in the preceding example,

$$I_z = \int_{r_2}^{r_1} 2\delta\pi\rho^3 d\rho = \frac{\delta\pi}{2}(r_1^4 - r_2^4),$$

as already found.

(7) *Find the moment of inertia for a homogeneous elliptical disc*

ANS. Let the semi-transverse axis Oa be a and the semi-conjugate axis Ob be b. Let a circle be described about the ellipse so that its radius Oa is equal to the semi-transverse axis a.



Then we have for the ratio of the mass of any element ee of the ellipse to that of the corresponding element EE of the circle

$$\frac{ee}{EE} = \frac{bb}{BB} = \frac{b}{a}.$$

Hence the moment of inertia of the ellipse relative to the principal axis OY is $\frac{b}{a}$ times that of the circle. In the same way the moment of inertia of the ellipse relative to the principal axis OX is $\frac{a}{b}$ times that of the circle.

We have then, from example (5), since $\delta\pi a^2$ is the mass of the circle and $\delta\pi ab = \overline{m}$ is the mass of the ellipse,

$$I_y = \overline{m}\frac{a^2}{4}, \qquad I_x = \overline{m}\frac{b^2}{4},$$

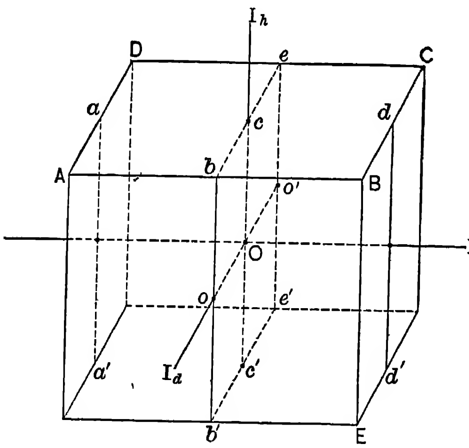
and for the axis OZ

$$I_z = \overline{m}\frac{a^2 + b^2}{4}.$$

Hence *the moment of inertia for a homogeneous elliptical disc is the same as for a particle of one eighth the mass of the ellipse at the extremities of the two principal axes, and a particle of one half the mass of the ellipse at its centre of mass.*

(8) *Find the moment of inertia for a homogeneous right parallelopipedon.*

ANS. We have seen, from example (3), that for a parallelogram we have one sixth of the mass at the middle of each side and two sixths the mass at the centre.



If this parallelogram moves parallel to itself it describes the parallelopipedon, and the centre and middle points describe straight lines, aa', bb', dd', ee', each of mass $m_1 = \frac{1}{6}\overline{m}$ and the straight line cc' of mass $m_2 = \frac{1}{3}\overline{m}$, where $\overline{m}$ is the mass of the parallelopipedon.

We have, then, two systems of three lines each, viz., the system aa', dd', each of mass $m_1 = \frac{1}{6}\overline{m}$, and cc' of mass $m_2 = \frac{1}{6}\overline{m}$, and the system bb', ee', each of mass $m_1 = \frac{1}{6}\overline{m}$, and cc' of mass $m_2 = \frac{1}{6}\overline{m}$.

From example (2), page 39, we have, for the first system $m_1 = \frac{1}{6}\overline{m}$ at the middle points of aa' and dd', and $\frac{1}{2}m_1 = \frac{1}{12}\overline{m}$ at c and c'. For the second system we have $m_1 = \frac{1}{6}\overline{m}$ at the middle points of bb' and ee', and $\frac{1}{2}m_1 = \frac{1}{12}\overline{m}$ at c and c'.

The moment of inertia for any axis is then the same as for a particle of one sixth the mass of the parallelopipedon at the middle point of each face.

Thus if $AB = b$, $BC = d$, and $BE = h$, we have for the axis I_b through the centre of mass O parallel to AB

$$I_b = \frac{\overline{m}}{12}(h^2 + d^2 \sin^2 A).$$

For the axis I_d through the centre of mass O parallel to BC

$$I_d = \frac{\overline{m}}{12}(h^2 + b^2 \sin^2 A).$$

For the axis I_h through the centre of mass parallel to BE

$$I_h = \frac{\overline{m}}{12}(b^2 + d^2).$$

The axis I_h is a principal axis. The axes of I_b and I_d are not principal axes. The principal axes in the middle section have the same position as for a parallelogram, page 41.

If the ends are rectangular, we have $A = 90°$, and I_b, I_h and I_d are the principal axes. In this case we have

$$I_b = \frac{\overline{m}}{12}(h^2 + d^2), \qquad I_d = \frac{\overline{m}}{12}(h^2 + b^2), \qquad I_h = \frac{\overline{m}}{12}(b^2 + d^2).$$

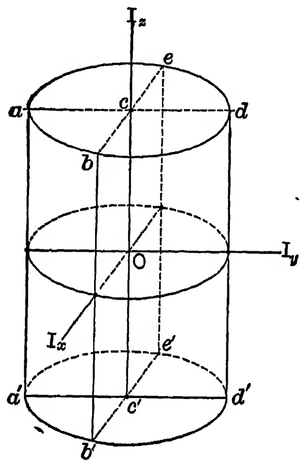
For a cube $b = d = h$ and

$$I_b = I_d = I_h = \frac{\overline{m}}{12} D^2,$$

where D is the diagonal of a face.

(9) *Find the moment of inertia for a homogeneous right cylinder.*

ANS. We have seen from examples (5) and (7) that for a circular or elliptical disc we have one eighth of the mass at the extremities of two principal axes and four eighths of the mass at the centre.



If the disc moves parallel to itself, it generates the cylinder, and the centre and extremities of the principal axes describe straight lines.

We have, then, two systems of three lines each, viz., the system aa', dd' each of mass $m_1 = \frac{1}{8}\overline{m}$, and cc' of mass $m_2 = \frac{4}{8}\overline{m}$, and the system bb', ee', each of mass $m_1 = \frac{1}{8}\overline{m}$, and cc' of mass $m_2 = \frac{4}{8}\overline{m}$.

From example (2), page 39, we have for the first system $m_1 = \frac{1}{8}\overline{m}$ at the middle points of aa' and dd', $\frac{4}{6}m_1 = \frac{1}{12}\overline{m}$ at c and c', and $\frac{4}{6}m_1 = \frac{1}{12}\overline{m}$ at the centre of mass O. For the second system we have also $\frac{1}{8}\overline{m}$ at the middle points of bb' and ee', $\frac{1}{12}\overline{m}$ at c and c' and $\frac{1}{12}\overline{m}$ at the centre of mass O.

The moment of inertia for any axis is, then, *the same as for a particle of one eighth the mass of the cylinder at the extremities of two principal axes of the middle section, and a particle of one sixth the mass at the centre of mass and at the centre of mass of each end.*

If l is the length and r the radius for circular base, we have for the principal axes through the centre of mass

$$I_x = I_y = \frac{\overline{m}}{4}\left(r^2 + \frac{l^2}{3}\right), \qquad I_z = \frac{\overline{m}r^2}{2}.$$

For a hollow circular cylinder, if r_1 is the outer and r_2 the inner radius, we have, from example (6),

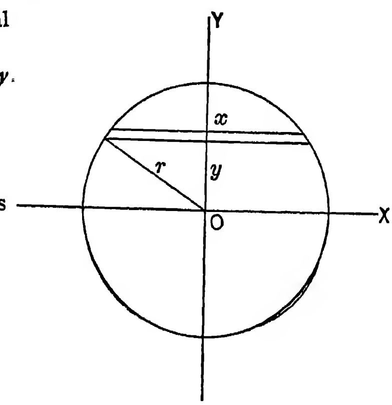
$$I_x = I_y = \frac{\overline{m}}{4}(r_1^2 + r_2^2) + \frac{\overline{m}l^2}{12}, \qquad I_z = \frac{\overline{m}}{2}(r_1^2 + r_2^2).$$

If the bases are ellipses of semi-transverse axis a and semi-conjugate axis b, we have

$$I_x = \frac{\overline{m}}{4}\left(b^2 + \frac{l^2}{3}\right), \qquad I_y = \frac{\overline{m}}{4}\left(a^2 + \frac{l^2}{3}\right), \qquad I_z = \frac{\overline{m}}{4}(a^2 + b^2).$$

(10) *Find the moment of inertia of a homogeneous sphere by Calculus.*

ANS. Any three diameters at right angles are central principal axes.



Let r be the radius, and take a circular element at a distance y from OX. The radius x of this element is given by

$$x^2 = r^2 - y^2.$$

The area of the element is then $\pi x^2 = \pi(r^2 - y^2)$. Its volume is $\pi(r^2 - y^2)dy$, and if δ is the density, its mass is

$$m = \delta\pi(r^2 - y^2)dy.$$

The moment of inertia of the element relative to OY is

$$\frac{mx^2}{2} = \frac{\delta\pi}{2}(r^2 - y^2)^2 dy.$$

Integrating between the limits $y = +r$ and $y = -r$, we have, since $\overline{m} = \frac{4}{3}\delta\pi r^3$ is the mass of the sphere, for the moment of inertia relative to any diameter

$$I = \tfrac{2}{5}\overline{m}r^2.$$

If we integrate between the limits $y = +r$ and $y = 0$, we have for the moment of inertia of a *hemisphere* relative to the axis OY perpendicular to the base at the centre, since the mass of the hemisphere is $\overline{m} = \frac{2}{3}\delta\pi r^3$,

$$I_y = \tfrac{2}{5}\overline{m}r^2.$$

The moment of inertia of the slice relative to OX is

$$\frac{mx^2}{4} + my^2 = \frac{\delta\pi}{4}(r^2 - y^2)^2dy + \delta\pi(r^2 - y^2)y^2dy.$$

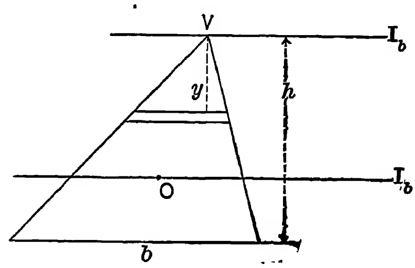
Integrating between the limits $y = +r$ and $y = 0$, we have for the moment of inertia of a *hemisphere* relative to any line OX in its base through the centre

$$I_x = \tfrac{2}{5}\overline{m}r^2.$$

The expression $I = \frac{2}{5}\overline{m}r^2$ evidently holds for any spheroid of revolution whose equatorial radius is r.

(11) *Find the moment of inertia for a homogeneous right cone or pyramid, by Calculus.*

ANS. Let A be the area of the base, h the height or altitude. Take any slice parallel to the base at a distance y from the vertex. Then the area of this slice is $\frac{y^2}{h^2}A$, its volume is $\frac{Ay^2dy}{h^2}$, and if δ is the density, its mass is



$$m = \frac{\delta Ay^2dy}{h^2}.$$

Let k_b be the radius of gyration of the base for any line in the plane of the base. Then the radius of gyration of the slice for any parallel line in its plane is $\frac{y}{h}k_b$. The moment of inertia of the slice relative to a parallel line through the vertex V is then

$$\frac{my^2k_b^2}{h^2} + my^2 = \frac{\delta Ak_b^2y^4dy}{h^4} + \frac{\delta Ay^4dy}{h^2}.$$

Integrating between the limits $y = h$ and $y = 0$, we have, since $\overline{m} = \frac{\delta Ah}{3}$ is the mass of the cone or pyramid, for an axis through the apex A at right angles to the axis of the cone or pyramid

$$I'_b = \tfrac{3}{5}\overline{m}(k_b^2 + h^2).$$

For a parallel axis through the centre of mass O

$$I_b = I'_b - \overline{m}(\tfrac{3}{4}h)^2 = \tfrac{3}{5}\overline{m}k_b^2 + \tfrac{3}{80}\overline{m}h^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let k_v be the radius of gyration of the base for the vertical axis of the cone or pyramid. Then the radius of gyration of the slice for this axis is $\frac{y}{h}k_v$. The moment of inertia of any slice relative to this axis is then

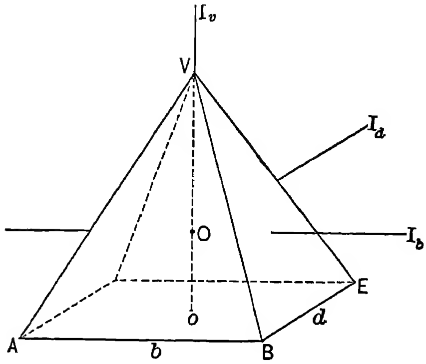
$$\frac{my^2k_v^2}{h^2} = \frac{\delta Ak_v^2y^4dy}{h^4}.$$

Integrating between the limits $y = h$ and $y = 0$, we have for the moment of inertia relative to the geometric axis

$$I_v = \tfrac{3}{5}\overline{m}k_v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) are general and hold good for any base. We have only to substitute for k_b and k_v their values in any case.

CASE 1.—For a right pyramid with parallelogram base we have (example 3) for a line through the centre of mass o of the base parallel to $AB = b$



$$k_b^2 = \frac{d^2 \sin^2 A}{12}.$$

Hence for a parallel line through the centre of mass O we have, from equation (1),

$$I_b = \overline{m}\,\frac{d^2 \sin^2 A}{20} + \frac{3}{80}\overline{m}h^2.$$

In the same way for an axis through O parallel to BE

$$I_a = \overline{m}\,\frac{b^2 \sin^2 A}{20} + \frac{3}{80}\overline{m}h^2.$$

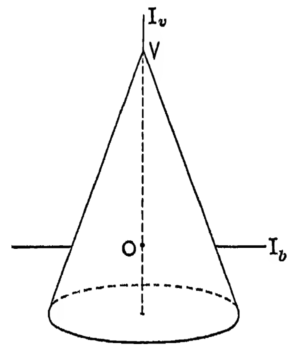
For the axis VO we have

$$k_v^2 = \tfrac{1}{12}(b^2 + d^2).$$

Hence from equation (2),

$$I_v = \frac{\overline{m}}{20}(b^2 + d^2).$$

CASE 2.—For a right cone with circular base we have (example 5) for any line in the plane of the base through the centre



$$k_b^2 = \frac{r^2}{4}.$$

Hence for a parallel line through the centre of mass O

$$I_b = \tfrac{3}{20}\overline{m}r^2 + \tfrac{3}{80}\overline{m}h^2.$$

For the axis VO we have

$$k_v^2 = \frac{r^2}{2}.$$

Hence

$$I_v = \tfrac{3}{10}\overline{m}r^2.$$

CASE 3.—For a right cone with elliptic base, if the semi-axes are a and b, we have (example 7)

$$k_a^2 = \frac{b^2}{4}, \quad k_b^2 = \frac{a^2}{4}, \quad k_v^2 = \frac{a^2 + b^2}{4}.$$

Hence

$$I_a = \tfrac{3}{20}\overline{m}b^2 + \tfrac{3}{80}\overline{m}h^2, \qquad I_b = \tfrac{3}{20}\overline{m}a^2 + \tfrac{3}{80}\overline{m}h^2, \qquad I_v = \tfrac{3}{20}\overline{m}(a^2 + b^2).$$

KINEMATICS OF A POINT. GENERAL PRINCIPLES.

CHAPTER I.

LINEAR AND ANGULAR DISPLACEMENT.

Kinematics.—As we have seen (page 1), we deal in mechanics with measurable relations of matter, space and time.

In the preceding pages we have given those measurable relations of matter and space only; of which we shall make future use.

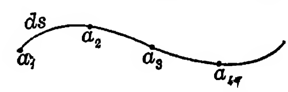
Let us now consider those measurable relations of space and time only, of which we shall also have to make future use.

That branch of science which treats of the measurable relations of space and time only, that is, of pure motion, is called KINEMATICS. It adds to the ideas of pure geometry the idea of motion.

We shall first consider the motion of a point and then the motion of a rigid body.

Path of a Point.—Two points are said to be *consecutive* when they are so close together that no third point can be taken between them. The line joining the successive consecutive positions of a point during its motion is called its PATH.

Let a_1, a_2, a_3, etc., represent the successive consecutive positions of a point. Then the line $a_1a_2a_3$, etc., is the path.



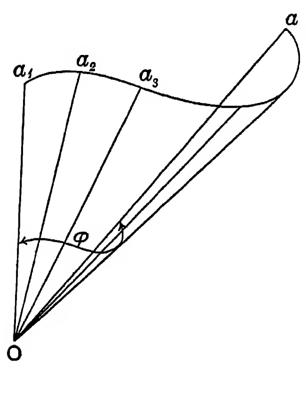
We always denote the length of the path or the distance described by the point by the letter s.

The distance between any two consecutive points, as $a_1\,a_2$, $a_2\,a_3$, etc., is then indefinitely small. This indefinitely small distance we denote by ds.

This portion of the path ds between two consecutive points we consider as a straight line and call an *element* of the path. The direction of any element is that of a tangent to the path.

Any path, then, we consider as made up of straight-line elements, each of length ds, and each tangent to the path, while the entire length of the path is the sum of all the elements and is denoted by $s = \Sigma ds$.

If the time of describing the path is denoted by t, then the indefinitely small time of describing an element of the path is denoted by dt.

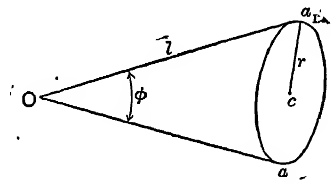


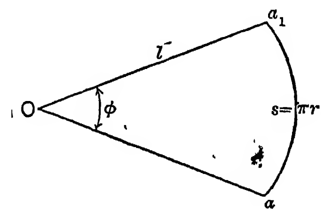
Angle described by a Point.—Let a point move in any path whatever from the initial position a_1 to the final position a.

Then the distance s from a_1 to a, *measured along the path*, is the DISTANCE DESCRIBED by the point relative to a_1. Take any point O as pole, and draw Oa_1, Oa_2 . . . Oa from O to every consecutive point of the path. The sum of all the indefinitely small elementary angles $a_1Oa_2 + a_2Oa_3 +$ etc. gives the angle swept over by the radius vector Oa in passing from the position Oa_1 to the final position Oa. We call this the ANGLE DESCRIBED by the point relative to O.

Denote its magnitude by ϕ. Then the indefinitely small elementary angles a_1Oa_2, a_2Oa_3, etc., are denoted by $d\phi$.

Example.—Let a point move in a circle of radius r ft. through a half circumference, from a_1 to a. Take the pole O in the perpendicular OC through the centre C of the circle, so that $Oa_1 = Oa = l$ ft. The successive positions of the radius vector are then elements of a cone. If we develop this cone, we have a

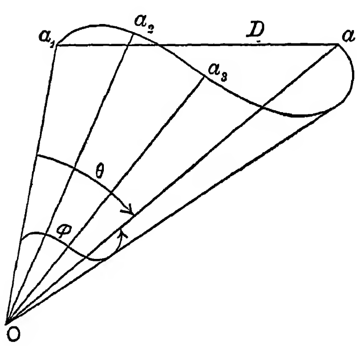




circular sector a_1Oa, whose radius is l ft., length of arc $s = \pi r$ ft., and whose developed angle $a_1Oa = \phi$ is the angle described. Hence

$$l\phi = \pi r, \quad \text{or} \quad \phi = \frac{\pi r}{l} \text{ radians.}$$

Linear and Angular Displacement.—Let a_1 and a, as before, be the initial and final positions of a point moving in any path.



Then, as already defined, the distance s from a_1 to a, measured along the path, is the distance described relative to a_1, and the angle ϕ swept over by the radius vector is the angle described relative to O.

Now draw the straight line a_1a from the initial position a_1 to the final position a. This line a_1a is the LINEAR DISPLACEMENT of the moving point. We denote its length by D. If it is indefinitely small, the two points a_1 and a become consecutive, and the displacement $D = ds$ *is then an element of the path.*

Denote the angle a_1Oa *subtended at O by the linear displacement D* by θ. This angle is the ANGULAR DISPLACEMENT of the moving point relative to O. *If it is indefinitely small, we have* $d\theta \doteq d\phi$.

Linear displacement is then measured in units of length, as feet, and angular displacement in radians.

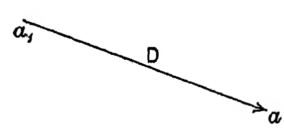
Example.—In the preceding example we have seen that the distance described is $s = \pi r$ feet, and the angle described is $\phi = \frac{\pi r}{l}$ radians.

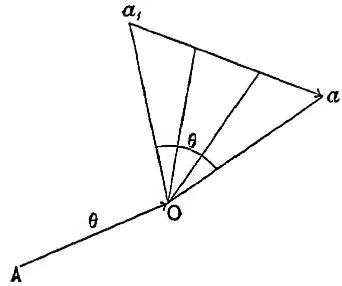
The linear *displacement* is, however, $D = 2r$ ft. in a direction from a_1 to a_1 and the angular *displacement* θ is given by

$$l \sin\frac{\theta}{2} = r, \quad \text{or} \quad \sin\frac{\theta}{2} = \frac{r}{l}.$$

Line Representative of Linear and Angular Displacement.—We see, then, that the linear displacement $a_1a = D$ is a straight line. It has therefore both magnitude and direction.

Thus a straight line a_1a represents by its length the magnitude D ft. of any linear displacement, and the arrow indicates the direction. The linear displacement is thus completely represented in both magnitude and direction by a straight line and arrow.





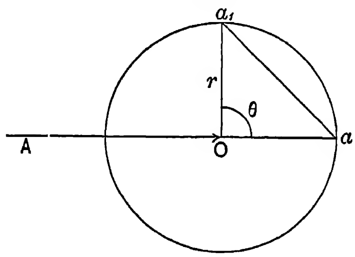
In the same way we can represent angular displacement. Thus if a_1 and a are the initial and final positions of a point and O the pole, so that $a_1Oa = \theta$ is the angular displacement, then a straight line AO through the pole at right angles to the plane of a_1Oa represents by its length the magnitude θ of the angular displacement, and the arrow indicates that if we look along this line in the direction of the arrow, the rotation of the radius vector is *always seen clockwise*. In the figure, for instance, from Oa_1 to Oa.

The line representative AO *coincides, therefore, with the axis about which the radius vector turns.*

By "direction" of angular displacement *we always mean the direction of its line representative as indicated by its arrow.*

Thus if the radius vectors Oa_1 and Oa in the figure lie in a vertical east and west plane, the direction of the angular displacement *is north*, because the axis is a north and south line, and if we look north along the axis we see rotation of the radius vector clockwise.

Example.—*Let a point move in a circle of radius r ft. in an east and west plane. The point starts from the top and moves eastward through a quadrant. Find the distance described, the angle described, the linear displacement, the angular displacement, for pole at centre.*



ANS.—The distance described is $s = \frac{\pi r}{2}$ ft. The angle described is $\phi = \frac{\pi}{2}$ radians The linear displacement is $a_1a = r\sqrt{2}$ ft., making an angle of 45° with the initial radius Oa_1.

The angular displacement is $\theta = \frac{\pi}{2}$ radians *north*, because the line representative AO must have its arrow pointing north in order that rotation of radius vector may be seen clockwise when we look in the direction of the arrow.

Note that "angle described" $\phi = \frac{\pi}{2}$ radians has magnitude only, while "angular displacement" has in this case the same magnitude *and also direction*, and is $\theta = \frac{\pi}{2}$ radians *north*.

Vector Quantities.—All quantities which have magnitude and direction so that they can be represented by straight lines and arrows are called VECTOR quantities.

Linear and angular displacement are thus vector quantities. In order that they may be known, therefore, magnitude *and direction* must be given. Magnitude alone is not sufficient.

Thus, in the preceding example, it is not sufficient to say that the linear displacement is $r\sqrt{2}$ ft. This is the magnitude only. The direction must also be stated.

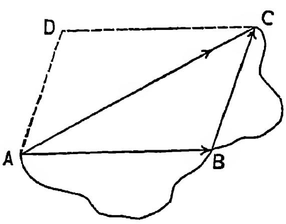
So, also, it is not sufficient to say that the angular displacement is $\theta = \frac{\pi}{2}$ radians. In this case this is the angle described, but it is only the magnitude of the angular displacement. The direction must also be stated. Thus the angular displacement is $\theta = \frac{\pi}{2}$ radians *north.*

Displacement in General.—The term "displacement" *always signifies linear displacement* unless otherwise specified.

CHAPTER II.

RESOLUTION AND COMPOSITION OF LINEAR DISPLACEMENTS.

Resolution and Composition of Linear Displacements.—Suppose a point to move by any path from A to B, so that AB is the line representative of the linear displacement. Then let it move by any path from B to C, so that BC is the line representative of the second displacement.



It is evident that AC is the line representative of the RESULTANT displacement, when the two displacements AB and BC are thus *successive*.

These two displacements AB and BC are called COMPONENT DISPLACEMENTS.

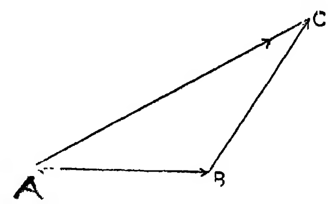
Suppose, however, that the two component displacements, instead of being successive, are *simultaneous*. That is, while the point goes from A to B, let the line AB move parallel to itself to DC, so that the end B arrives at C at the same instant that the point arrives at B. Then it is evident that AC is still the resultant displacement.

We see, then, that if a point has two component displacements AB, BC, either simultaneous or successive, the resultant displacement AC due to combining them is easily found. This combination is called COMPOSITION of displacements.

Also, if we have any given displacement AC, we can find the equivalent component displacements, simultaneous or successive, in any two directions we please. This is called RESOLUTION of displacement.

We can RESOLVE a displacement into equivalent components, or we can COMBINE the components into an equivalent resultant.

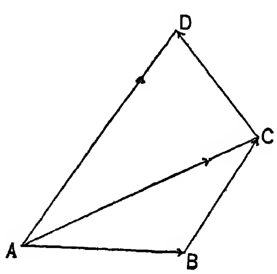
Triangle and Polygon of Linear Displacements.—We can express this composition or resolution as follows:



If two sides of a triangle, AB, BC, taken the *same way round*, as shown by the arrows, give the linear displacements, simultaneous or successive, of a point, then the third side, AC, taken the *opposite way round*, as shown by the arrow, gives the resultant displacement.

Inversely, any displacement AC can be resolved into two components AB and BC, simultaneous or successive, in any two desired directions, by completing the triangle ABC, and taking AB and BC the opposite way round.

This is called the principle of the "*triangle of displacements.*"

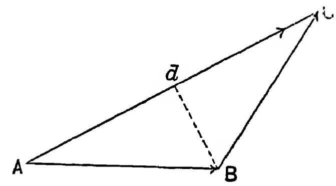


Again, if we had a third displacement, CD, the resultant of AC already found, and CD is AD But AC is the resultant of AB and BC. Hence AD is the resultant of AB, BC and CD, either simultaneous or successive.

Hence if *any number* of displacements, simultaneous or successive, are given by the sides of a polygon $A\,B\,C\,D$, etc., taken the *same way round*, the line AD which closes the polygon, taken the *other way round*, gives the resultant displacement.

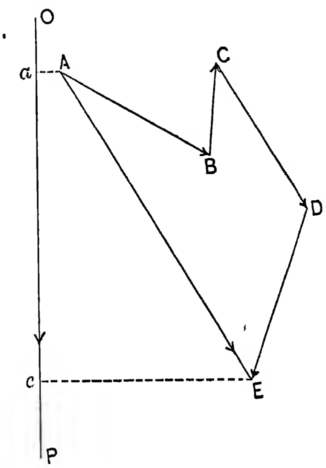
This is called the principle of the "*polygon of displacements.*" The triangle of displacements is evidently only a special case.

Rectangular Components.—When a displacement is resolved into two components at right angles, the components are RECTANGULAR COMPONENTS. Unless otherwise specified, when we speak of the components of any displacement, rectangular components are to be understood.



Component of the Resultant equal to the Algebraic Sum of the Components of the Displacements.—It is evident that the resultant of any two given displacements is equal to the algebraic sum of their components along the resultant.

For if AB and BC are the given displacements, the resultant AC is equal to the sum of the components Ad and dC.



So also for any number of displacements, the resultant AE is equal to the algebraic sum of the components of the displacements along AE.

The component in any direction of the resultant itself is then equal to the algebraic sum of the components of the displacements in the same direction.

Thus the projection ae of the resultant AE upon any line OP, that is, the component of AE along OP, is the same as the algebraic sum of the components of AB, BC, CD, DE along OP.

That is, *the component of the resultant in any direction is equal to the algebraic sum of the components of the displacements in that direction.*

Examples.—(1) *A point has three displacements, N. 60° E., 40 ft.; S. 50 ft.; W. 30° N., 60 ft. Find the resultant displacement.*

ANS $10\sqrt{3}$ ft. W.

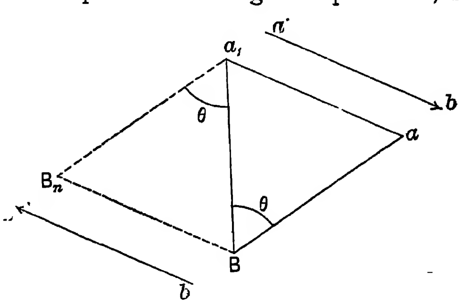
(2) *Three component displacements have magnitudes represented by 1, 2 and 3, and directions given by the sides of an equilateral triangle. Find the magnitude of the resultant.* ANS. $\sqrt{3}$.

(3) *Show that the resultant of two equal displacements of magnitude a, inclined 60°, is equal to the resultant of a and 2a inclined 120°.*

(4) *To a man in a balloon the starting-point bears N. 20° E., and is depressed 30° below the horizontal. A point at the same level as the starting-point and 10 miles from it is vertically below him. Find the component displacements of the balloon relative to the starting-point.*

ANS 9.39 miles south; 3.42 miles west; 5.77 miles high.

Relative Displacement.—Since the displacement of a point is change of position, it can only be determined by reference to some chosen point of reference (page 11).



Thus if a_1 and a are the initial and final positions of a moving point A, and B is some chosen point of reference, the positions a_1 and a *relative to B* are known, if we know the radius vectors Ba_1 and Ba and the angle $a_1Ba = \theta$.

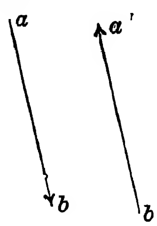
To an observer at B the point A is seen to move from a_1 to a through the angle θ, and a_1a is then the displacement of A *relative to B*.

A line ab, then, parallel and equal to a_1a *with arrow at b* gives this displacement a_1a in magnitude and direction, and it *also indicates* by the end letters that the line ab with arrow at b is the displacement of A *relative to B*.

Now to an observer on the moving point A we see, by completing the parallelogram, that the point B would appear to the observer on A to move from B to B_n, through the same angle θ, and BB_n parallel, equal and opposite to a_1a gives the displacement of B *relative to A*.

A line ba, then, parallel and equal to BB_n, with *arrow at a*, gives the displacement BB_n in magnitude and direction, and *it also indicates* by the end letters that the line ba with arrow at a is the displacement of B *relative to A*.

Hence any change in the relative position of any two points A and B may be regarded as a displacement of A relative to B, as indicated by the line ab with arrow at b, or as an equal and opposite displacement of B relative to A, as indicated by the line ba with arrow at a.

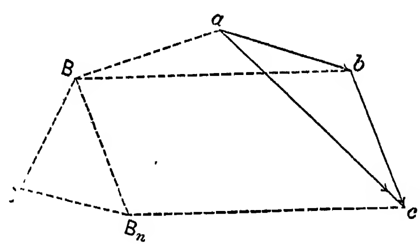


Notation.—The student should notice carefully the preceding notation. A relative displacement is given in magnitude and direction by a straight line and arrow. The letter at the arrow end denotes the point of reference; that at the other end the moving point.

Thus the line ab with arrow at b gives the magnitude and direction of the displacement of A relative to B. The equal parallel line with arrow at a gives the displacement of B relative to A.

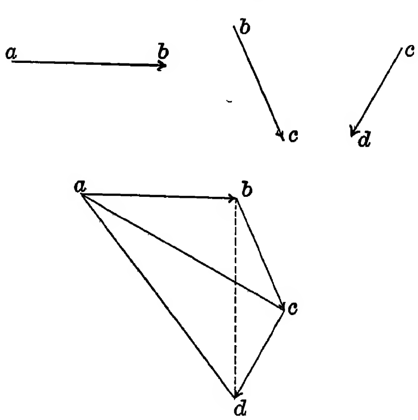
Triangle and Polygon of Relative Displacements.—Except for notation we have changed nothing, and the principles already established still hold.

Thus let a moving point A have a displacement ab relative to some point B, so that it moves from a to b, and at the same time, or afterwards, let the point B have the displacement BB_n relative to some point C. Then the line bc through b, parallel and equal to BB_n, gives the displacement of B relative to C, and it is evident that A moves from a to c, and hence *ac is the resultant displacement of A relative to C.*



Hence if two sides ab, bc of a triangle abc, taken the same way round, give the displacement of A relative to B, and B relative to C, either simultaneous or successive, then the third side, ac, will give the resultant displacement of A relative to C if taken the opposite way round, from a to c. If taken the same way round, from c to a, it gives the displacement of C relative to A.

This is the principle of the *triangle of relative displacements.*



Again, let ab, bc, cd, etc., be the line representatives of the displacements of A relative to B, B relative to C, and C relative to D, etc. Then if we lay off these displacements we obtain the polygon $abcd$. As we have just seen, ac gives the displacement of A relative to C. The line ad gives then the displacement of A relative to D. Also any line in the polygon, as bd, gives the displacement of B relative to D.

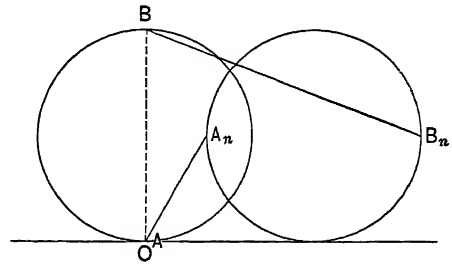
Hence if any number of relative displacements, simultaneous or successive, are given by the sides of a polygon $abcd$, etc., the line which closes the polygon, taken the opposite way round, gives the displacement of the first point relative to the last; taken the same way round, the displacement of the last relative to the first.

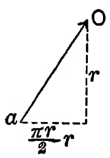
This is the principle of the *polygon of relative displacements.* The triangle of relative displacements is evidently a special case.

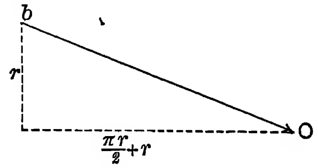
Examples.—(1) *A circle of radius r rolls on a horizontal plane until it turns through a quarter revolution. Find the displacement of the point of the circle initially in contact with the plane relative to the point diametrically opposite.*

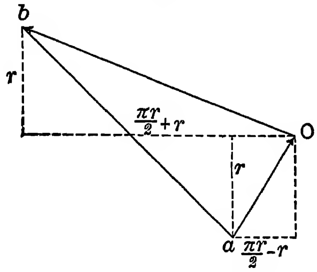
ANS. Let O be the initial point of contact on the plane, and A the corresponding point of the circle, initially at O, and B the point of the circle diametrically opposite.

When the circle rolls through a quadrant, A has moved to A_n, and B to B_n. The displacement of A relative to the fixed point O of the plane is given by the line from A to A_n.









A parallel and equal line aO with arrow at O gives, then the displacement of A relative to O.

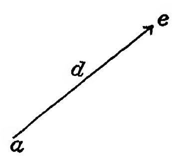
The displacement of B relative to O is the line BB_n. A parallel and equal line bO with arrow at O gives then the displacement of B relative to O. The same line Ob with arrow at b gives the displacement of O relative to B.

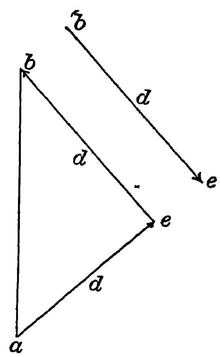
If, then, we lay off these displacements in order, aO, Ob, the closing line ab gives the displacement of A relative to B. This displacement we then easily find to be $2r\sqrt{2}$, making an angle of 45° with the vertical. The same line ba taken in the opposite direction gives the displacement of B relative to A.

(2) *Two railway trains A and B run, one northeast a distance d, the other southeast the same distance. Find the displacement of A relative to B.*

ANS. The displacement of A relative to some fixed point E on the earth is represented by the line ae with arrow at e. The displacement of B relative to the same point E is represented by the line be with arrow at e. The same line with arrow at b gives, then, the displacement of E relative to B.

Laying off these displacements ae, eb in order, we find ab, the displacement of A relative to B, to be $d\sqrt{2}$ in a direction north.





(3) *A point A moves 30 ft. in a given direction relative to a fixed point P. Another point, B, moves relative to P 40 ft. in a direction at right angles to the direction of A. Find the displacement of A relative to B.*

ANS. 50 ft. in a direction inclined to the direction of A by an angle whose tangent is $\frac{4}{3}$.

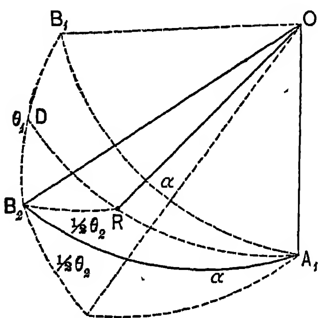
(4) *The displacement of a point A relative to a point B is a distance 6 ft. south, and relative to a point C 5 ft. west. If C is initially a distance 4 ft. south of B, find the final position of C relative to B.*

ANS. Distance of C's final position from B is $5\sqrt{5}$ ft., and the direction from B to the final position of C is east of south by an angle whose tangent is $\frac{1}{2}$.

CHAPTER III.

RESOLUTION AND COMPOSITION OF ANGULAR DISPLACEMENTS.

Resolution and Composition of Finite Successive Angular Displacements about Different Axes.—Let O be a fixed point or pole, and let OA_1 and OB_2 be two given axes, intersecting at O and making the angle $A_1OB_2 = \alpha$.



Suppose successive finite angular displacements about these axes in the following order:

1st. An angular displacement θ_1 about OA_1. 2d. An angular displacement θ_2 about OB_2.

It is required to find the resultant angular displacement.

CONSTRUCTION.—Draw through O an axis OB_1 making the angle $A_1OB_1 = \alpha$ with OA_1, and the angle $B_1 OB_2 = \theta_1$ with OB_2. Let the axis OA_1 and OB_1 be rigidly fixed relative to each other, so that when the angular displacement θ_1 takes place about OA_1, the axis OB_1 will turn into the position OB_2.

Take the distances $OA_1 = OB_1 = OB_2$. Then if we have the angular displacement θ_1 about OA_1, and next θ_2 about OB_2, the points B_1 and A_1 move on the surface of a sphere of radius OA_1.

Join A_1B_1, B_1B_2 and B_2A_1 by great circles of this sphere. Then the arc $B_1A_1 = \alpha$, arc $B_2A_1 = \alpha$, arc $B_1B_2 = \theta_1$, and in the spherical triangle $B_1A_1B_2$ the spherical angle at A_1 is θ_1.

Bisect this angle by a great circle A_1D meeting B_1B_2 at D. Draw a great circle B_2R through B_2 inclined to B_2A_1 by the spherical angle $\frac{1}{2}\theta_2$ and meeting A_1D at R.

Then the resultant angular displacement is about the axis OR.

PROOF.—Draw the arc B_2R' making the same angle, $\frac{1}{2}\theta_2$, with B_2A_1 on the other side, and make the arc $B_2R' = B_2R$. Then A_1R' will equal A_1R, and the angle $B_2A_1R' = \frac{1}{2}\theta_1 = B_2A_1R$.

Now when the angular displacement θ_1 takes place about OA_1, the axis OB_1 moves to OB_2 through the angle $B_1A_1B_2 = \theta_1$, and the axis OR, if rigidly fixed relative to OA_1, will move to OR' through the angle $RA_1R' = \theta_1$.

Next, when the angular displacement θ_2 takes place about OB_2 through the angle $R'B_2R = \theta_2$, the point R' evidently moves back to R.

Hence the axis OR has the same position before and after the angular displacements. The resultant angular displacement is then about the axis OR.

MAGNITUDE AND POSITION.—For the position of this resultant axis and the magnitude θ of the resultant angular displacement about it we have in the spherical triangle A_1RB_2 the side $A_1B_2 = \alpha$, the angle $A_1B_2R = \frac{1}{2}\theta_2$ and the angle $RA_1B_2 = \frac{1}{2}\theta_1$. Also the angle $B_2RA_1 = 180° - \frac{1}{2}\theta$.

Hence we have for the magnitude of the resultant angular displacement θ the equation

$$\cos \tfrac{1}{2}\theta = \cos \tfrac{1}{2}\theta_1 \cos \tfrac{1}{2}\theta_2 - \sin \tfrac{1}{2}\theta_1 \sin \tfrac{1}{2}\theta_2 \cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

Also for the direction of OR

$$\frac{\sin ROA_1}{\sin \frac{1}{2}\theta_2} = \frac{\sin ROB_2}{\sin \frac{1}{2}\theta_1} = \frac{\sin \alpha}{\sin \frac{1}{2}\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II})$$

From (I) we can find θ, and from (II) we can find the angles ROA_1 and ROB_2 which the axis OR makes with the axes OA_1 and OB_2.

Examples.—(1) *The telescope of a transit instrument initially horizontal and pointing north is first turned to an altitude of* 60° *and then turned to the west. Find the resultant angular displacement.*

ANS. We have $\theta_1 = 60°$, $\theta_2 = 90°$, $\alpha = 90°$. Hence, from (I), $\cos \frac{1}{2}\theta = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$, or $\theta = 104° \, 28' \, 39''$, or $\theta = 1.823$ radians.

Hence $\sin \frac{1}{2}\theta = \frac{1}{2}\sqrt{\frac{5}{2}}$, and we have, from (II), for the position of the resultant axis

$$\sin ROA_1 = \frac{\sin 90°}{\sin \frac{1}{2}\theta} \sin 45° = 2\sqrt{\tfrac{1}{5}}, \quad \text{or} \quad ROA_1 = 63° \, 26' \, 5.8'';$$

$$\sin ROB_2 = \frac{\sin 90°}{\sin \frac{1}{2}\theta} \sin 30° = \sqrt{\tfrac{2}{5}}, \quad \text{or} \quad ROB_2 = 39° \, 13' \, 53.4''.$$

(2) *In the preceding example let the successive displacements be taken in reverse order.*

ANS. We have then $\theta_1 = 90°$, $\theta_2 = 60°$, $\alpha = 90°$. Hence $\cos \frac{1}{2}\theta = \frac{1}{2}\sqrt{\frac{3}{2}}$ and $\sin \frac{1}{2}\theta = \frac{1}{2}\sqrt{\frac{5}{2}}$, just as before. But for the position of the resultant axis

$$\sin ROA_1 = \sqrt{\tfrac{2}{5}} \quad \text{and} \quad \sin ROB_2 = 2\sqrt{\tfrac{1}{5}}.$$

Resolution and Composition of Angular Displacements in General.—We have just seen how to find the résultant for finite successive angular displacements about intersecting axes.

For all other cases of angular displacement about intersecting axes we can combine and resolve angular displacements just like linear displacements, as explained in the preceding chapter.

This can be shown as follows:

All other cases fall under the head of

(*a*) Finite or indefinitely small successive or simultaneous angular displacements about the same axis;

(*b*) Indefinitely small successive or simultaneous angular displacements about different intersecting axes.

(*c*) Finite simultaneous angular displacements about different intersecting axes.

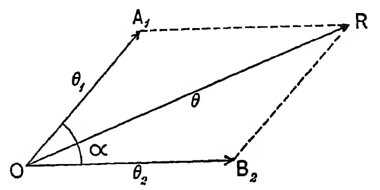
Let us consider these cases in order.

(*a*) FINITE OR INDEFINITELY SMALL SUCCESSIVE OR SIMULTANEOUS ANGULAR DISPLACEMENTS ABOUT THE SAME AXIS.—If the axis of all the angular displacements is the same, the plane of rotation of the radius vector does not change, and all the line representatives lie in the same straight line. The resultant is then given by the algebraic sum of the line representatives, and this evidently holds whether the angular displacements are successive or simultaneous, finite or indefinitely small.

(*b*) INDEFINITELY SMALL SUCCESSIVE OR SIMULTANEOUS ANGULAR DISPLACEMENTS ABOUT DIFFERENT INTERSECTING AXES.—Let the angular displacements be indefinitely small and successive in the order θ_1, θ_2. Then we have $\sin \frac{1}{2}\theta_1 = \frac{1}{2}\theta_1$, $\sin \frac{1}{2}\theta_2 = \frac{1}{2}\theta_2$, $\sin \frac{1}{2}\theta = \frac{1}{2}\theta$. Hence, from equations (II),

$$\theta : \theta_1 : \theta_2 :: \sin \alpha : \sin ROB_2 : \sin ROA_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let OA_1 be the line representative of θ_1, OB_2 the line representative of θ_2, and the angle $A_1OB_2 = \alpha$. Complete the parallelogram and draw OR. Then in the triangle ROB_2 we have



$$OR : \theta_1 : \theta_2 : : \sin\alpha : \sin ROB_2 : \sin ROA_1. \quad (2)$$

Comparing (2) with (1) we see that $OR = \theta$, or the resultant is the third side of the triangle.

Hence we have the "*triangle of angular displacements*," just as for linear displacements, page 54.

The same evidently holds for simultaneous angular displacements indefinitely small.

(*c*) FINITE SIMULTANEOUS ANGULAR DISPLACEMENTS ABOUT DIFFERENT INTERSECTING AXES.—The same evidently holds for finite simultaneous angular displacements about the same or different axes. For we can divide each finite displacement into a number of indefinitely small displacements, and treat each pair as in the previous case.

Hence for all cases except finite successive angular displacements about different intersecting axes, which case we have discussed on page 58, we can combine and resolve angular displacements by means of their line representatives, just like linear displacements in the preceding chapter.

We have then the "*triangle and polygon of angular displacements*," just as for linear displacements, page 55.

Also, just as on page 55, the component of the resultant in any direction is equal to the algebraic sum of the components of the displacements in that direction.

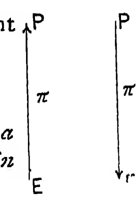
Also, we can find *relative* angular displacement in precisely the same way (page 56).

Examples.—(1) *A body has two simultaneons rotations of 2 radians and 4 radians about axes inclined 60° to each other. Find the resultant.*

ANS. $2\sqrt{7}$ radians about an axis inclined to the greater component at an angle whose sine is $\frac{\sqrt{3}}{2\sqrt{7}}$.

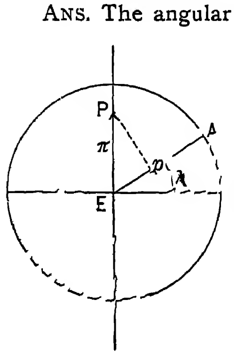
(2) *A sphere with one of its superficial points fixed has two simultaneous rotations, one of 8 radians about a tangent line and one of 15 radians about a diameter. Find the axis of the resultant angular displacement, the resultant angular displacement, and the number of revolutions about the resultant axis.*

ANS. Axis inclined to greater component at an angle whose tangent is $\frac{8}{15}$. Resultant angular displacement 17 radians. Number of revolutions $\frac{17}{2\pi}$ = about 2.75 revolutions.



(3) *A pendulum suspended from a point in the prolonged polar axis of the earth swings in a plane through the axis. Find the angular displacement of this plane relative to the earth in 12 hours.*

ANS. The angular displacement of the earth relative to the plane is $EP = \pi$ radians north. That is, if we look north along the polar axis the rotation is seen clockwise. The angular displacement of the plane relative to the earth is then $PE = \pi$ radians south. That is, an observer on the earth looking *south* along the polar axis sees the plane rotate clockwise through 180° in 12 hours.



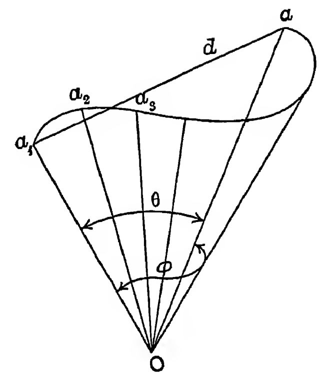
(4) *Let the pendulum be suspended from a point in the prolonged radius of the earth at a place of latitude λ, and swing in a meridian plane. Find the angular displacement of the plane relative to the earth in 12 hours.*

ANS. The line representative of the angular displacement of the earth relative to the plane is still $EP = \pi$ radians north. The component of this along the radius of the place is $Ep = \pi \sin\lambda$ radians. The angular displacement of the plane relative to the earth is then pE. That is, an observer at A looking towards the centre of the earth would see the plane shift clockwise through $\pi\sin\lambda$ radians in 12 hours.

CHAPTER IV.

LINEAR AND ANGULAR SPEED AND VELOCITY.

Mean Linear and Angular Speed.—Let a point move in any path from the initial position a_1 to the final position a in the time t.



Let s be the distance described (page 51). Then $\frac{s}{t}$, or the *distance described per unit of time*, is the MEAN LINEAR SPEED.

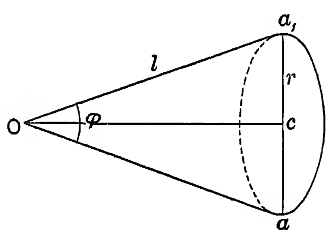
Let ϕ be the *angle described* (page 51). Then $\frac{\phi}{t}$, or the *angle described per unit of time*, is the MEAN ANGULAR SPEED.

Linear speed is then measured in feet per second, and angular speed in radians per second.

Example.—Let a point move in a circle of radius r ft. through a half circumference, from a_1 to a, in $t = 2$ seconds.

Then the mean linear speed is $\frac{s}{t} = \frac{\pi r}{t}$ ft. per sec.

Take the pole O in the perpendicular OC through the centre C of the circle, so that $Oa_1 = Oa = l$ ft. Then (page 52) the angle described is $\phi = \frac{\pi r}{l}$ radians, and the mean angular speed is $\frac{\phi}{t} = \frac{\pi r}{lt}$ radians per sec.



Mean Linear and Angular Velocity.—In the preceding figure $a_1a = D$ is the linear displacement (page 52), and the angle $a_1Oa = \theta$ is the angular displacement (page 52).

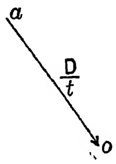
Let t be the time of moving from a_1 to a. Then $\frac{D}{t}$, or the *linear displacement per unit of time*, is the MEAN LINEAR VELOCITY, and $\frac{\theta}{t}$, or the *angular displacement per unit of time*, is the mean angular velocity relative to O.

Linear velocity is measured, then, like linear speed, in feet per second, and angular velocity, like angular speed, in radians per second.

Example.—In the preceding example we have seen that the mean linear speed is $\frac{s}{t} = \frac{\pi r}{t}$ ft. per sec., and the mean angular speed is $\frac{\phi}{t} = \frac{\pi r}{lt}$ radians per sec.

The mean linear velocity is, however, $\frac{D}{t} = \frac{2r}{t}$ ft. per sec. in a direction from a_1 to a, and the mean angular velocity for pole at O is $\frac{\theta}{t}$ radians per sec., where θ is given by $l \sin \frac{1}{2}\theta = r$, or $\sin \frac{1}{2}\theta = \frac{r}{l}$, *and has also direction* as explained in the next article.

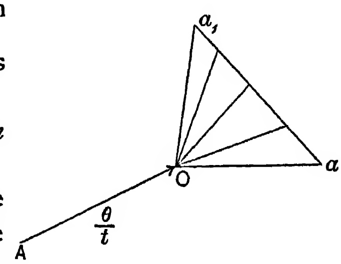
Line Representatives of Mean Linear and Angular Velocity.—Since mean linear velocity is linear displacement per unit of time, we can represent it by a straight line, just like linear displacement itself.



Thus the straight line ao represents by its length the magnitude $\frac{D}{t}$ of any mean linear velocity, and the arrow represents the direction.

In the same way, since mean angular velocity is angular displacement per unit of time, we can represent it by a straight line like angular displacement itself.

Thus the straight line AO at right angles to the plane of a_1Oa represents by its length the magnitude $\frac{\theta}{t}$ of any mean angular velocity, and looking in the direction of the arrow from A to O the rotation of the radius vector is always seen clockwise.



By direction of angular velocity we always mean the direction of its line representative.

Thus if the initial and final positions Oa_1 and Oa of the radius vector lie in a vertical east and west plane a_1Oa, the direction of the mean angular velocity is *north*, because in such case the line representative AO would point north, so that looking in the direction of the arrow from A to O we should see the rotation of the radius vector clockwise.

Distinction between Mean Speed and Velocity.—Let the initial position of a point at the time t_1 be at a distance s_1 measured along the path from any fixed point in the path, taken as an origin or point of reference, and at any other time t greater than t_1 let the distance measured along the path from this same fixed point be s. Then $(s - s_1)$ is the distance described in the interval of time $(t - t_1)$, and the mean linear speed is

$$\frac{s - s_1}{t - t_1}.$$

In similar notation we have $(\phi - \phi_1)$, the angle described in the interval of time $(t - t_1)$, and the mean angular speed is

$$\frac{\phi - \phi_1}{t - t_1}.$$

If s is greater than s_1, the distance described is away from the origin and the mean linear speed is positive. If s is less than s_1, the distance described is towards the origin and the linear speed is negative. So, also, if ϕ is greater than ϕ_1, the angle described is away from the initial line of reference and the mean angular speed is positive. If ϕ is less than ϕ_1, the angle described is towards the initial line of reference and the mean angular speed is negative.

Thus if the distance from the origin is increasing the linear speed is positive, if decreasing it is negative. If the angle described from the initial radius is increasing the angular speed is positive, if decreasing it is negative.

Mean linear speed, then, is mean time-rate of distance described, and mean angular speed is mean time-rate of angle described. Both possess magnitude and sign, according to whether the distance or angle described increases or decreases with the time, but both are independent of direction of the path.

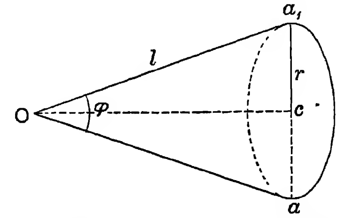
Such quantities which have sign and magnitude but are independent of direction are called SCALAR quantities. They cannot be represented by straight lines.

But, as we have seen, mean linear velocity $\frac{D}{t}$ is mean time-rate of linear displacement,

and mean angular velocity $\frac{\theta}{t}$ is mean time-rate of angular displacement. Both possess not only magnitude but direction. Such quantities are VECTOR quantities. They can be represented by straight lines.

We see, then, that mean speed and velocity, *although measured in the same units, are quantities of a different kind*, and the number of units are in general different for each.

Example.—Let a point move in a circle of radius $r = 10$ feet. Let the plane of the circle be an east and west plane. Let the point move from the top a_1 eastward through a half circumference, from a_1 to a, in the time $t = 3$ seconds. Take the pole O in the perpendicular OC through the centre C of the circle, so that $Oa_1 = Oa = l = 20$ ft.



Then the distance described is $s = \pi r = 31.416$ ft., and the mean linear speed is $\frac{s}{t} = +10.472$ ft per sec.

The angle described is (page 52) $\phi = \frac{\pi r}{l} = 1.5708$ radians, and the mean angular speed is $\frac{\phi}{t} = +0.5236$ radians per sec.

On the other hand, the linear displacement is $D = 2r = 20$ ft. in the direction from a_1 to a, and the mean linear velocity is $\frac{D}{t} = 6.66$ ft. per sec. *in the same direction.*

The angular displacement for pole at O is given by $l \sin \frac{1}{2}\theta = r$, or $\sin \frac{1}{2}\theta = \frac{r}{l} = \frac{1}{2}$, or $\theta = \frac{\pi}{3}$ radians *north*, and the mean angular velocity is $\frac{\theta}{t} = \frac{\pi}{9}$ radians per sec. *north.* We see, then, that the mean linear speed and velocity, although measured in the same units, are in this case not only different in magnitude, but one is independent of direction, while the other has direction. So, also, for mean angular speed and velocity.

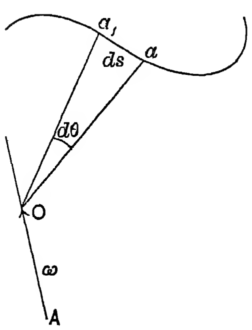
Instantaneous Linear and Angular Speed and Velocity.—The *limiting magnitude* of the mean linear or angular speed when the interval of time is indefinitely small is the INSTANTANEOUS linear or angular speed.

These are SCALAR quantities having sign and magnitude but independent of direction, just like mean linear and angular speed.

The *limiting magnitude and direction* of the mean linear or angular velocity when the interval of time is indefinitely small is the INSTANTANEOUS linear or angular velocity.

Instantaneous linear and instantaneous angular velocity, then, are vector quantities, having both magnitude *and direction*, and can therefore be represented by straight lines, just like mean linear and mean angular velocity.

Thus, let the interval of time be indefinitely small and denoted by dt.



Then the initial and final positions a_1 and a of a moving point will become consecutive points of the path, so that the distance a_1a can be denoted by ds.

We see, then, that in this case the distance described is ds. That the linear displacement is ds in magnitude, and its direction is tangent to the path, so that a_1a is its line representative. That is, in this case, *distance described is the magnitude of the linear displacement*.

If, then, we denote the instantaneous linear speed by v, we can write

$$v = \frac{ds}{dt},$$

and this will also be the magnitude of the instantaneous linear velocity, the direction of which is always tangent to the path.

Similarly, in the indefinitely small time dt, the indefinitely small angle described is a_1Oa, which can be denoted by $d\phi$. The indefinitely small angular displacement $d\theta$ will be the same in magnitude, and its direction will be at right angles to the plane of rotation a_1Oa, so that AO is its line representative. That is, in this case, *angle described* ($d\phi$) *is the magnitude of the angular displacement* ($d\theta$).

If, then, we denote the instantaneous angular speed by ω, we can write

$$\omega = \frac{d\theta}{dt},$$

and this will also be the magnitude of the instantaneous angular velocity, the direction of which, as indicated by its line representative AO, is always at right angles to the plane of rotation, so that looking in the direction of its arrow we see the rotation of the radius vector clock-wise. The line representative AO coincides, therefore, with the *axis* about which the radius vector rotates.

Examples.—(1) *Let the distance described by a moving point be given by the equation*

$$s = 7t + 8t^2, \quad \ldots \quad (1)$$

where s is the number of feet described in any number of seconds t.

Then for any other number of seconds t_1 we can write

$$s_1 = 7t_1 + 8t_1^2. \quad \ldots \quad (2)$$

Suppose t_1 to be less than t. Then subtracting (2) from (1) we have for the distance $s - s_1$ described in the interval of time $t - t_1$

$$s - s_1 = 7(t - t_1) + 8(t^2 - t_1^2) = 7(t - t_1) + 8(t + t_1)(t - t_1).$$

The mean linear speed is, then, for this interval of time

$$\frac{s - s_1}{t - t_1} = 7 + 8(t + t_1). \quad \ldots \quad (3)$$

We see from (3) that as the interval of time $t - t_1$ decreases, t_1 approaches equality with t and the mean linear speed given by (3) approaches the limiting value $7 + 16t$. We have then for the instantaneous speed, or the magnitude of the instantaneous velocity,

$$v = \frac{ds}{dt} = 7 + 16t. \quad \ldots \quad (4)$$

In order that the linear velocity may be completely known we must specify, in addition to its magnitude as given by (4), its direction, which is always tangent to the path at the instant.

Students familiar with the Calculus will note that (4) is obtained directly from (1) by differentiating s with reference to t.

If in (4) we make $t = 2$, we have $v = 39$ ft. per sec. This does not mean that the point has described 39 feet in the preceding second, nor that it will describe 39 feet in the next second. But it means that *at the instant* (2 seconds from the start) it is moving at such a rate that, if that rate did not change, the point *would* describe 39 feet in the next second. It is the instantaneous speed.

(2) *Let a point move in a circle in a vertical east and west plane, and let the angle described by the radius vector from a fixed point to the moving point be given by the equation*

$$\theta = 7t + 8t^2,$$

where θ is the number of radians described in any number of seconds t, the origin being at the centre. Find, as in the preceding example, the mean and instantaneous angular speed and velocity.

ANS. The mean angular speed is

$$\frac{\theta - \theta_1}{t - t_1} = 7 + 8(t + t_1).$$

The instantaneous angular speed is

$$\omega = \frac{d\theta}{dt} = 7 + 16t$$

This is the magnitude of the instantaneous angular velocity, whose direction is *north* if the point is moving eastward in the plane from the top point.

CHAPTER V.

LINEAR AND ANGULAR VELOCITY.

Speed and Velocity in General.—The terms "speed" and "velocity" always signify *instantaneous linear* speed and velocity unless otherwise specified. The terms linear and angular speed and velocity always signify *instantaneous* linear and angular speed and velocity unless otherwise specified.

Uniform and Variable Linear Velocity.—When the linear velocity has the same magnitude *and direction* whatever the interval of time, it is UNIFORM. If either magnitude *or direction* change, it is VARIABLE.

If, then, the velocity is uniform, the line representative has always the same magnitude and direction; and since velocity is tangent to the path, we have *uniform speed* in a straight line. The velocity in this case is the same as the mean velocity for any interval of time.

If only the direction changes, we have uniform speed in a curve. In this case the magnitude of the velocity is the same as the *mean speed* for any interval of time.

If only the magnitude changes, we have variable speed in a straight line.

If both magnitude and direction change, we have variable speed in a curve.

Examples.—(1) *A point moves with uniform speed in a circle of radius $r = 6$ ft. and makes a half revolution in the time $t = 3$ sec. Find the mean speed, the mean velocity, the instantaneous speed and the instantaneous velocity.*

ANS. The mean speed is $\frac{\pi r}{t} = 2\pi$ ft. per sec., and the mean velocity is $\frac{2r}{t} = 4$ ft. per sec. in the direction of the diameter through the starting-point.

Since the speed is uniform, the instantaneous speed must be the same as the mean speed, or 2π ft. per sec.

The magnitude of the instantaneous velocity is the same as the instantaneous speed, or $v = 2\pi$ ft. per sec., but the direction at any instant is tangent to the path at the position of the point at that instant.

(2) *Criticise the statement, "a point moves in a circle with uniform velocity."*

ANS. A point can move in a circle or in any path with *uniform speed*. But if the *velocity* is uniform, it can only move in a straight line with uniform speed. A point cannot move in a curve with uniform velocity.

Uniform and Variable Angular Velocity.—*By direction of angular velocity we always mean the direction of its line representative* (page 62).

When the angular velocity has the same magnitude *and direction* whatever the interval of time, it is UNIFORM. If either magnitude *or direction* change, it is VARIABLE.

If, then, the angular velocity is uniform, the line representative has always the same magnitude and direction, and we have uniform angular speed in an unchanging plane. The angular velocity in such case is the same as the mean angular velocity for any interval of time.

If only the direction changes, we have uniform angular speed, but a changing plane of rotation. In this case the magnitude of the angular velocity is the same as the mean angular speed for any interval of time.

If only the magnitude changes, we have variable angular speed in an unchanging plane.

If both magnitude and direction change, we have variable angular speed and a changing plane of rotation.

Examples.—(1) *A point moves with uniform angular speed in a circle of radius $r = 6$ ft. in a vertical east and west plane, and makes a half revolution in the time $t = 2$ sec. Find the mean angular speed, the mean angular velocity, the instantaneous angular speed and velocity, origin at the centre.*

ANS. The mean angular speed is $\frac{\pi}{t} = \frac{\pi}{2}$ radians per sec. The mean angular velocity, if the point moves eastward from the top, is also $\frac{\pi}{2}$ radians per sec. *north.*

Since the angular speed is uniform, the instantaneous angular speed is the same as the mean speed. The magnitude of the instantaneous angular velocity is the same, but its direction, if the point moves eastward from the top, is *north.*

(2) *Criticise the statement, "a point moves in a circle the plane of which is constantly changing, with uniform angular velocity."*

ANS. A point can move in a circle the plane of which is changing, with uniform *angular speed.* But if the angular *velocity* is uniform, its line representative does not change either in magnitude or direction, and the plane therefore cannot change. A point cannot move in a changing plane with uniform angular velocity.

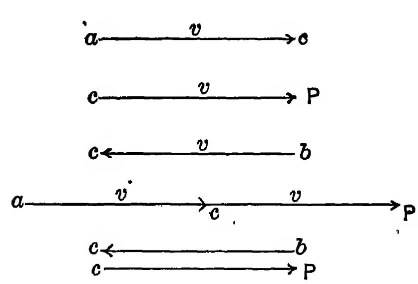
Resolution and Composition of Linear Velocity.—Since linear velocity is linear displacement per unit of time when the interval of time is indefinitely small (page 63), it has magnitude and direction, and can be represented by a straight line, just like linear displacement itself. Therefore all the principles of Chapter II, page 54, which hold good for linear displacements hold good also for linear velocities.

We have, then, the "*triangle and polygon of velocities,*" and can combine and resolve velocities just like displacements.

We also have relative velocity with similar notation as for displacements (page 55).

Examples.—(1) *A ship sails N. 30° E. with a speed of 10 miles an hour. Find its velocity east and north.*

ANS. 5 miles per hour east, $5\sqrt{3}$ miles per hour north.

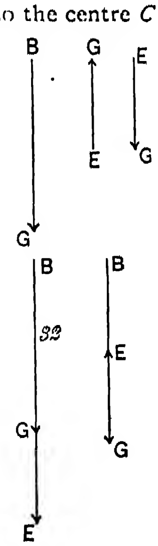


(2) *Find the vertical velocity of a train moving up a 1-per-cent gradient at a speed of 30 miles per hour.*

ANS. 0.3 miles per hour.

(3) *A circle rolls on a horizontal plane. Its centre moves with a velocity v towards the east. Find the velocity of the top and bottom points relative to the plane.*

ANS. The velocity of the top point a relative to the centre C is given by the line $ac = v$ towards the east; of the centre C relative to the plane by $cP = v$ towards the east; of the bottom point b relative to the centre C by $bc = v$ towards the west.



If, then, we lay off ac and cP in order, we have the velocity of a relative to the plane P given by $aP = 2v$ towards the east. The top point has, relative to the earth, twice the velocity of the centre.

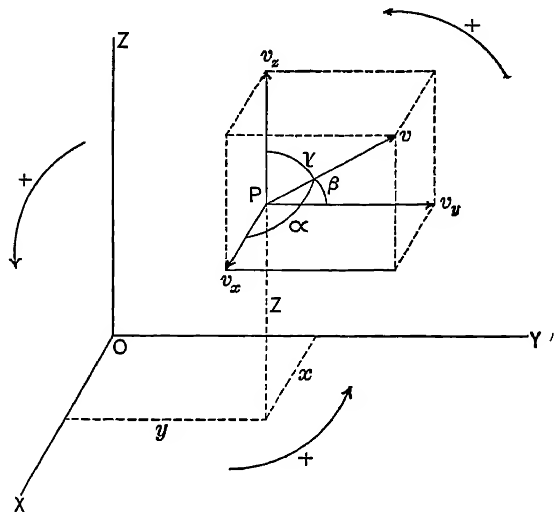
If we lay off b relative to C, and C relative to P, we have b relative to P, zero. The bottom point is at rest relative to the earth.

(4) *A ball let fall in an elevator has a velocity relative to the ground of 32 feet per sec., while the elevator has a velocity relative to the ground of 12 feet per sec. Find the velocity of the ball relative to the elevator when it is rising and falling.*

ANS. The velocity of the ball relative to the ground is given by $BG = 32$ downwards. If elevator is rising, we have elevator relative to ground given by $EG = 12$ upwards. If elevator is falling, $EG = 12$ downwards.

In the first case, laying off B relative to G, and G relative to E, we have B relative to E 44 feet per sec. down. In the second case, laying off B relative to G, and G relative to E, we have B relative to E 20 ft. per sec. downwards.

Rectangular Components of Velocity.—Let a point P be given by its co-ordinates x, y, z, and let it have the velocity v, whose line representative makes the angles α, β, γ, with axes through P parallel to the co-ordinate axes OX, OY, OZ, respectively.



Let the components of v parallel to these axes be v_x, v_y, v_z, respectively.

Then we have

$$v_x = \frac{dx}{dt} = v\cos\alpha, \qquad v_y = \frac{dy}{dt} = v\cos\beta,$$

$$v_z = \frac{dz}{dt} = v\cos\gamma.$$

Analytic Determination of Resultant Velocity.—If the point P has several simultaneous velocities, v_1, v_2, v_2, etc., making the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, etc., then we have for the resultant components

$$\left.\begin{aligned} v_x &= \frac{dx}{dt} = v_1\cos\alpha_1 + v_2\cos\alpha_2 + v_3\cos\alpha_3 + \ldots = \Sigma v\cos\alpha, \\ v_y &= \frac{dy}{dt} = v_1\cos\beta_1 + v_2\cos\beta_2 + v_3\cos\beta_3 + \ldots = \Sigma v\cos\beta, \\ v_z &= \frac{dz}{dt} = v_1\cos\gamma_1 + v_2\cos\gamma_2 + v_3\cos\gamma_3 + \ldots = \Sigma v\cos\gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

In these summations we must take components in the directions OX, OY, OZ positive, in the opposite directions negative.

We have, then, for the magnitude of the resultant velocity v

$$v = +\sqrt{v_x^2 + v_y^2 + v_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This resultant passes, of course, through P, and its direction cosines are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v}, \qquad \cos\gamma = \frac{v_z}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have also, from (3),

$$v = v_x\cos\alpha + v_y\cos\beta + v_z\cos\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In determining these cosines we take v always positive and measure all angles α, β, γ in such directions that their projections on the co-ordinate planes XY, YZ, ZX shall be positive from OX around to OY, from OY around to OZ, from OZ around to OX, as indicated by the arrows in the figure.

If all the velocities are in one plane, as, for instance, the plane of XY, we make $\gamma = 90°$, and hence $v_z = 0$ in (1), (2) and (3).

Examples.—(1) *A point has the component velocities in the same plane XY, $v_1 = 12$, $v_2 = 24$, $v_3 = 36$, $v_4 = 48$ ft. per sec., making angles with OX of $\alpha_1 = +60°$, $\alpha_2 = +150°$, $\alpha_3 = +240°$, $\alpha_4 = 330°$. Find the resultant velocity.*

ANS. We have $\alpha - \beta = 90°$. Hence $\beta_1 = -30°$, $\beta_2 = +60°$, $\beta_3 = +150°$, $\beta_4 = +240°$. Hence

$$v_x = 6 - 12\sqrt{3} - 18 + 24\sqrt{3} = +8.784 \text{ ft. per sec.},$$

$$v_y = +6\sqrt{3} + 12 - 18\sqrt{3} - 24 = -32.784 \text{ ft. per sec.},$$

The resultant is $v = +33.94$ ft. per sec., making the angle α with OX given by

$$\cos\alpha = \frac{+8.784}{+33.94} = 0.2588, \quad \text{or} \quad \alpha = \text{about } 75°.$$

(2) *A point has the component velocities $v_1 = 40$, $v_2 = 50$, $v_3 = 60$ ft. per sec., making the angles $\alpha_1 = +60°$, $\beta_1 = +60°$, γ_1 obtuse, $\alpha_2 = +60°$ $\beta_2 = +60°$, γ_2 acute; $\alpha_3 = +60°$, $\beta_3 = +30°$. Find the resultant.*

ANS. We find the angles γ by the equation, page 13,

$$\cos^2\gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta). \quad \text{Hence } \gamma_1 = +135°, \ \gamma_2 = +45°, \ \gamma_3 = 90°.$$

We have, then, $v_x = 20 + 25 + 30 = +75$ ft. per sec., $v_y = 20 + 25 + 30\sqrt{3} = +96.96$ ft. per sec., $v_z = -\frac{40}{\sqrt{2}} + \frac{50}{\sqrt{2}} + 0 = +7.071$ ft. per sec. The resultant is $v = 119.6$ ft. per sec., and its direction cosines are $\cos\alpha = \frac{+75}{119.6}$, $\cos\beta = \frac{+96.96}{119.6}$, $\cos\gamma = \frac{+7.071}{119.6}$, or $\alpha = 51°\,10'$, $\beta = 35°\,50'$, $\gamma = 86°\,37'$.

Resolution and Composition of Angular Velocity.—Since angular velocity is angular displacement per unit of time *when the interval of time is indefinitely small* (page 63), it has magnitude and direction and can be represented by a straight line, just like angular displacement itself. But since the time is indefinitely small, and therefore the angular displacement indefinitely small, we have the case (*b*), page 59. Therefore all the principles of Chapter II, page 54, hold good also for angular velocities whether simultaneous or successive, and we have the "triangle and polygon" of angular velocities, and can combine and resolve them just like linear displacements (page 55).

We have also relative angular velocity with similar notation as for linear displacements (page 56).

Examples.—(1) *A point on a sphere is rotating uniformly about a diameter at the rate of 10 radians per minute. Find the component angular velocity about another diameter inclined 30° to the first.*

ANS. $5\sqrt{3}$ radians per min.

(2) *A point has two simultaneous or successive angular velocities of 2 radians per sec. and 4 radians per sec. about axes inclined 60° to each other. Find the resultant.*

ANS. $2\sqrt{7}$ radians per sec. about an axis inclined to the greater component at an angle whose sine is $\frac{\sqrt{3}}{2\sqrt{7}}$.

(3) *A sphere with one of its superficial points fixed has two angular velocities, either simultaneous or successive, one of 8 radians per sec. about a tangent line, and one of 15 radians per sec. about a diameter. Find the resultant angular velocity.*

ANS. 17 radians per sec. about an axis inclined to the greater component at an angle whose tangent is $\frac{8}{15}$.

(4) *A pendulum suspended from a point in the prolonged polar axis of the earth swings in a plane through the axis. Find the angular velocity of this plane relative to the earth. See example (3), page 60.*

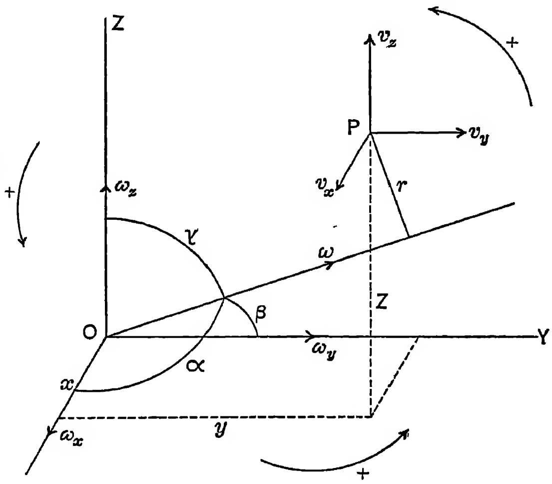
ANS. 2π radians per day south. That is, an observer looking south along the polar axis sees the plane rotate *clockwise* at this rate.

(5) *Let the pendulum be suspended from a point in the prolonged radius of the earth at a place of latitude λ. Find the angular velocity of the plane of swing relative to the earth. See example (4), page 60.*

ANS $2\pi \sin \lambda$ radians per day in a direction towards the centre of the earth. That is, an observer looking towards the centre of the earth would see the plane rotate clockwise at this rate.

The time of a complete revolution of the plane would then be $\frac{2\pi}{2\pi \sin \lambda} = \frac{1}{\sin \lambda}$ days. If λ is 60°, the time of revolution would be $\frac{2}{\sqrt{3}} = 1.155$ days. At the equator $\sin \lambda = 0$, and the plane does not rotate relatively to the earth. At the poles $\sin \lambda = 1$, and the plane makes a revolution in 1 day.

Rectangular Components of Angular Velocity.—Let a point P be given by its co-ordinates x, y, z, and let it have the angular velocity ω, whose line representative passes through O and makes the angles α, β, γ with the axes OX, OY, OZ, respectively.



Then the components of ω along the axes are $\omega_x = \omega \cos \alpha$, $\omega_y = \omega \cos \beta$, $\omega_z = \omega \cos \gamma$.

These components are positive in the directions OX, OY, OZ. Since rotation is always seen clockwise when we look along a line representative in the direction of its arrow, we have, then, positive rotation in the plane XY from OX around to OY; in the plane YZ from OY around to OZ; in the plane ZX from OZ around to OX, as shown by the arrows in the figure.

Analytic Determination of Resultant Angular Velocity.—If the point P has several simultaneous angular velocities, ω_1, ω_2, ω_3, etc., all passing through O and making the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, etc., then we have for the resultant components

$$\left.\begin{aligned} \omega_x &= \omega_1 \cos \alpha_1 + \omega_2 \cos \alpha_2 + \omega_3 \cos \alpha_3 + \ldots = \Sigma\omega \cos \alpha, \\ \omega_y &= \omega_1 \cos \beta_1 + \omega_2 \cos \beta_2 + \omega_3 \cos \beta_3 + \ldots = \Sigma\omega \cos \beta, \\ \omega_z &= \omega_1 \cos \omega_1 + \omega_2 \cos \gamma_2 + \omega_3 \cos \gamma_3 + \ldots = \Sigma\omega \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

In these summations we must take components in the directions OX, OY, OZ positive, in the opposite direction negative.

We have, then, for the magnitude of the resultant

$$\omega = + \sqrt{\omega_x^2 + \omega_y^2 + w_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This resultant passes, of course, through O, and its direction cosines are

$$\cos \alpha = \frac{\omega_x}{\omega}, \qquad \cos \beta = \frac{\omega_y}{\omega}, \qquad \cos \gamma = \frac{\omega_z}{\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, we have also, from (3),

$$\omega = \omega_x \cos \alpha + \omega_y \cos \beta + \omega_z \cos \gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In determining the cosines, we take ω always positive and measure all angles α, β, γ in such directions that their projections on the co-ordinate planes XY, YZ, ZX shall be posi-

tive from OX around to OY, from OY around to OZ, from OZ around to OX, as indicated by the arrows in the figure.

If all the velocities are in one plane, as, for instance, the plane of XY, we make $\gamma = 90°$ and hence $\omega_z = 0$ in (1), (2) and (3).

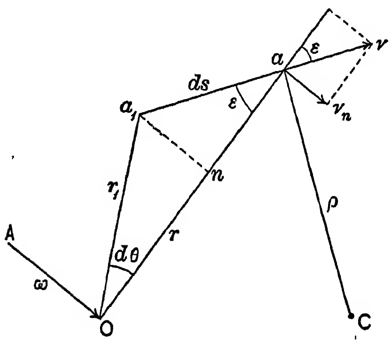
Example.—(1) *A point has the component angular velocities in the same plane XY, $\omega_1 = 12$, $\omega_2 = 24$, $\omega_3 = 36$, $\omega_4 = 48$ radians per sec., making angles with OX, $\alpha_1 = +60°$, $\alpha_2 = +150°$, $\alpha_3 = +240°$, $\alpha_4 = +330°$ Find the resultant velocity.* (*See example (1), page 68*)

ANS. $\omega_x = +8.784$ radians per sec.; $\omega_y = -32.784$ radians per sec.; $\omega = +33.99$ radians per sec., $\cos\alpha = 0.2588$. or $\alpha =$ about $75°$.

(2) *A point has the component angular velocities $\omega_1 = 40$, $\omega_2 = 50$, $\omega_3 = 60$ radians per sec., making the angles $\alpha_1 = +60°$, $\beta_1 = +60°$, γ_1 obtuse; $\alpha_2 = +60°$, $\beta_2 = +60$, γ_2 acute; $\alpha_3 = +60°$, $\beta_3 = +30°$. Find the resultant.*

ANS. See example (2), page 68.

Linear in Terms of Angular Velocity.—Let a_1 and a represent two consecutive positions of a point moving in any path. Then the distance $a_1a = ds$.



Take any point O as a pole, and draw the radius vectors $Oa_1 = r_1$ and $Oa = r$. Then the indefinitely small angle a_1Oa is $d\theta$.

Drop the perpendicular a_1n from a_1 upon Oa, and denote the angle of inclination a_1an between the radius vector Oa and a_1a by ϵ. Then

$$a_1n = ds \, . \sin\epsilon.$$

But if the points a_1 and a are consecutive, we have also

$$a_1n = r_1 d\theta.$$

Hence we have

$$r_1 d\theta = ds \, . \sin\epsilon. \quad \quad (1)$$

But for consecutive points $na = dr$ and

$$r_1 = On = r - dr.$$

Hence equation (1) becomes

$$r d\theta = dr \, . \, d\theta + ds \, . \sin\epsilon. \quad \quad (2)$$

As $d\theta$ in this equation decreases, $rd\theta$ approaches the limiting value $ds \, . \sin\epsilon$. We have then at the limit

$$r d\theta = ds \, . \sin\epsilon. \quad \quad (3)$$

If we divide by the indefinitely small time dt in passing from a_1 to a, we have

$$r\frac{d\theta}{dt} = \frac{ds}{dt} \, . \sin\epsilon.$$

But (page 64) $\frac{d\theta}{dt}$ is the magnitude of the angular velocity of the point relative to O, which we have denoted by ω, and (page 63) $\frac{ds}{dt}$ is the magnitude of the linear velocity,

which we have denoted by v. Also, $v \sin \epsilon$ is the component of the velocity v at right angles to the radius vector r in the plane of v and r.

Let us denote this normal component by v_n. Then we have

$$rw = v \sin \epsilon = v_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence *the product of the radius vector r for any point by the angular velocity ω gives the linear velocity v_n of the point normal to the radius vector in the plane of v and r.*

The line representative of the angular velocity ω is a straight line AO through the pole O at right angles to the plane a_1Oa of v and r, so that, looking in the direction of its arrow, the rotation of the radius vector is seen clockwise.

Equation (4) is general, whatever the path or wherever the pole O may be taken.

If the pole O is taken at the centre of curvature C, so that Oa is the radius of curvature ρ, then $\epsilon = 90^\circ$ and we have

$$\rho\omega = v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the pole O is taken anywhere in the plane through ρ perpendicular to v, we still have $\epsilon = 90^\circ$, and in this case

$$r\omega = v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

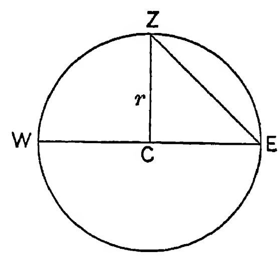
Hence in this case *the product of the radius vector by the angular velocity gives the linear velocity itself.*

Example.—*Let the distance described by a moving point be given by the equation*

$$s = 7t + 8t^2 + 2t^3,$$

where s is the number of feet described in any number of seconds t.

Let the path be a circle in an east and west plane, and let the point start from the top point Z and move eastward. Let the radius of the circle be $r = \frac{124}{\pi} = 39.47$ ft.



At the end of $t = 2$ sec. find the distance and angle described; the linear and angular displacement, the mean linear and angular speed, the mean linear and angular velocity; the instantaneous linear aad angular velocity. The origin is taken at C and the starting-point at Z in the figure.

ANS Insert $t = 2$ in the equation, and we have for the distance described $s = 62$ ft.

We have $r\theta = s$ Hence $\theta = \frac{s}{r} = \frac{62\pi}{124} = \frac{\pi}{2}$ radians The point, then, has moved in 2 seconds through a quadrant from Z to E.

The linear displacement is then from Z to E, or $D = r\sqrt{2}$ ft in a direction making an angle of 45° with the initial radius vector CZ.

The magnitude of the angular displacement is the same as the angle described, or $\frac{\pi}{2}$ radians, and its *direction is north.*

We find, just as in example (1), page 64, the mean linear speed

$$\frac{s - s_1}{t - t_1} = 7 + 8(t + t_1) + 2(t^2 + tt_1 + t_1^2).$$

For $t_1 = 0$ and $t = 2$ this gives

$$\frac{s - s_1}{t - t_1} = 31 \text{ ft. per sec.}$$

We have for the mean angular speed

$$r\frac{\theta - \theta_1}{t - t_1} = 31, \quad \text{or} \quad \frac{\theta - \theta_1}{t - t_1} = \frac{31}{r} = \frac{\pi}{4} \text{ radians per sec.}$$

The mean linear velocity is the linear displacement per unit of time, or $\frac{r\sqrt{2}}{t} = \frac{r}{\sqrt{2}}$ ft. per sec. in the direction from Z to E.

The mean angular velocity is the angular displacement per unit of time, or $\frac{\pi}{2t} = \frac{\pi}{4}$ radians per sec., *direction north.*

The instantaneous linear velocity is

$$v = \frac{ds}{dt} = 7 + 16t + 6t^2.$$

For $t = 2$ this gives $v = 63$ ft. per sec.

The instantaneous angular velocity is given in magnitude by $r\omega = v$, or $\omega = \frac{v}{r} = \frac{63\pi}{124}$ radians per sec., and *its direction is north.*

CHAPTER VI.

LINEAR AND ANGULAR RATE OF CHANGE OF SPEED. LINEAR ACCELERATION.

Change of Speed.—When the speed of a point varies in magnitude, the difference for any interval of time between the final and initial instantaneous speeds is the change of speed for that interval of time.

Mean Rate of Change of Speed.—The change of speed per unit of time is the MEAN RATE OF CHANGE OF SPEED.

Thus, if the linear speed of a point at any time t is v, and at an earlier time t_1 it is v_1, then the change of linear speed is $(v - v_1)$ for the interval of time $(t - t_1)$, and the mean rate of change of linear speed is

$$\frac{v - v_1}{t - t_1}.$$

This is positive if v is greater than v_1, or if the linear speed increases with the time, and negative if v is less than v_1, or if the linear speed decreases with the time.

Similarly, if the angular speed of a point at any time t is ω, and at an earlier time, t_1, it is ω_1, then the change of angular speed is $(\omega - \omega_1)$ for the interval of time $(t - t_1)$, and the mean rate of change of angular speed is

$$\frac{\omega - \omega_1}{t - t_1}.$$

This is positive if ω is greater than ω_1, or if the angular speed increases with the time, and negative if ω is less than ω_1, or if the angular speed decreases with the time.

Rate of change of speed, whether linear or angular, is, then, like speed itself, a SCALAR quantity having magnitude and sign but independent of direction (page 62).

The unit of rate of change of linear speed is evidently one unit of linear speed per unit of time, or 1 ft.-per-sec. per sec.

The unit of rate of change of angular speed is one unit of angular speed per unit of time, or 1 radian-per-sec. per sec.

Instantaneous Rate of Change of Speed.—The limiting value of the mean rate of change of speed when the interval of time is indefinitely small is the INSTANTANEOUS RATE OF CHANGE OF SPEED.

We have, then, the instantaneous rate of change of linear speed given by

$$\frac{dv}{dt},$$

and the instantaneous rate of change of angular speed given by

$$\frac{d\omega}{dt}.$$

If the instantaneous rate of change of speed is zero, the speed itself is uniform and therefore the same as the mean speed for any interval of time, which is also uniform.

If the instantaneous rate of change of speed does not vary in magnitude, it is uniform and therefore the same as the mean rate of change for any interval of time, which is also uniform.

If the instantaneous rate of change of speed varies in magnitude with the time, it is variable. In such case the mean rate of change is also variable.

Examples.—(1) *Let the distance described by a moving point be given by the equation*

$$s = 7t + 8t^2,$$

where s is the number of feet described in any number of seconds t

This is the same equation as in example (1), page 64, and we find, as before in that example, the instantaneous linear speed

$$v = \frac{ds}{dt} = 7 + 16t. \quad \text{(1)}$$

For any other time, t_1, we can write

$$v_1 = 7 + 16t_1. \quad \text{(2)}$$

Suppose t_1 less than t. Then subtracting (2) from (1), we have for the change of speed for any interval of time $(t - t_1)$

$$v - v_1 = 16(t - t_1). \quad \text{(3)}$$

The mean rate of change of speed is then

$$\frac{v - v_1}{t - t_1} = 16 \text{ ft.-per-sec. per sec.}$$

Since this does not change with the time, it is uniform and is the same as the instantaneous rate of change of speed, $\frac{dv}{dt}$.

(2) *Let the instantaneous linear speed of a point be given by*

$$v = 7 + 16t + 6t^2. \quad \text{(1)}$$

For any other less time, t_1, we have

$$v_1 = 7 + 16t_1 + 6t_1^2.$$

Subtracting (2) from (1), we have for the change of speed for any interval of time $(t - t_1)$

$$v - v_1 = 16(t - t_1) + 6(t^2 - t_1^2) = 16(t - t_1) + 6(t + t_1)(t - t_1).$$

The mean rate of change of speed is then

$$\frac{v - v_1}{t - t_1} = 16 + 6(t + t_1). \quad \text{(3)}$$

This, we see, is variable and changes with the time. As the interval of time $(t - t_1)$ decreases, t_1 approaches equality with t and the mean rate of change of speed given by (3) approaches the limiting value $16 + 12t$. We have, then, for the instantaneous rate of change of speed

$$\frac{dv}{dt} = 16 + 12t.$$

This, we see, changes with the time and is therefore variable.

(3) *Let the angular speed of a point be given by*

$$\omega = 7 + 16t + 6t^2.$$

where ω is the angular speed in radians per sec. for any instant t.

Then we have, just as before, the change of angular speed for any interval of time $(t - t_1)$

$$\omega - \omega_1 = 16(t - t_1) + 6(t^2 - t_1^2).$$

The mean rate of change of angular speed is, then,

$$\frac{\omega - \omega_1}{t - t_1} = 16 + 6(t + t_1),$$

and the instantaneous rate of change of angular speed is

$$\frac{d\omega}{dt} = 16 + 12t.$$

Mean Linear Acceleration of a Point.—Let a_1 and a be two positions of a point moving in any path and passing from a_1 to a in the time interval $t - t_1$. Let v_1 and v be the corresponding instantaneous linear velocities.

FIG. (a).

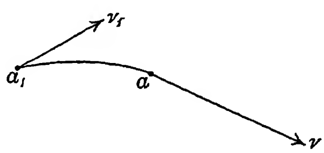
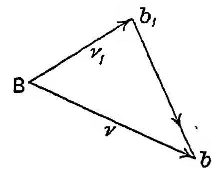


FIG. (b).



Each velocity, Fig. (*a*), is tangent to the path at the corresponding point, and is equal in magnitude to the speed at that point, so that $(v - v_1)$ is the *change of speed*, and $\frac{v - v_1}{t - t_1}$ is the *mean rate of change of speed* (page 73).

In Fig. (*b*) take any point B as a pole and lay off Bb_1 and Bb, the line representatives of v_1 and v, and join b_1 and b by a straight line.

Then the straight line b_1b is the line representative of the *change of velocity*, and $\frac{b_1b}{t - t_1}$ is the *mean rate of change of velocity* for the interval of time $t - t_1$.

This is called the MEAN LINEAR ACCELERATION of the point for the interval of time.

It is therefore a vector quantity given in magnitude and direction by a straight line parallel to b_1b, having the direction from b_1 to b as shown by the arrow in Fig. (*b*), and equal in magnitude to $\frac{b_1b}{t - t_1}$.

Mean linear acceleration can be defined, then, as *mean time-rate of change of velocity*, whether that change takes place in the direction of the velocity or not.

The unit of mean linear acceleration is evidently one unit of linear speed per unit of time, or 1 ft.-per-sec. per sec., just as for rate of change of linear speed (page 73).

Instantaneous Linear Acceleration.—The limiting magnitude and direction of the mean linear acceleration when the interval of time is indefinitely small is the INSTANTANEOUS LINEAR ACCELERATION.

FIG. (a)

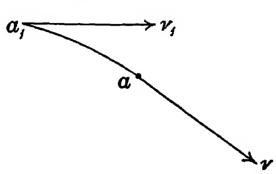
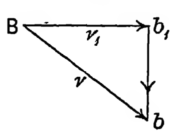


FIG. (b).



Thus if the interval of time dt is indefinitely small, the two positions a_1 and a of the moving point, Fig. (*a*), are consecutive.

In Fig. (*b*), then, $\frac{b_1b}{dt}$ is the instantaneous linear acceleration.

It is therefore a vector quantity given in magnitude and direction by a straight line parallel to b_1b, having the direction from b_1 to b as shown by the arrow in Fig. (*b*), and equal in magnitude to $\frac{b_1b}{dt}$.

Instantaneous linear acceleration can then be defined as *limiting time-rate of change of velocity, whether that change* $\left(\frac{b_1b}{dt}\right)$ *take place in the direction of the velocity or not.*

Its unit is evidently one unit of speed per unit of time, or 1 ft.-per-sec. per sec., just as for mean linear acceleration.

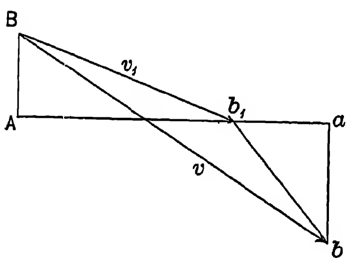
We shall always denote instantaneous linear acceleration by the letter f.

The term "acceleration" always signifies instantaneous linear acceleration unless otherwise specified.

We shall see later how to determine it in magnitude and direction (page 78); it is sufficient to note here that *if the path is a straight line,* b_1b coincides with v, and we have $f = \frac{dv}{dt}$. That is, *the magnitude of the acceleration in this case is the instantaneous rate of change of speed* (page 73).

Resolution and Composition of Linear Acceleration.—Since acceleration is thus time-rate of change of velocity when the interval of time is indefinitely small, and can be represented by a straight line, it follows that all the principles of Chap. II, page 55, for displacements hold good also for accelerations. We can then combine and resolve linear accelerations just like displacements (page 56), and we have the triangle and polygon of accelerations just as for displacements.

We have also relative acceleration with similar notation as for displacement (page 56).



Let Bb_1 and Bb be the initial and final velocities v_1 and v of a point in an indefinitely small time dt, so that $\frac{b_1b}{dt}$ is the acceleration f. Draw BA and ba at right angles to any line Aa through b_1 in any given direction.

Then $\frac{b_1a}{dt}$ is the component f_a in the direction Aa of the acceleration f. But

$$\frac{b_1a}{dt} = \frac{Aa - Ab_1}{dt},$$

and Aa and Ab_1 are the components in the direction Aa of the velocities v and v_1

Hence *the component acceleration in any direction is equal to the time-rate of change of velocity in that direction.*

Examples.—(1) *A ball let fall in an elevator has an acceleration downwards relative to the ground of 32 ft.-per-sec. per sec., while the elevator has an acceleration relative to the ground of 12 ft.-per-sec. per sec. Find the acceleration of the ball relative to the elevator when the acceleration of the elevator is upwards and downwards.*

ANS. Solution precisely the same as for example (4), page 66. When acceleration of elevator is upwards, speed of the elevator increases if it is going up, decreases if it is going down, and acceleration of ball relative to elevator is 44 ft.-per-sec. per sec. downwards.

When acceleration of elevator is downwards, speed of the elevator increases if it is going down, decreases if it is going up, and acceleration of ball relative to elevator is 20 ft.-per-sec per sec. downwards.

(2) *Two trains move in straight lines making an angle of 60°. The one, A, is increasing its speed at the rate of 4 ft.-per-min. per min. The other, B, has the brakes on and is losing speed at the rate of 8 ft.-per-min. per min. Find the relative accelerations.*

ANS. A relative to B, $4\sqrt{7}$ ft.-per-min. per min, inclined to the direction of A at an angle whose sine is $\sqrt{\frac{3}{7}}$.

(3) *The initial and final velocities of a moving point during an interval of two hours are 8 miles per hour E 30° N., and 4 miles per hour N. Find the change of velocity and the mean acceleration.*

ANS. $4\sqrt{3}$ miles per hour W.; $2\sqrt{3}$ miles-per-hour per hour W.

(4) *A point has the simultaneous accelerations, 60 ft.-per-sec. per sec. N.; 88 ft.-per-sec. per sec. W. 30° S., 60 ft.-per-sec. per sec. E. 30° S. Find the magnitude and direction of the resultant.*

ANS. 28 ft.-per-sec. per sec. W. 30° S.

Rectangular Components of Acceleration.—The same equations and conventions hold for the components of an acceleration in three rectangular directions as for velocity (page 66). We have therefore only to replace v in the equations and figure of page 67 by f with the proper subscripts.

Analytic Determination of Resultant Acceleration.—So, also, the same equations and conventions hold for finding the resultant acceleration as for finding the resultant velocity, page 67. We have therefore only to replace v in equations (1), (2), (3), (4), page 67, by f with the proper subscripts.

Examples.—*Students should solve examples (1) and (2), page 68, for accelerations instead of velocities.*

Tangential and Central Acceleration.—We have seen (page 75) that if v_1 and v are the velocities at a_1 and a, Fig. (a), and if the interval of time dt is indefinitely small, so that a_1 and a are consecutive points of the path, then,

FIG. (a).

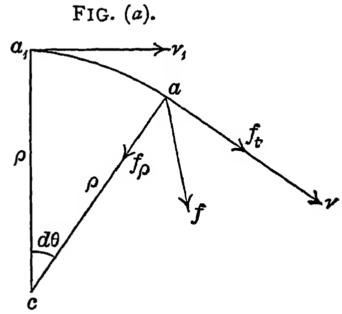
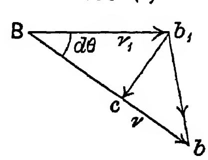


FIG. (b).



in Fig. (b), $\dfrac{b_1b}{dt}$ is the instantaneous linear acceleration f at a.

This has magnitude and direction, can be represented by a straight line, and combined and resolved just like displacement.

We can therefore resolve the acceleration $f = \dfrac{b_1b}{dt}$ at a into a component $\dfrac{cb}{dt}$ along the velocity v, and $\dfrac{b_1c}{dt}$ at right angles to the velocity v. Since v is tangent to the path at a, we call the first component, $\dfrac{cb}{dt}$, the TANGENTIAL ACCELERATION and denote it by f_t, the subscript t denoting that the acceleration is along v and therefore *tangent* to the path. The second component, $\dfrac{b_1c}{dt}$, is perpendicular to v and must evidently *always act towards the centre of curvature C of the path* along the radius of curvature $aC = \rho$. We therefore call it the CENTRAL ACCELERATION and denote it by f_ρ, the subscrpit ρ denoting that the acceleration is along the radius of curvature ρ and therefore towards the centre of curvature.

Now, since the points a_1 and a are consecutive, we have $cb = v - v_1 = dv$, and hence

$$f_t = \frac{cb}{dt} = \frac{dv}{dt}. \qquad (1)$$

But $\dfrac{dv}{dt}$ is the instantaneous rate of change of linear speed (page 73).

Hence *the magnitude of the tangential linear acceleration f_t at any instant is the instantaneous rate of change of linear speed $\dfrac{dv}{dt}$ at that instant.*

We see also from Fig. (b) that, since v_1 and v are perpendicular to the radius of curvature

Ca_1 and Ca in Fig. (a), the indefinitely small angle $a_1Ca = d\theta$ is equal to b_1Bb. We have, then, when b_1 and b are consecutive, $b_1c = v_1d\theta = vd\theta$, and hence

$$f_\rho = \frac{b_1c}{dt} = v\frac{d\theta}{dt}.$$

But $\frac{d\theta}{dt}$ is the angular velocity ω *for the centre of curvature* C, and if ρ is the radius of curvature, we have (page 71) $\rho\omega = v$. Therefore

$$f_\rho = v\omega = \rho\omega^2 = \frac{v^2}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence *the magnitude of the central acceleration* f_ρ *at any instant is given by the product* $v\omega$ *of the linear velocity* v *and the angular velocity* ω *for the centre of curvature at that instant; or by the product* $\rho\omega^2$ *of the radius of curvature* ρ *and the square of the angular velocity* ω^2 *for the centre of curvature at that instant; or by the quotient* $\frac{v^2}{\rho}$ *of the square of the linear velocity* v^2 *divided by the radius of curvature* ρ *at that instant.*

The direction of the central acceleration is *always towards the centre of curvature.*

Since the acceleration f is the resultant of f_ρ and f_t, we have for the magnitude of f

$$f = \sqrt{f_t^2 + f_\rho^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for its direction cosine with f_ρ

$$\cos\alpha = \frac{f_\rho}{f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Example.—*A point moves in a vertical circle of radius $r = 21$ ft. in an east and west plane with a velocity given at any instant by*

$$v = 7 + 16t + 6t^2,$$

where t is the interval of time from the start. At the end of $t = 2$ sec. the point is at the top of the circle moving eastward. Find ω, f_t and f_ρ at this instant.

ANS. We find, just as in example (2), page 74,

$$f_t = \frac{dv}{dt} = 16 + 12t,$$

or for $t = 2$ sec. $f_t = 40$ ft.-per-sec. per sec. towards the east. Also, for $t = 2$ sec. $v = 63$ ft. per sec. east, and hence at this instant $f_\rho = \frac{v^2}{r} = 189$ ft.-per-sec. per sec. towards the centre, and $\omega = \frac{f_\rho}{v} = 3$ radians per sec. *north.*

The acceleration f at the instant is, then,

$$f = \sqrt{f_t^2 + f_\rho^2} = 63\sqrt{10} \text{ ft.-per-sec. per sec.},$$

making with the radius through the top point an angle whose cosine is $\frac{f_\rho}{f} = \frac{3}{\sqrt{10}}$.

Uniform and Variable Acceleration.—If the acceleration f has the same magnitude and direction, whatever the time, it is UNIFORM. If either magnitude or direction change, it is VARIABLE.

If f is zero, there is no change of velocity and we have uniform speed in a straight line, or uniform velocity.

If f_ρ is zero but f_t is not, we have variable speed in a straight line. The velocity is variable (page 65).

If f_t is zero but f_ρ is not, we have uniform speed in some curved path. The velocity is variable (page 65). If at the same time f_ρ is constant in magnitude, *the curved path is a circle.* For $f_\rho = \frac{v^2}{\rho}$, and if f_ρ is constant and v is constant, ρ must also be constant.

If f_t and f_ρ are both variable, we have variable speed in some curved path. The velocity is variable (page 65).

Example.—*Criticise the statement, " a point moves in a circle with uniform acceleration."*

ANS.—If the acceleration is uniform, it does not change either in magnitude or direction. But if a point moves in a circle, we must have a radial acceleration f_ρ at every instant. The acceleration is therefore not uniform. If the magnitude of f_ρ is constant, then we have uniform speed in the circle. If not, we have varying speed in the circle. A point can move in a circle with *constant* central acceleration, but not with *uniform* acceleration.

The Hodograph.—Let a point moving in any path have the consecutive positions a_1, a_2, a_3, etc., and let the corresponding velocities be v_1, v_2, v_3, etc., Fig. (a). These velocities are tangent to the path at a_1, a_2, a_3, etc., and are equal in magnitude to the speed at these points.

FIG. (a).

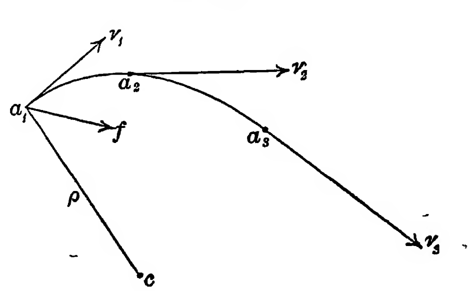
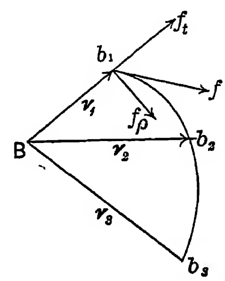


FIG. (b).



Now from any point B, Fig. (b), draw the line representatives Bb_1, Bb_2, Bb_3 of v_1, v_2, v_3, etc. The extremities of these lines will form a polygon $b_1b_2b_3$, and if the points a_1, a_2, a_3 are consecutive, the points b_1, b_2, b_3 will also be consecutive, and the polygon $b_1b_2b_3$ becomes a curve also. Thus as the point a describes the path $a_1a_2a_3$, Fig. (a), we can conceive a point b to describe a curve $b_1b_2b_3$, Fig. (b).

This curve is called the HODOGRAPH. The point B is the POLE of the hodograph; the points b_1, b_2, b_3 of the hodograph are the *points corresponding* to a_1, a_2, a_3 of the path.

When the point a moves from a_1 to a_2 in the indefinitely small time dt, the corresponding point b moves in the same time from b_1 to b_2. Now in such case $\frac{a_1a_2}{dt}$ is the velocity in the path, and $\frac{b_1b_2}{dt}$ is, then, the corresponding velocity in the hodograph.

But we have seen (page 75) that $\frac{b_1b_2}{dt}$ is the acceleration f at a_1.

Hence *any radius vector Bb in the hodograph is the line representative of the velocity v at the corresponding point of the path, and the velocity at any point of the hodograph is the acceleration f at the corresponding point of the path.*

Let $Ca = \rho$ be the radius of curvature, and ω the angular velocity of the point relative to C, so that the radius of curvature ρ turns about C with the angular velocity ω.

Then, since v is perpendicular to ρ, the angular velocity of Bb in Fig. (b) is also ω, and hence $v\omega = f_\rho$, where f_ρ is the acceleration perpendicular to v and therefore along the radius of curvature ρ towards the centre of curvature C.

We have then

$$f_\rho = v\omega,$$

just as already found (page 78).

Examples.—(1) *A point moves with uniform velocity. What is the hodograph?*

ANS. A point.

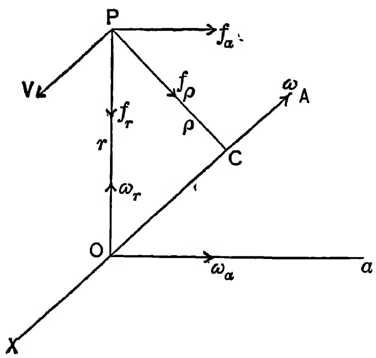
(2) *A point moves with uniform acceleration. What is the hodograph?*

ANS. A straight line.

(3) *A point moves with uniform speed v in any plane path. What is the hodograph?*

ANS. A circle of radius v.

Axial and Radial or Deflecting and Deviating Acceleration.—Let a point P have the angular velocity ω about an axis CA through the centre of curvature C, so that $PC = \rho$ is the radius of curvature. Let $v = \rho\omega$ be the velocity of P. Then $f_\rho = v\omega$.



We can resolve ω at any point O of the axis CA into a component ω_r about an axis OP through the point P, and a component ω_a about an axis Oa at right angles to OP. We can also resolve the central acceleration f_ρ into a component f_r along OP, and a component f_a parallel to Oa.

The component ω_r does not affect the velocity of P, since it passes through P. Hence, if r is the distance OP, we have

$$v = r\omega_a,$$

and the plane of rotation is the plane POX. The component ω_r *causes this plane to rotate* about the axis OP.

We see, then, that if a point rotates with angular velocity ω about an axis through the centre of curvature, we can resolve the motion at any point of that axis into rotation *in a rotating plane.*

If the radius of curvature *in this plane* is r, we have $v = r\omega_a$. The central acceleration *in this plane* is then

$$f_r = v\omega_a = r\omega_a^2. \qquad (1)$$

We call this the *radial* or *deflecting* acceleration because it is in the direction of the radius and causes change of direction of v in the plane of rotation POX.

But the rotation ω_r about OP causes the plane of rotation POX to change direction. We have, then, the acceleration at right angles to the plane of rotation POX

$$f_a = v\omega_r = r\omega_a\omega_r. \qquad (2)$$

We call this the *axial* or *deviating* acceleration because it is in the direction of the axis of rotation Oa and causes deviation of the plane of rotation POX. The radial or deflecting acceleration f_r then causes change of direction of v in the plane of rotation, that is P moves in a curve in this plane. The axial or deviating acceleration f_a causes change of this plane of rotation.

If f_r is zero, the path is a straight line in a rotating plane.

If f_a is zero, the path is a curve in a constant plane.

If f_a and f_r are zero, the path is a straight line. In all cases, since f_a and f_r are components of the central acceleration f_ρ, we have

$$f_\rho = \sqrt{f_a^2 + f_r^2}. \qquad (3)$$

Example.—*A point moves in a vertical circle of radius $r = 21$ ft. in an east and west plane with a velocity given at any instant by*

$$v = 7 + 16t + 6t^2,$$

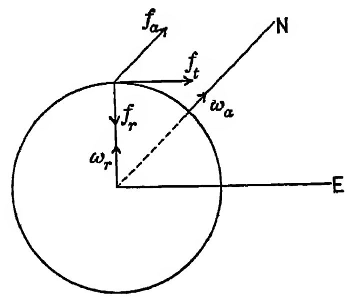
where t is the interval of time from the start. At the end of $t = 2$ sec. the point is at the top of the circle moving east. At the same instant the plane of the circle rotates about the vertical diameter with an angular velocity of 2 radians per sec. upwards. Find ω, f_t, ρ, and f_ρ and f.

ANS. We have, just as in example, page 78, for rotation in the plane the tangential acceleration

$$f_t = \frac{dv}{dt} = 16 + 12t,$$

or for $t = 2$ sec. $f_t = 40$ ft.-per-sec. per sec. towards the east. Also for $t = 2$ sec. $v = 63$ ft. per sec. east, and hence at this instant the deflecting or radial acceleration is $f_r = \frac{v^2}{r} = 189$ ft.-per-sec. per sec. towards the centre and $\omega_a = \frac{f_r}{v} = 3$ radians per sec. north.

We have also due to rotation of the plane, since $\omega_r = 2$ radians per sec., the axial or deviating acceleration $f_a = v\omega_r = 126$ ft. per sec. north.



Hence

$$\omega = \sqrt{\omega_a^2 + \omega_r^2} = \sqrt{13} \text{ radians per sec.},$$

making an angle with r in the plane of r and ω_a whose cosine is

$$\frac{\omega_r}{\omega} = \frac{2}{\sqrt{13}}.$$

The radius of curvature ρ is given by $\rho\omega = v$, or $\rho = \frac{v}{\omega} = \frac{63}{\sqrt{13}}$ ft.

The central acceleration is

$$f_\rho = \sqrt{f_a^2 + f_r^2} = 227.14 \text{ ft.-per-sec. per sec.}$$

The total acceleration is

$$f = \sqrt{f_t^2 + f_\rho^2} = 230.64 \text{ ft.-per-sec. per sec.}$$

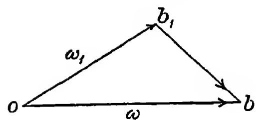
The direction cosines with f_t, f_a and f_r are

$$\cos\alpha + \frac{f_t}{f} = + \frac{4}{23}, \quad \cos\beta = \frac{f_a}{f} = + \frac{126}{230}, \quad \cos\gamma = - \frac{f_r}{f} = - \frac{189}{230}.$$

CHAPTER VII.

ANGULAR ACCELERATION.

Mean Angular Acceleration of a Point.—Let ω_1 and ω be the initial and final angular velocities of a point for any interval of time $t - t_1$.



Take any point O as a pole and lay off Ob_1 and Ob, the line representatives of ω_1 and ω, and join b_1b by a straight line.

Then the straight line b_1b is the line representative of the *change of angular velocity*, and $\frac{b_1b}{t - t_1}$ is the mean rate of change of angular velocity for the interval of time $t - t_1$.

This is called the MEAN ANGULAR ACCELERATION of the point for the interval of time.

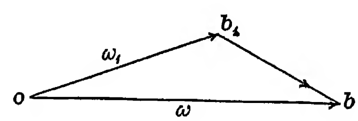
It is therefore a vector quantity given in magnitude and direction by a straight line parallel to b_1b, having the direction from b_1 to b as shown by the arrow in the figure, and equal in magnitude to $\frac{b_1b}{t - t_1}$.

Mean angular acceleration can be defined, then, as *mean time-rate of change of angular velocity*, whether that change takes place in the direction of the angular velocity or not.

The unit of mean angular acceleration is evidently one unit of angular speed per unit of time, or 1 radian-per-sec. per sec., just as for rate of change of angular speed (page 73).

Instantaneous Angular Acceleration.—The limiting magnitude and direction of the mean angular acceleration when the interval of time is indefinitely small is the INSTANTANEOUS angular acceleration.

Thus if the interval of time is dt or indefinitely small, then $\frac{b_1b}{dt}$ is the instantaneous angular acceleration.



It is therefore a vector quantity given in magnitude and direction by a straight line parallel to b_1b, having the direction from b_1 to b as shown by the figure and equal in magnitude to $\frac{b_1b}{dt}$.

Instantaneous angular acceleration can then be defined as *limiting time-rate of change of angular velocity*, whether that change take place in the direction of the velocity or not.

Its unit is evidently one unit of angular speed per unit of time, or 1 radian-per-sec. per sec., just as for mean angular acceleration.

We shall always denote instantaneous angular acceleration by the letter α.

The term "angular acceleration" always signifies instantaneous angular acceleration unless otherwise specified.

We shall see later how to determine it in magnitude and direction (page 84); it is sufficient to note here that if the point moves in an unchanging plane, b_1b coincides with

ω, and we have $\alpha = \frac{d\omega}{dt}$. That is, the magnitude of the angular acceleration in this case is the instantaneous rate of change of angular speed (page 73).

Resolution and Composition of Angular Acceleration.—Since angular acceleration is time-rate of change of angular velocity when the interval of time is indefinitely small, it has magnitude and direction and can be represented by a straight line, just like angular velocity (page 64). Therefore all the principles of Chapter II, page 54, hold good also for angular accelerations whether simultaneous or successive, and we have the "triangle and polygon" of angular accelerations, and can combine and resolve them just like linear displacement.

We have also relative angular acceleration with similar notation as for linear displacements (page 55).

Example.—(1) *A sphere has angular acceleration about a diameter at the rate of 10 radians-per-sec. per sec. Find the component angular acceleration about another diameter inclined 30° to the first.*

ANS.—$5\sqrt{3}$ radians-per-sec. per sec.

(2) *A body has two simultaneous or successive angular accelerations of 2 radians-per-sec. per sec. and 4 radians-per-sec. per sec. about axes inclined 60° to each other. Find the resultant.*

ANS. $2\sqrt{7}$ radians-per-sec. per sec. about an axis inclined to the greater component at an angle whose sine is $\frac{\sqrt{3}}{2\sqrt{7}}$.

(3) *A sphere with one of its superficial points fixed has two angular accelerations, either simultaneous or successive, one of 8 radians-per-sec. per sec. about a tangent line, and one of 15 radians-per-sec. per sec. about a diameter. Find the resultant.*

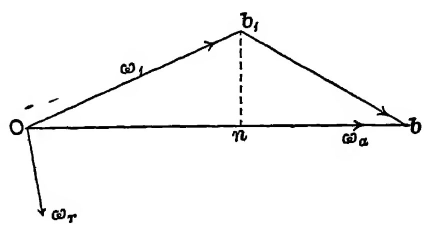
ANS. 17 radians-per-sec. per sec. about an axis inclined to the greater component at an angle whose tangent is $\frac{8}{15}$.

Rectangular Components of Angular Acceleration.—The same equations and conventions hold for the components of an angular acceleration in three rectangular directions as for angular velocity, page 69. We have therefore only to replace ω in the equations and figure of page 69 by α with the proper subscripts.

Analytic Determination of Resultant Angular Acceleration.—So, also, the same equations and conventions hold for finding the resultant angular acceleration as for finding the resultant angular velocity, page 69. We have therefore only to replace ω in equations (1), (2), (3) and (4), page 69, by α with the proper subscripts.

Examples.—*Students should solve examples (1) and (2), page 70, for angular accelerations instead of velocities.*

Axial and Normal Angular Acceleration.—We have seen that if ω_1 and ω_a are the initial and final angular velocities of a point during an indefinitely small interval of time dt, then $\frac{b_1 b}{dt}$ is the instantaneous angular acceleration α.



We can resolve this angular acceleration $\alpha = \frac{b_1 b}{dt}$ into a component $\frac{nb}{dt}$ along ω_a, and $\frac{b_1 n}{dt}$ normal to ω_a. Since ω_a coincides with the axis of rotation of the radius vector, we call the first component $\frac{nb}{dt}$ the AXIAL angular acceleration and denote it by α_a, the subscript a denoting that the acceleration is along the axis. The second component, $\frac{b_1 n}{dt}$ normal to ω_a, we denote by α_n, and call it the NORMAL angular acceleration.

Now let ω_r be the angular velocity of the line representative $Ob = \omega_a$ about an axis perpendicular to the plane of b_1Ob.

Then, just as on page 80, we have

$$\alpha_a = \frac{d\omega_a}{dt}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$\alpha_n = \omega_a\omega_r. \quad \ldots \ldots \ldots \ldots \quad (2)$$

But $\frac{d\omega_a}{dt}$ is the instantaneous rate of change of angular speed.

Hence *the magnitude of the axial angular acceleration α_a is the instantaneous rate of change of angular speed.*

The magnitude of the normal angular acceleration α_n *is the product $\omega_a\omega_r$ of the angular velocity ω_a about the axis and the angular velocity ω_r of the axis itself.*

Since the acceleration α is the resultant of α_a and α_n, we have for the magnitude of α

$$\alpha = \sqrt{\alpha_a^2 + \alpha_n^2}, \quad \ldots \ldots \ldots \ldots \quad (3)$$

and for its direction cosine with the axis

$$\cos \alpha = \frac{\alpha_a}{\alpha}. \quad \ldots \ldots \ldots \ldots \quad (4)$$

Uniform and Variable Angular Acceleration.—If the acceleration α has the same magnitude and direction whatever the time, it is UNIFORM. If either magnitude or direction change, it is VARIABLE.

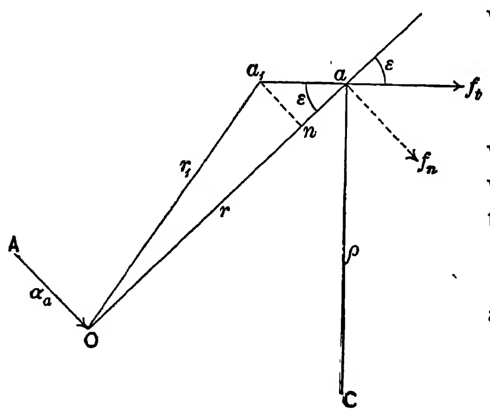
If α is zero, there is no change of angular velocity, and we have uniform angular speed in an unchanging plane.

If α_n is zero but α_a is not, we have variable angular speed in an unchanging plane.

If α_a is zero but α_n is not, we have uniform angular speed in a changing plane.

If α_a and α_n are both variable, we have variable angular speed in a changing plane.

Linear in Terms of Angular Acceleration.—Let O be any pole, and $Oa = r$ the radius vector to any point a moving in any path.



We have proved, on page 71, that

$$r\omega = \sin \epsilon,$$

where ω is the angular velocity and v is the linear velocity making the angle ϵ with Oa. We can therefore write

$$rd\omega = dv \sin \epsilon,$$

and dividing by dt,

$$r\frac{d\omega}{dt} = \frac{dv}{dt}\sin \epsilon.$$

But $\frac{d\omega}{dt}$ is the magnitude of the axial angular acceleration α_a, and $\frac{dv}{dt}$ is the magnitude of the tangential acceleration f_t.

Hence we have

$$r\alpha_a = f_t \sin \epsilon = f_n, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where f_n is the component of f normal to r.

Hence *the product of the radius vector r by the axial angular acceleration α_a gives the linear acceleration f_n normal to the radius vector in the plane of f_t (or v) and r.*

If the pole O is taken at the centre of curvature C so that Ba is the radius of curvature ρ, then $\epsilon = 90°$, and we have

$$\rho\alpha_a = f_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

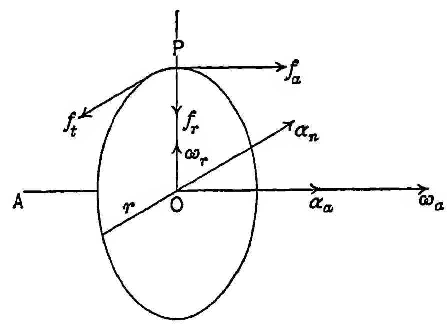
If the pole O is taken anywhere in the plane through ρ perpendicular to f_t, we still have $\epsilon = 90°$. and in this case

$$r\alpha_a = f_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, in this case, *the product of the radius vector by the axial angular acceleration gives the tangential acceleration.*

If, then, a point P has the angular acceleration α_a about an axis AO, so that the radius vector $OP = r$ is in a plane perpendicular to f_t through the radius of curvature, we have

$$r\alpha_a = f_t.$$



If at the same time the plane of rotation has the angular acceleration α_n, we have

$$r\alpha_n = f_a. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But, as we have seen, page 84, if ω_a is the angular velocity about the axis and ω_r is the angular velocity of the axis about OP, we have

$$\alpha_n = \omega_a\omega_r.$$

Hence we have

$$f_a = r\omega_a\omega_r,$$

just as already found on page 80.

We have already found (page 80) the radial acceleration

$$f_r = r\omega_a^2.$$

The acceleration f is, then, the resultant of f_r, f_a and f_t, or of f_ρ and f_t, since f_ρ is the resultant of f_r and f_a.

Example.—*A point moves in a vertical circle of radius $r = 2$ ft. in an east and west plane with an angular velocity given at any instant by*

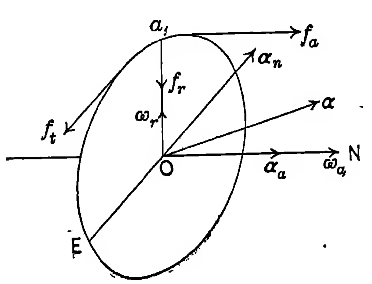
$$\omega_a = 2t + 4t^2.$$

At the end of $t = \frac{1}{2}$ sec the point is at the top of the circle moving east, and at the same instant the plane of the circle is rotating about the radius through the point with the angular velocity $\omega_r = 3$ radians per sec.

Find α_a, α_n, α, v, f_r, f_a, f_ρ, f_t and f.

ANS. We find, just as in example (2), page 74, the axial angular acceleration

$$\alpha_a = \frac{d\omega_a}{dt} = 2 + 8t.$$



For $t = \frac{1}{2}$ sec. we have, then, $\alpha_a = 6$ radians-per-sec. per sec. *north.*

The angular velocity for $t = \frac{1}{2}$ sec. is $\omega_a = 2$ radians per sec. *north.*

The normal angular acceleration is

$$\alpha_n = \omega_a\omega_r = 6 \text{ radians-per-sec. per sec. } \textit{west}$$

if ω_r has the direction given in the figure.

The angular acceleration α is then given by

$$\alpha = \sqrt{\alpha_a^2 + \alpha_n^2} = 6\sqrt{2} \text{ radians-per-sec. per sec.},$$

and its direction cosine with the axis is

$$\cos\beta = \frac{\alpha_a}{\alpha} = \frac{1}{\sqrt{2}}.$$

The linear velocity is $v = r\omega_a = 4$ ft. per sec. *east.*

The tangential acceleration is $f_t = r\alpha_a = 12$ ft -per-sec. per sec. *east.*

The radial acceleration is $f_r = \frac{v^2}{r} = r\omega_a^2 = v\omega_a = 8$ ft.-per-sec. per sec. towards the centre C.

The axial acceleration is $f_a = r\omega_a\omega_r = 12$ ft.-per-sec. per sec. *north*

The central acceleration is $f_p = \sqrt{f_r^2 + f_a^2} = 4\sqrt{13}$ ft.-per-sec. per sec., and its direction cosine with the radius is

$$\cos\gamma = \frac{f_r}{f_p} = \frac{2}{\sqrt{13}}.$$

The resultant acceleration f is given by

$$f = \sqrt{f_t^2 + f_r^2 + f_a^2} = \sqrt{22} \text{ ft.-per-sec. per sec.},$$

and its direction cosines with f_t, f_a and f_r are

$$\cos\alpha = \frac{f_t}{f} = \frac{3}{\sqrt{22}}, \quad \cos\beta = \frac{f_a}{f} = \frac{3}{\sqrt{22}}, \quad \cos\gamma = \frac{f_r}{f} = \frac{2}{\sqrt{22}}.$$

Homogeneous Equations.—We have already called attention (page 4) to the fact that the units in all numeric equations are always understood, and when these units are inserted the equation must be HOMOGENEOUS, that is, every term must express a quantity of the same kind. When this is not the case the equation is impossible, and some error must have been made in its derivation.

Thus suppose that the result of some investigation is expressed by

$$s + at = bv,$$

where a and b are numbers only, s is a number of feet, t a number of seconds, and v a number of feet per second. Without reference to the various steps of the investigation by which this result has been reached, we can say at once, from inspection, that it is incorrect and some error has been committed. For we cannot have a number of feet plus a number of seconds equal a number of feet per second.

If, however, a stands for a number of feet per second, and b stands for a number of seconds, the equation is homogeneous and possible. For if we insert the units, we now have

$$s\,(\text{ft.}) + a\left(\frac{\text{ft.}}{\text{sec.}}\right) \times t\,(\text{sec.}) = b\,(\text{sec.})\,v\left(\frac{\text{ft.}}{\text{sec.}}\right).$$

or

$$s\,(\text{ft.}) + at\,(\text{ft.}) = bv\,(\text{ft.}).$$

Here all the terms are quantities of the same kind and the equation is homogeneous. The relations expressed by it are possible. It does not follow that the equation is necessarily correct. It may still have been incorrectly derived. But it is not impossible and we cannot reject it upon simple inspection.

The student should form the habit of thus testing all equations. It will often prevent

him from making errors in a long investigation, and save him the trouble of going over every step in order to locate some error which may have been otherwise committed.

Examples.—(1) *In the equation*

$$\theta = 7t + 8t^2 + 2t^3,$$

what must be the units of 7, 8 and 2?

ANS. $7\,\frac{\text{radians}}{\text{sec.}}$, $8\,\frac{\text{rad.}}{\text{sec.}^2}$, $2\,\frac{\text{rad.}}{\text{sec.}^3}$, or 7 radians per sec., 8 rad. per sec.^{2}, 2 rad. per sec.^{3}

(2) *In the example, page 71, we have given the equation*

$$s = 7t + 8t^2 + 2t^3.$$

What must be the units 7, 8 and 2?

ANS. $7\,\frac{\text{ft.}}{\text{sec.}}$, $8\,\frac{\text{ft.}}{\text{sec.}^2}$, $2\,\frac{\text{ft.}}{\text{sec.}^3}$ or 7 ft. per sec., 8 ft. per sec.^{2}, 2 ft. per sec.^{3}.

The student should go over the solution of this example on page 71 and test thus every equation in it.

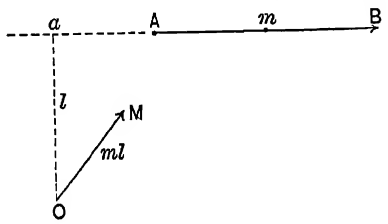
CHAPTER VIII.

MOMENTS. MOMENT OF DISPLACEMENT, VELOCITY AND ACCELERATION.

Moment of a Vector Quantity.—The product of a vector quantity by the length of the perpendicular let fall from any point upon the direction of the line representative of the vector is called the MOMENT of the vector relative to that point.

The perpendicular is called the LEVER-ARM, and the point is the CENTRE OF MOMENTS.

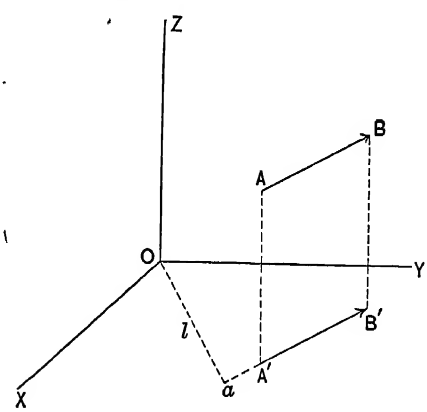
Thus let AB be the line representative of any vector whose magnitude is m. Take any point O as a centre of moments and draw the *lever-arm*, or perpendicular Oa upon the direction of AB produced if necessary. Let the length of this perpendicular or *lever-arm* be l. Then the product ml gives the magnitude of the *moment* of the vector AB relative to O.



This moment is itself a vector quantity, that is it has both magnitude *and direction* and can therefore be itself represented by a straight line.

Thus the line representative of the moment is a straight line OM at right angles to the plane of BAO, whose magnitude is ml, and so directed by its arrow that when we look along it in the direction of its arrow, the rotation indicated by the direction of AB relative to O is seen clockwise.

Resolution and Composition of Moments.—We can then combine and resolve moments and we have component and resultant moments and the triangle and polygon of moments, just as for displacements (page 55). Also, the component of the resultant in any direction is equal to the agebraic sum of the components in that direction.



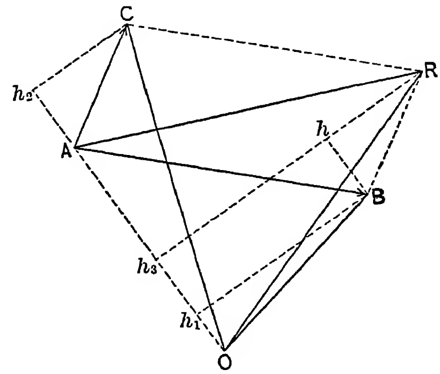
Moment about an Axis.—Let OZ be a given axis and AB a vector. Take any plane XY perpendicular to the axis OZ intersecting this axis at O. Project AB upon this plane in $A'B'$. Now AB can be resolved at A into a component parallel to $A'B'$ and a component parallel to the axis OZ.

This latter component gives no rotation about OZ. The moment of AB about the axis OZ is, then, the moment of $A'B'$ about O.

Hence *the moment of a vector relative to any axis is the same as the moment of the component in a plane perpendicular to the axis, relative to the point of intersection of that axis and plane.*

Moment of Resultant.—Let AB and AC be two components. Completing the parallelogram, we have the resultant AR.

Choose any point O in the plane of BAC. Then the moment of AB relative to O is the length of AB multiplied by the perpendicular from O on AB, or twice the area of the triangle ABO.



In the same way, the moment of AC relative to O is twice the area of the triangle ACO, and the moment of the resultant AR relative to O is twice the area of the triangle ARO.

Draw a line through O and A, and drop the perpendiculars Bh_1, Ch_2, Rh_3. Also draw Bh parallel to OA. Then $Ch_2 = Rh$, and $Ch_2 + Bh_1 = Rh_3$.

The area of the triangle $ACO = AO \times \frac{1}{2}Ch_2$, the area of the triangle $ABO = AO \times \frac{1}{2}Bh_1$, the area of the triangle $ARO = AO \times \frac{1}{2}Rh_3$.

We have, then, the area of ARO equal to the sum of the areas of ACO and ABO.

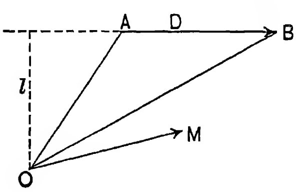
If we had a third component, the same would hold for the resultant of this and R, and so on.

If we had taken O inside the angle CAB, the moments of AC and AB would have been opposite in direction and we should have found the area of ARO equal to the difference of the areas ACO and ABO.

Hence *for any number of components the moment of the resultant is equal to the algebraic sum of the moments of the components.*

We have dealt thus far with linear and angular displacements, velocities and accelerations, and have seen that they are all vector quantities. Each and all of them, then, can have a moment relative to a point or axis, and the preceding principles apply to all alike.

Moment of Displacement.—Let AB be a linear displacement D. Then the moment Dl is twice the area of the triangle AOB.



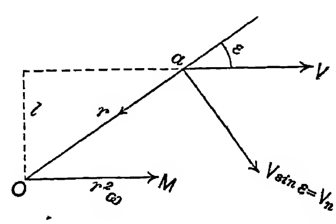
Hence *the moment of a linear displacement relative to any point O is twice the area swept through by the radius vector OA in moving from OA to OB in the plane AOB.*

The unit of moment of linear displacement is therefore one square unit of length, or one square foot.

The line representative is OM at right angles to the plane of AOB, so that looking in its direction, as indicated by the arrow, the rotation of the radius vector is seen clockwise.

Moment of Velocity.—If AB, in the preceding figure, represents a linear velocity v, then the moment vl relative to O is *twice the areal velocity of the radius vector OA.*

The unit of moment of a linear velocity is then the square of the unit of length per unit of time, or one square foot per second.

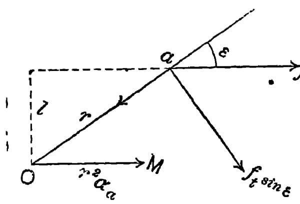


The line representative OM is as before. If r is the radius vector and ω the angular velocity relative to O, we have (page 71) $r\omega = v \sin \epsilon = v_n$, or $v_n = r\omega$, and the lever-arm $l = r \sin \epsilon$. Therefore the moment

$$vl = r^2\omega = v_n r. \qquad (1)$$

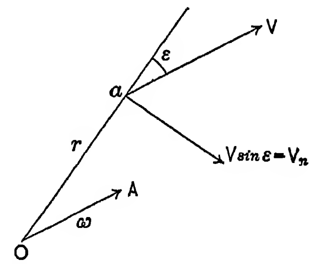
Moment of Acceleration.—If AB, represents a linear acceleration f, then the moment fl relative to O is *twice the areal acceleration of the radius vector OA.*

The unit of moment of a linear acceleration is then the square of the unit of length per unit of time squared, or one square ft.-per-sec. per sec.



The line representative OM is as before. If r is the radius vector and α_a the axial angular acceleration, we have (page 84) $r\alpha_a = f_t \sin \epsilon = f_n$. The lever arm $l = r \sin \epsilon$. Therefore the moment $f_t l$ *of the tangential linear acceleration* is

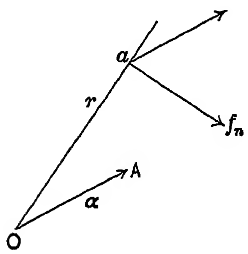
$$f_t l = r^2\alpha_a = f_n r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$



Moment of Angular Velocity.—If OA represents an angular velocity ω, then, as we have seen (page 71), the moment $r\omega$ relative to any point a gives the linear velocity $v_n = v \sin \epsilon$ of that point at right angles to the plane of the axis OA and the radius vector Oa, in such a direction that, looking along OA in its direction, the rotation of the radius vector is seen clockwise.

The moment of an angular velocity is, then, a linear velocity, and its unit is one foot per second.

Moment of Angular Acceleration.—If OA represents an angular acceleration, α then, as we have seen, page 84, the moment $r\alpha$ relative to any point a gives the linear acceleration f_n of that point at right angles to the plane of the axis OA and the radius vector Oa, in such a direction that, looking along OA in its direction, the rotation of the radius vector is seen clockwise.



The moment of an angular acceleration is, then, a linear acceleration, and its unit is one foot-per-sec. per sec.

Example.—*The distance in feet s described by a point in any time t sec. is given by*

$$s = a + bt^2.$$

The path is a circle of radius r feet in a vertical east and west plane. The point when $t = 0$ is at the top and it moves towards the east.

(a) *State the units of a and b.* (b) *Find the linear velocity, the tangential, radial and resultant acceleration at any instant.* (c) *The angular velocity and acceleration at any instant.* (d) *The areal velocity and acceleration of the radius at any instant*

At the end of 3 secs. find (e) *the distance described, the mean speed, the angular and linear displacement.*

ANS. (a) The unit of a must be 1 ft., of b 1 ft.-per-sec. per sec.

(b) $v = \frac{ds}{dt} = 2bt$ ft. per sec. tangent to the path at the instant;

$f_t = \frac{dv}{dt} = 2b$ ft.-per-sec. per sec. tangent to the path at the instant;

$f_p = \frac{v^2}{r} = \frac{4b^2t^2}{r}$ ft.-per-sec. per sec. towards the centre;

$f = \sqrt{f_t^2 + f_p^2} = \sqrt{4b^2 + \frac{16b^4t^4}{r^2}}$ ft.-per-sec. per sec., making an angle with the radius whose tangent is $\frac{f_t}{f_p} = \frac{2r}{4bt^2}$.

(c) $\omega = \frac{v}{r} = \frac{2bt}{r}$ radians per sec. *north;* $\alpha_a = \frac{f_t}{r} = \frac{2b}{r}$ radians-per-sec. per sec. *north.*

(d) $\frac{rv}{2} = brt$ sq. ft. per sec. *north;* $\frac{r}{2}f_t = br$ sq.-ft.-per-sec. per sec. *north.*

(e) When $t = 0$, $s_1 = a$. When $t = 3$, $s = a + 9b$. Hence the distance described is

$$s - s_1 = 9b \text{ ft.}$$

The mean speed $= \frac{9b}{3} = 3b$ ft. per sec.

Angular displacement $\theta - \theta_1 = \frac{s - s_1}{r} = \frac{9b}{r}$ radians. Linear displacement $= 2r \sin \frac{\theta - \theta_1}{2} = 2r \sin \frac{9b}{2r}$ in a direction from the initial to the final position.

KINEMATICS OF A POINT. APPLICATION OF PRINCIPLES.

CHAPTER I.

MOTION OF A POINT. CONSTANT AND VARIABLE RATE OF CHANGE OF SPEED.

Rate of Change of Speed Zero.—As we have seen, page 79, if the tangential acceleration f_t is zero, the resultant acceleration f is the same as the central acceleration f_ρ and therefore always at right angles to the velocity v. In such case we have in general uniform speed in some curve, and $f = f_\rho = \frac{v^2}{\rho}$ (page 78). If the central acceleration $f_\rho = f$ is constant in magnitude, then, since v is also constant in magnitude, we must have ρ constant in magnitude and the path is a circle. If f_ρ is zero, then ρ is infinity or the path is a straight line.

Whatever the path may be, let s_1 be the initial distance of the moving point measured along the path from any given fixed point of the path taken as origin, and s the final distance of the moving point from that origin, measured in the same way, at the end of any interval of time t, so that the distance described in this interval of time t is $s - s_1$.

Now if the tangential acceleration f_t is zero, the speed in the path does not change and hence the instantaneous speed at any instant is equal to the mean speed for any interval of time. We have, then,

$$v = \frac{s - s_1}{t}, \quad \text{or} \quad vt = s - s_1. \qquad (1)$$

This equation is general whatever the path, provided f_t is zero. If s is greater than s_1, v is positive. If s is less than s_1, v is negative. Hence a positive (+) value for v denotes that the distance from the origin increases as the interval of time increases, a negative (−) value for v denotes that the distance from the origin decreases as the interval of time increases, without regard to direction of motion.

Rate of Change of Speed Constant.—If the tangential acceleration f_t is constant in magnitude, we have in general motion in a curved path with uniform rate of change of speed. If in this case the central acceleration $f_\rho = 0$, the path is a straight line.

Whatever the path may be, let v_1 and v be the initial and final speeds for any interval of time t. Then, since for constant f_t the instantaneous rate of change of speed at any instant is equal to the mean rate of change of speed for any interval of time, we have

$$f_t = \frac{v - v_1}{t}. \qquad (2)$$

The value of f_t is positive (+) when v is greater than v_1, that is when the speed increases as the interval of time increases, and negative (—) when the speed decreases as the interval of time increases, without regard to direction of motion.

From (2) we have

$$v = v_1 + f_t t. \qquad (3)$$

The average speed during the interval t is

$$\frac{v + v_1}{2} = v_1 + \tfrac{1}{2} f_t t. \qquad (4)$$

The distance described $(s - s_1)$ between the initial and final positions, measured along the path, is the mean speed multiplied by the time, or

$$s - s_1 = \frac{v + v_1}{2} \cdot t = v_1 t + \tfrac{1}{2} f_t t^2. \qquad (5)$$

Inserting the value of t from (2) we have also

$$s - s_1 = \frac{v^2 - v_1^2}{2 f_t}. \qquad (6)$$

Hence

$$v^2 = v_1^2 + 2 f_t (s - s_1). \qquad (7)$$

These equations are general whatever the path, provided f_t is constant. In applying them f_t is positive (+) when the speed increases, and negative (—) when the speed decreases, during the interval of time t, without regard to direction of motion.

So, also, v is positive (+) when the distance from the origin increases, and negative (—) when the distance from the origin decreases, during the interval of time t, without regard to direction of motion.

Rate of Change of Speed Variable.—If the rate of change of speed is variable, we have from (1), in Calculus notation, for the velocity at any instant

$$v = \frac{ds}{dt}, \qquad (8)$$

and from (2) for the tangential acceleration at any instant

$$f_t = \frac{dv}{dt} = \frac{d^2 s}{dt^2}, \qquad (9)$$

and from (8) for the distance described

$$s - s_1 = \int_0^t v\,dt. \qquad (10)$$

We have also, from (9),

$$f_t ds = v dv,$$

and hence

$$\int_{s_1}^{s} f_t ds = \frac{v^2}{2}. \qquad (11)$$

These equations are general, and the preceding equations can be deduced from them.

Thus if f_t is zero, we have uniform speed and

$$v = \frac{s - s_1}{t}.$$

If f_t is constant, we have, by integrating (9).

$$v = f_t t + C,$$

where C is the constant of integration. When $t = 0$, v equals v_1 and hence $C = v_1$. Hence

$$v = v_1 + f_t t.$$

Inserting this value of v in (8) and integrating, we have

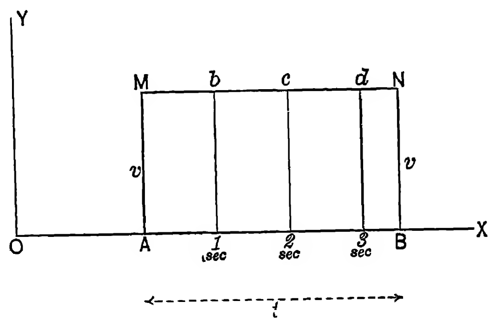
$$s = v_1 t + \tfrac{1}{2} f_t t^2 + C.$$

But when $t = 0$ we have $s = s_1$, hence $C = s_1$. Therefore

$$s - s_1 = v_1 t + \tfrac{1}{2} f_t t^2.$$

Graphic Representation of Rate of Change of Speed.—If we represent intervals of time by distances laid off horizontally along the axis of x, and the corresponding speeds by ordinates parallel to the axis of y, we shall have in general a curve for which the change of y with x will show the law of change of speed with the time.

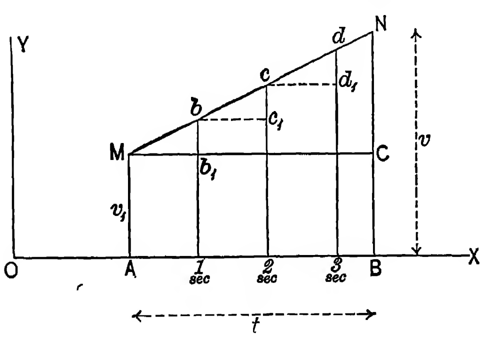
(a) RATE OF CHANGE OF SPEED ZERO.—Lay off from A along AB equal distances, so that the distances from A to 1, 1 to 2, 2 to 3, etc., are all equal and represent each one second of time, and let AB represent the entire time t.



Then at A, 1, 2, 3, and B erect the perpendiculars AM, $1b$, $2c$, $3d$, BN, and let the length of each represent the speed at the corresponding instant.

Since there is no change of speed, these perpendiculars will all be of equal length, we shall have $AM = 1b = 2c = 3d = BN = v$, and the speed at any interval of time will be given by the ordinate at that instant to the line MN parallel to AB.

The space described in any time is given by $s - s_1 = vt$. This is evidently given by the area $AMNB$ in the diagram. Therefore *the area corresponding to any time gives the space described in that time.*



(b) RATE OF CHANGE OF SPEED CONSTANT.—Lay off as before the time along AB, and at A, 1, 2, 3, B, the corresponding speeds, so that AM is the initial speed v_1, and BN the final speed v. Draw MC, bc', cd', parallel to AB.

Then bb' will be the change of speed in the first sec., cc' the change of speed in the next sec., and so on. Since these are to be constant, NM is a straight line, the ordinate to which at any instant will give the speed at that instant.

The rate of change of speed is then $\frac{bb'}{1 \text{ sec.}} = \frac{cc'}{1 \text{ sec.}} = \frac{dd'}{1 \text{ sec.}} = f_t$. But $\frac{bb'}{1 \text{ sec.}} = \frac{v - v_1}{t} = f_t$. Hence *the rate of change of speed is the tangent of the angle NMC which the line MN makes with the horizontal.* Hence $f_t = \frac{NC}{t}$, or $NC = f_t t$.

The distance described in the time t is, from equation (5), given by $\frac{v + v_1}{2} t$. But this

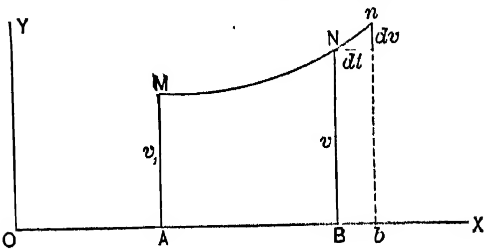
is the area of $AMNB$. Therefore *the area corresponding to any time gives the space described in that time.*

We have then directly from the figure, since $NC = f_t t$,

$$s - s_1 = \frac{v + v_1}{2} t = v_1 t + \tfrac{1}{2} NC \times t = v_1 t + \tfrac{1}{2} f_t t^2.$$

If v_1 is greater than v, a will be negative, and the line MN is inclined below the horizontal MC.

[(*c*) **Rate of Change of Speed Variable**].—If the rate of change of speed is not constant, we shall have in general a curve MNn. The tangent to this curve at any point N makes an angle with the axis of X, whose tangent is $\frac{dv}{dt} = f_t$, equation (9), or the rate of change of speed. The elementary area $BNnb = vdt = ds$, equation (8), and the total area $AMNB = \int_{t=0}^{t} vdt = s - s_1$, equation (10).

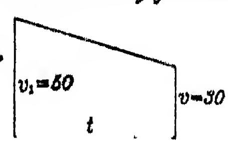


When $\frac{dv}{dt} = 0$, or $\frac{d^2s}{dt^2} = 0$, or $f_t = 0$, the tangent to the curve is horizontal at the corresponding point, and we have the speed at that point a maximum or minimum according as the curve is concave or convex to the axis of X.

Examples.—(1) *The speed of a point changes from 50 to 30 ft. per sec. in passing over 80 ft. Find the constant rate of change of speed and the time of motion.*

ANS. $f_t = \frac{v^2 - v_1^2}{2(s - s_1)} = \frac{900 - 2500}{2 \times 80} = -10$ ft.-per-sec. per sec. The minus sign indicates decreasing speed. $t = \frac{v - v_1}{f_t} = \frac{30 - 50}{-10} = 2$ sec.

(2) *Draw a figure representing the motion in the preceding example, and deduce the results directly from it*



Ans.—Average speed $= \frac{50 + 30}{2} = 40$ ft. per sec. Hence $40t = 80$, or $t = 2$ sec. Also $f_t = \frac{30 - 50}{2} = -10$ ft.-per-sec. per sec.

(3) *A point starts from rest and moves with a constant rate of change of speed. Show that this rate is numerically equal to twice the number of units of distance described in the first second.*

ANS. We have $t = 1$ and $v_1 = 0$; hence, from eq. (5), $\frac{s - s_1}{1 \text{ sec.}^2} = \frac{1}{2} f_t$, or $f_t = \frac{2(s - s_1)}{1 \text{ sec.}^2}$, which is *numerically* equal to $2(s - s_1)$.

(4) *In an air-brake trial, a train running at 40 miles an hour was stopped in 625.6 ft. If the rate of change of speed was constant during stoppage, what was it?*

ANS. From eq. (6) we have for $v = 0$, $s - s_1 = 625.6$, and $v_1 = \frac{40 \times 5280}{60 \times 60}$, $f_t = -\frac{v_1^2}{2 \times 625.6} = -\frac{(40 \times 5280)^2}{(60 \times 60)^2 \times 2 \times 625.6} = -2.75$ ft.-per-sec. per sec. The (−) sign shows retardation.

(5) *A point starts with a speed v_1 and has a constant rate of change of speed $-f_t$. When will it come to rest, and what distance does it describe?*

ANS. From eq. (3), when $v = 0$, we have $v_1 - f_t t = 0$, or $t = \frac{v_1}{f_t}$. From eq. (5), $s - s_1 = \frac{v_1}{2} t = \frac{v_1^2}{2f_t}$.

(6) *A point describes 150 ft. in the first three seconds of its motion and 50 ft. in the next two seconds. If the rate of change of speed is constant, when will it come to rest? When will it have a speed of 30 ft. per sec.?*

ANS. From eq. (5) we have for $s - s_1 = 150$ and $t = 3$, $150 = 3v_1 + \frac{9}{2} f_t$; and for $s - s_1 = 200$ and $t = 5$, $200 = 5v_1 + \frac{25}{2} f_t$. Combine these two equations and we have $f_t = -10$ ft.-per-sec. per sec., and $v_1 = 65$ ft. per sec. From eq. (3), if $v = 0$, we have $65 - 10t = 0$, or $t = 6.5$ sec. From eq. (3) we also have, if $v = 30$, $30 = 65 - 10t$, or $t = 3.5$ sec.

(7) *A point whose speed is initially 30 meters per sec. and is decreasing at the rate of 40 centimeters-per-sec. per sec., moves in its path until its speed is 240 meters per minute. Find the distance traversed and the time.*

ANS. We have $v_1 = 30$ and $v = 4$ meters per sec., and $f_t = -0.4$ meter-per-sec. per sec. From eq. (6), $s - s_1 = \dfrac{16 - 900}{-0.8} = 1105$ meters. From (3) we have $4 = 30 - 0.4t$, or $t = 65$ sec.

(8) *A point has an initial speed of v_1 and a variable rate of change of speed given by $+ kt$, where k is a constant. What is the speed and distance described at the end of a time t?*

ANS. From eq. (9) we have $f_t = \dfrac{dv}{dt} = kt$, and integrating, $v = \dfrac{kt^2}{2} + C$. If, when $t = 0$, we have $v = v_1$, we obtain $C = v_1$, and hence $v = v_1 + \dfrac{kt^2}{2}$.

From eq. (8), $ds = vdt = v_1dt + \dfrac{kt^2dt}{2}$. Integrating, $s = v_1t + \dfrac{kt^3}{6} + C$. If, when $t = 0$, we have $s = 0$, we obtain $C = 0$, and hence $s = v_1t + \dfrac{kt^3}{6}$.

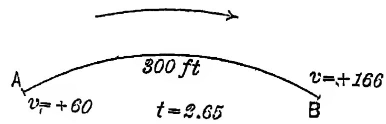
(9) *A point has an initial speed of 60 ft. per sec. and a rate of change of speed of + 40. ft.-per-sec. per sec. Find the speed after 8 sec.; the time required to traverse 300 ft.; the change of speed in traversing that distance; the final speed.*

ANS From eq. (3) we have $v = 60 + 40 \times 8 = 380$ ft. per sec. From eq. (5) we have $300 = 60t + 20t^2$, or $t = \pm\sqrt{\dfrac{69}{4}} - \dfrac{3}{2} = +\ 2.65$ sec. or $-\ 5.65$ sec. The first value only applies.

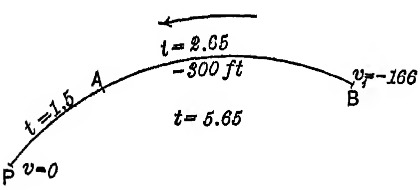
From eq. (3) we have $v - v_1 = 40t = 20(\pm\sqrt{69} - 3) = +\ 106$ ft. per sec. or $-\ 226$ ft. per sec. The first value only applies.

We have for the final speed $v = +\ 166$ ft. per sec. or $-\ 166$ ft. per sec. The first value only applies.

That is, the point starts from A with the speed $v_1 = 60$ ft per sec. and describes the path $AB = 300$ ft. in $t = 2.65$ sec., the speed at B being $v = 166$ ft. per sec.



In order to interpret the negative values obtained, we observe that $v = -\ 166$ ft. per sec. means that the distance from the origin decreases. Take A as origin and let the point then start from B towards A with the speed $v_1 = -\ 166$ ft. per sec. Then from eq. (3) we have $v = -\ 166 + 40t$. We see that when $t = 4.15$ sec., $v = 0$, and the point has passed to some point P, where the speed is zero. This point is the *turning-point*. For t greater than 4.15 sec. v becomes positive; that is, the point moves back towards B, and arrives at a point A, where the speed is $v = +\ 60$ in the time given by $60 = 40t$, or $t = 1.5$ sec. The entire time from B to P and back to A is $t = 5.65$ sec. This is the time given by the negative value of t in the example; that is, it is the *time before* the start, during which the point moves from B to P and back to A. The change of speed $v - v_1$ is $+\ 60 + 166 = +\ 226$, which is the negative value in the example.



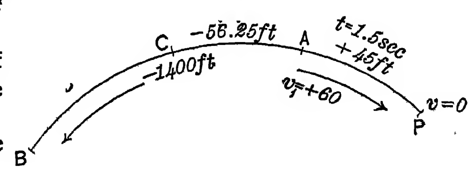
For the space BA described *between the initial and final positions* we have

$$\frac{v + v_1}{2}t, \quad \text{or} \quad \frac{+\ 60 - 166}{2} \times 5.65 = -\ 300 \text{ ft.},$$

the ($-$) sign showing that the distance is on the other side of the origin from the case of the example.

We see, then, that our equations are general if we have regard to the signs of v, v_1, s, s_1, and f_t.

(10) *If the motion in example (9) is retarded, find (a) the distance described from the starting- to the turning-point; (b) the distance described from the starting-point after 10 sec., the speed acquired and the distance between the final and initial positions; (c) the distance described during the time in which the speed changes to $-$ 90 ft. per sec., and this time; (d) the time required by the moving point to return to the starting-point.*



ANS. The initial speed is $v_1 = +\ 60$ ft. per sec., the rate of change of speed is $f_t = -\ 40$ ft -per-sec. per sec. Let the time count from the start at A, so that $s_1 = 0$, when $t = 0$.

(*a*) We have from eq. (6) for the distance from A to the turning-point P, where $v = 0$,

$$s = \frac{-\ v_1^2}{2f_t} = \frac{-\ 3600}{-\ 80} = +\ 45 \text{ ft.}$$

(*b*) From eq. (5) we have for the distance between the initial and final positions after 10 sec.

$$AB = s = v_1t + \frac{1}{2}f_tt^2 = 60 \times 10 - 20 \times 100 = -1400 \text{ ft.}$$

The minus sign shows distance on left of A. The total distance described is then 1490 ft. The speed acquired is given by eq. (5) $-1400 = \frac{v + 60}{2} \times 10$, or $v = -340$ ft. per sec. The minus sign shows motion from A towards B.

(*c*) From eq. (6) the distance between the initial and final positions, when the speed is -90 ft. per sec., is $AC = s = \frac{v^2 - v_1^2}{2f_t} = \frac{8100 - 3600}{-80} = -56.25$ ft. The minus sign shows that C is on the left of A. The total distance described from the start is then $56.25 + 90 = 146.25$ ft. The time, from eq. (5), is

$$-56.25 = \frac{-90 + 60}{2}t, \quad \text{or} \quad t = 3.75 \text{ sec.}$$

(*d*) The time to reach the turning-point, as we have seen, is 1.5 sec. The time to return is, from eq. (5)' $45 = \frac{60}{2}t = 1.5$ sec. The entire time to go and return is then 3 sec.

Rate of Change of Angular Speed Zero.—As we have seen, page 84, if the angular acceleration α is zero, there is no change of angular velocity. The angular velocity ω is then uniform, its line representative has always the same magnitude and direction, and we have uniform angular speed in an unchanging plane.

In this case, if θ_1 is the initial angle of the radius vector from a fixed line in the plane, and θ is the final angle, we have

$$\omega = \frac{\theta - \theta_1}{t}, \quad \text{or} \quad \omega t = \theta - \theta_1. \qquad (1)$$

[Compare equation (1), page 91.]

This equation is general. If θ is greater than θ_1, ω is positive. If θ is less than θ_1, ω is negative. Hence a positive (+) value for ω denotes that the angle from the initial radius vector increases as the interval of time increases, a negative (—) value for ω denotes that the angle from the initial position of the radius vector decreases as the interval of time increases, without regard to direction of rotation.

Rate of Change of Angular Speed Uniform—Motion in a Plane.—As we have seen, page 84, if α_n is zero, we have $\alpha = \alpha_a$ coinciding with ω, and hence rotation in a plane with varying angular speed. If, in this case, α_a is uniforn, we have uniformly varying angular speed in an unchanging plane. We have then (compare with equations (2) to (7), page 92)

$$\alpha = \frac{\omega - \omega_1}{t}, \qquad (2)$$

where ω_1 and ω are the initial and final angular velocities. The value of α is (+) when the angular velocity increases, and (—) when it decreases during the time.

From (2) we have, just as on page 92,

$$\omega = \omega_1 + \alpha t. \qquad (3)$$

The average angular speed is

$$\frac{\omega + \omega_1}{2} = \omega_1 + \tfrac{1}{2}\alpha t. \qquad (4)$$

The angle described in the time t is

$$\theta - \theta_1 = \frac{\omega + \omega_1}{2}t = \omega_1 t + \tfrac{1}{2}\alpha t^2. \qquad (5)$$

Inserting the value of t from (2), we have

$$\theta - \theta_1 = \frac{\omega^2 - \omega_1^2}{2\alpha}. \quad (6)$$

Hence

$$\omega^2 = \omega_1^2 + 2\alpha(\theta - \theta_1). \quad (7)$$

It will be noted that all these equations are precisely similar to equations (2) to (7), page 92. We have only to replace s, v, f_t by θ, ω, α.

In applying these equations α is positive when the angular speed increases, and negative when it decreases. So, also, ω is positive (+) when the angle from the initial position of the radius vector increases as the interval of time increases, and negative (—) when the angle from the initial position of the radius vector decreases as the interval of time increases, without regard to direction of rotation.

Rate of Change of Angular Speed Variable.—If the rate of change of angular speed is variable, we have from (1), in Calculus notation for the angular velocity at any instant,

$$\omega = \frac{d\theta}{dt}, \quad (8)$$

and from (2), for the axial angular acceleration at any instant,

$$\alpha_a = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \quad (9)$$

and from (8), for the angle described,

$$\theta - \theta_1 = \int_\theta^t \omega dt. \quad (10)$$

We have also, from (9),

$$\alpha_a d\theta = \omega d\omega,$$

and hence

$$\int_{\theta_1}^{\theta} \alpha_a d\theta = \frac{\omega^2}{2}. \quad (11)$$

These equations are general, and the preceding equations can be deduced from them.

Thus, if α_a is zero, we have uniform angular speed and

$$\omega = \frac{\theta - \theta_1}{t}.$$

If α_a is constant, we have, by integrating (9),

$$\omega = \alpha_a t + C,$$

where C is the constant of integration. When $t = 0$, $\omega = \omega_1$, and hence $C = \omega_1$. Hence

$$\omega = \omega_1 + \alpha_a t.$$

Inserting this value of ω in (8) and integrating, we have

$$\theta = \omega_1 t + \tfrac{1}{2}\alpha_a t^2 + C.$$

But when $t = 0$ we have $\theta = \theta_1$, hence $C = s_1$. Therefore

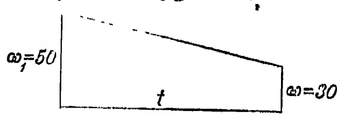
$$\theta - \theta_1 = \omega_1 t + \tfrac{1}{2}\alpha_a t^2.$$

Graphic Representation.—The graphic representation is precisely the same as on page 93. Thus we can represent intervals of time by distances laid off horizontally, and the corresponding angular speeds by distances laid off vertically, and thus obtain the same diagrams as for linear speed given on page 93.

Examples.—(1) *The angular speed of a point moving in a plane about some assumed point changes from 50 to 30 radians per sec. in passing through 80 radians. Find the constant rate of change of angular speed and the time of motion.*

ANS. $\alpha = \dfrac{\omega^2 - \omega_1^2}{2(\theta - \theta_1)} = -10$ radians-per-sec. per sec. The minus sign denotes decreasing speed. $t = \dfrac{\omega - \omega_1}{\alpha} = 2$ sec.

(2) *Draw a figure representing the motion in the preceding example, and deduce the results directly from it.*



ANS. Average speed $= \dfrac{50 + 30}{2} = 40$ radians per sec. Hence $40t = 80$, or $t = 2$ sec. Also $\alpha = \dfrac{30 - 50}{2} = -10$ radians-per-sec. per sec.

(3) *A point moving in a plane has an initial speed of 60 radians per sec. about an assumed point and a rate of change of speed of + 40 radians-per-sec. per sec. Find the speed after 8 sec.; the time required to describe 300 radians; the change of speed while describing that angle; the final speed.*

ANS. See example (9), page 95.

(4) *If the motion in the last example is retarded, find (a) the angular revolution from the start to the turning-point; (b) the angle described from the start after 10 sec.; the speed acquired and the angle between the final and initial positions; (c) the angle described during the time in which the speed changes to − 90 radians per sec., and this time; (d) the time required by the moving point to return to the initial position.*

ANS. See example (10), page 95.

(5) *A point moving in a plane describes about a fixed point angles of 120 radians, 228 radians and 336 radians in successive tenths of a second. Show that this is consistent with uniform rate of change of angular speed, and find this rate.*

ANS. $\alpha = 10800$ radians-per-sec. per sec.

(6) *Two points A and B move in the circumference of a circle with uniform angular speeds ω and ω'. The angle between them at the start is a. Find the time of the nth meeting, the angles described by A and B, and the interval of time between two successive meetings.*

ANS. Time of the nth meeting, $t_n = \dfrac{\pm a + (n-1)2\pi}{\omega \pm \omega'}$.

Angle described by A is $\theta = \omega t_n$.

" " " B is $\mp a$.

Interval of time between two successive conjunctions is

$$t_2 - t_1 = t_3 - t_2 = \frac{2\pi}{\omega \pm \omega'},$$

where we take the (+) or (−) sign for a according as B is in front of or behind A at start, and (+) or (−) sign for ω' according as the points move in opposite or the same directions.

(7) *What is the angular speed of a fly-wheel 5 ft. in diameter which makes thirty revolutions per minute, and what is the linear velocity of a point on its circumference? Also find its linear central acceleration.*

ANS. π radians per sec.; 2.5π ft. per sec., tangent to circ.; $2.5\pi^2$ ft.-per-sec. per sec.

(8) *Find the linear and angular speed of a point on the earth's equator, taking radius 4000 miles; also the linear central acceleration.*

ANS. 1535.9 ft. per sec.; $\dfrac{\pi}{12}$ radians, or 15° per hour; 0.112 ft.-per-sec. per sec.

(9) *The angular speed of a wheel is $\frac{3}{4}\pi$ radians per sec. Find the linear speed of points at a distance of 2 ft., 4 ft. and 10 ft. from the centre, also the linear central acceleration.*

ANS. $\frac{3}{2}\pi$, 3π, 7.5π ft. per sec.

$\frac{9}{8}\pi^2$, $\frac{9}{4}\pi^2$, $\frac{45}{8}\pi^2$ ft.-per-sec. per sec.

(10) *If the linear speed of a point at the equator is v, find the speed linear and angular at any latitude λ.*

ANS. $v \cos \lambda$; $\dfrac{\pi}{12}$ radians per hour, or 15° per hour.

(11) *A point moves with uniform velocity v. Find at any instant its angular speed about a fixed point whose distance from the path is a.*

ANS. $\frac{va}{r^2}$ radians per sec., where r is the radius vector. Uniform velocity means uniform speed in a straight line (page 65).

Hence the angular speed of a point moving with uniform speed in a straight line is inversely proportional to the square of the distance of the point from a fixed point not in the line.

(12) *The speed of the periphery of a mill-wheel 12 feet in diameter is 6 feet per sec. How many revolutions does the wheel make per sec.?*

ANS. $\frac{1}{2\pi}$ revolutions.

(13) *Deduce the equivalent of longitude for one minute of time and for one second of time.*

ANS. 15′ to 1 min., 15″ to 1 sec.

(14) *The diameter of the earth is nearly 8000 miles. Required the circumference at the equator and the linear speed at latitude 60°.*

ANS. 25000 miles; 521 miles per hour.

(15) *The wheel of a bicycle is 52 inches in diameter and performs 5040 revolutions in a journey of 65 minutes. Find the speed in miles per hour; the angular speed of any point about the axle; the areal velocity of a spoke; the relative velocity of the highest point with respect to the centre.*

ANS. 12 miles per hour; 8.12 radians per sec.; 19.06 sq ft. per sec., 12 miles per hour.

(16) *In going 120 yards the front wheel of a carriage makes six revolutions more than the hind wheel. If each circumference were a yard longer, it would make only four revolutions more. Find the circumference of each wheel.*

ANS. 4 yards and 5 yards

(17) *If the velocity of a point is resolved into several components in one plane, show that its angular speed about any fixed point in the plane is the sum of the angular speeds due to the several components.*

(18) *A point moves with uniform speed v in a circle of radius r. Show that its angular speed about any point in the circumference is* $\frac{v}{2r}$.

(19) *A point starting from rest moves in a circle with a constant rate of change of angular speed of 2 radians-per-sec. per sec. Find the angular speed at the end of 20 sec. and the angular displacement; also the linear speed and distance described and the number of revolutions; also the linear tangential acceleration and the central linear acceleration at the end of 20 sec.*

ANS. 40 rad. per sec.; 400 radians; $40r$ ft. per sec.; $400r$ ft.; $\frac{400}{2\pi}$ revolutions; $2r$ ft.-per-sec. per sec. tangential acceleration; $1600r$ ft.-per-sec. per sec. central acceleration.

(20) *A point moving with uniform rate of change of angular speed in a circle is found to revolve at the rate of $8\frac{1}{2}$ revolutions in the eighth second after starting and $7\frac{1}{2}$ revolutions in the thirteenth second after starting. Find its initial angular speed and its uniform rate of change of angular speed; also the initial linear speed and rate of change of speed; also the initial central acceleration.*

ANS. 20.2π radians per sec.; -0.4π radians-per-sec. per sec.; $20\,2\pi r$ ft. per sec.; $-0\,4\pi r$ ft.-per-sec. per sec.; $408.04\pi^2 r$ ft.-per-sec. per sec.

(21) *A point starts from rest and moves in a circle with a uniform rate of change of angular speed of 18 radians-per-sec. per sec. Find the time in which it makes the first, second and third revolutions.*

ANS. $\frac{\sqrt{2\pi}}{3}$, $\frac{2\sqrt{\pi}-\sqrt{2\pi}}{3}$, $\frac{\sqrt{6\pi}-2\sqrt{\pi}}{3}$ secs.

CHAPTER II.

UNIFORM ACCELERATION.

Uniform Acceleration—Motion in a Straight Line.—If the central acceleration f_n of a moving point is zero, the path must be a straight line and the resultant acceleration f is equal to the tangential acceleration f_t. If, then, the acceleration f of a moving point is uniform, so that it does not change in magnitude or direction, and if its direction coincides with that of the velocity, we have motion in a straight line in a given direction with uniform rate of change of speed, and equations (2) to (6), page 92, apply.

In applying these equations we should note that sign *now indicates direction.* If v or f are positive (+), they act away from the origin; if negative (92), they act towards the origin.

The most common instance of such motion is that of a body falling near the earth's surface. All points of such a body move in parallel straight lines with the same speed at any instant, so that the body has motion of translation only, and may be considered as a point.

The acceleration due to gravity is known to be practically uniform and is always denoted by the letter g.

Value of g.—The value of g is usually given in feet-per-sec. per sec. or in centimeters-per-sec. per sec.

It has been determined by much careful experiment and found to vary with the latitude λ and the height h above sea-level.

The general value is given by

$$g = 32.173 - 0.0821 \cos 2\lambda - 0.000003h,$$

where h is the height above sea-level in feet, and g is given in feet-per-sec. per sec., or

$$g = 980.6056 - 2.5028 \cos 2\lambda - 0.000003h,$$

where h is the height above sea-level in centimeters, and g is given in centimeters-per-sec. per sec.

It will be seen that the value of g increases with the latitude, and is greatest at the poles and least at the equator. It also decreases as the height above sea-level increases.

The following table gives the value of g at sea-level in a few localities:

	Latitude.	g F. S. Units.	g C. S. Units.
Equator	0° 0′	32.091	978.10
New Haven	41 18	32.162	980.284
Latitude 45°	45 0	32.173	980.61
Paris	48 50	32.183	980.94
London	51 40	32.182	980.889
Greenwich	51 29	32.191	981.17
Berlin	52 30	32.194	981.25
Edinburgh	55 57	32.203	981.54
Pole	90 0	32.255	983.11
United States	49 0	32.162	980.26
	25 0	32.12	979.00

For calculations where great accuracy is not required it is customary to take $g = 32$ ft.-per-sec. per sec. or $g = 981$ cm.-per-sec. per sec.

For the United States $g = 32\frac{1}{6}$ is a good average value and is therefore very often used. In exact calculations the value of *g for the place* must be used.

Body Projected Vertically Up or Down.—For a body projected vertically upwards *in vacuo*, taking the origin at the starting-point, we have then v_1 positive, since it is away from the origin, and g negative, since it is towards the origin. If the body is projected vertically downwards, taking still the origin at the starting-point, we have v_1 positive and g also positive, since both act away from the origin. We have then simply to replace f_t in equations (2) to (7), page 92, by $\pm g$, taking the plus sign for the falling body and the minus sign for the rising body.

When the final velocity for a body projected upwards is zero, we have from (3), page 92, for the time of rising to the highest or turning point, by making $v = 0$ and $f_t = -g$,

$$T = \frac{v_1}{g}.$$

For the time of rising to the highest or turning point and then returning to the starting-point we make $s - s_1 = 0$ in (5), page 92, and $f_t = -g$, and obtain

$$\frac{2v_1}{g} = 2T.$$

Hence the times of rising and returning are equal for motion *in vacuo*.

The distance from the starting-point to the turning-point is found from (6), page 92, by making $v = 0$ and $f_t = -g$, to be $\frac{v_1^2}{2g}$.

The distance $\frac{v_1^2}{2g}$ or $\frac{v^2}{2g}$ is called the *height due to the velocity* v_1 or v; that is, the distance a body must fall from rest in order to acquire the velocity v_1 or v.

When the distances in rising and falling are equal we have $s - s_1 = 0$ or $\frac{v_1^2}{2g} = \frac{v^2}{2g}$ or $v_1 = v$. That is, for motion *in vacuo* the velocity of return is equal to the velocity of projection.

If the time of rising is less than $T = \frac{v_1}{g}$, the displacement $s - s_1$ is equal to the distance described. But if the time is greater than $T = \frac{v_1}{g}$, the body reaches the turning-point and then falls from rest, and the entire distance described is

$$\text{distance described} = \frac{v_1^2}{2g} + \tfrac{1}{2}g(t - T)^2 = \frac{v_1^2}{g} - s = \frac{v_1^2 + v^2}{2g}.$$

The student will note that the motion is supposed to take place *in a vacuum*. That is, the effect of the resistance of the air is neglected. As a matter of fact this resistance has a great influence, and hence the following examples have little practical value except as illustrating the application of the equations.

Examples.—(Unless otherwise specified $g = 32.2$-ft.-per-sec. per sec. or 981 cm.-per-sec. per sec. All bodies supposed to move in vacuum.)

(1) *A point moves with a uniform velocity of 2 ft. per sec. Find the distance from the starting-point at the end of one hour.*

ANS. 7200 ft. Motion in a straight line.

(2) *Two trains have equal and opposite uniform velocities and each consists of 12 cars of 50 ft. They are observed to take 18 sec. to pass. Find their velocities.*

ANS. 22.73 miles per hour.

(3) *Two points move with uniform velocities of 8 and 15 ft. per sec. in directions inclined 90°. At a given instant their distance is 10 ft. and their relative velocity is inclined 30° to the line joining them. Find* (*a*) *their distance when nearest ;* (*b*) *the time after the given instant at which their distance is least.*

ANS. (*a*) 5 ft.; (*b*) $\frac{5}{17}\sqrt{3}$ sec.

(4) *A body is projected vertically upwards with a velocity of 300 ft. per sec. Find* (*a*) *its velocity after 2 sec. ;* (*b*) *its velocity after 15 sec. ;* (*c*) *the time required for it to reach its greatest height ;* (*d*) *the greatest height reached ;* (*e*) *its displacement at the end of 15 sec. ;* (*f*) *the space traversed by it in the first 15 sec. ;* (*g*) *its displacement when its velocity is 200 ft. per sec. upwards ;* (*h*) *the time required for it to attain a displacement of 320 ft.*

ANS. (*a*) 235.6 ft. per sec.; (*b*) 183 ft. per sec. downwards; (*c*) 9.3 sec.; (*d*) 1397.5 ft.; (*e*) 877.5 ft. upwards; (*f*) 1917.5 ft.; (*g*) 776.3 ft. upwards; (*h*) 1.13 sec. in ascending, 17.5 sec. in descending.

(5) *A ball is projected upwards from a window half way up a tower 117.72 meters high, with a velocity of 39.24 m. per sec. Find the time and speed* (*a*) *with which it passes the top of the tower ascending ;* (*b*) *the same point descending ;* (*c*) *reaches the foot of the tower.*

ANS. (*a*) 2 sec.; 19.62 m. per sec.; (*b*) 6 sec.; 19.62 m. per. sec.; (*c*) $(4 + 2\sqrt{7})$ sec.; $19.62\sqrt{7}$ m. per sec.

(6) *A stone is dropped into a well and the splash is heard in 3.13 sec. If sound travels in air with a uniform velocity of 332 meters per sec., find the depth of the well.*

ANS. 44.1 meters.

(7) *If in the preceding example the time until the splash is heard is T and the velocity of sound in air is V, find the depth.*

$$\text{ANS. Depth} = \frac{V}{g}\left[(Tg + V) - \sqrt{V(2Tg + V)}\right] = \left\{\left(\frac{V^2}{2g} + VT\right)^{\frac{1}{2}} - \frac{V}{(2g)^{\frac{1}{2}}}\right\}^2.$$

(8) *Show that a body projected vertically upwards requires twice as long a time to return to its initial position as to reach the highest point of its path, and has on returning to its initial position a speed equal to its initial speed.*

(9) *A stone projected vertically upwards returns to its initial position in 6 sec. Find* (*a*) *its height at the end of the first second, and* (*b*) *what additional speed would have kept it 1 sec. longer in the air.*

ANS. (*a*) 80.5 ft.; (*b*) 16.1 ft. per sec.

(10) *A body let fall near the surface of a small planet is found to traverse 204 ft. between the fifth and sixth seconds. Find the acceleration.*

ANS. 20.4 ft.-per-sec. per sec.

(11) *A particle describes in the nth second of its fall from rest a space equal to p times the space described in the* ($n - 1$)*th second. Find the whole space described.*

ANS $\dfrac{g(1 - 3p)^2}{8(1 - p)^2}$.

(12) *A body uniformly accelerated, and starting without initial velocity, passes over b feet in the first p seconds. Find the time of passing over the next b ft.*

ANS. $p(\sqrt{2} - 1)$ sec.

(13) *A ball is dropped from the top of an elevator 4.905 meters high. Acceleration of gravity is 9.81 meters-per-sec. per sec. Find the times in which it will reach the floor* (*a*) *when the elevator is at rest ;* (*b*) *when it is moving with a uniform downward acceleration of 9.81 m.-per-sec. per sec. ;* (*c*) *when moving with a uniform downward acceleration of 4.905 m.-per-sec. per sec. ;* (*a*) *when moving with a uniform upward acceleration of 4.905 m.-per-sec. per sec.*

ANS. (*a*) 1 sec.; (*b*) ∞; (*c*) $\sqrt{2}$ sec.; (*d*) $\sqrt{\frac{2}{3}}$ sec.

Uniform Acceleration—Motion in a Curve.—If the acceleration f is uniform, so that it does not change either in magnitude or direction, and if its direction does not coincide with that of the velocity, we have motion in a curve with uniform acceleration.

A common case of such motion is that of a body projected with any given velocity in any given direction at the surface of the earth, neglecting the resistance of the air. In such case the accelération due to gravity is practically uniform, acts downward and is equal to g ft.-per-sec. per sec. The curve or path in such case is called the TRAJECTORY.

FIG. (*a*).

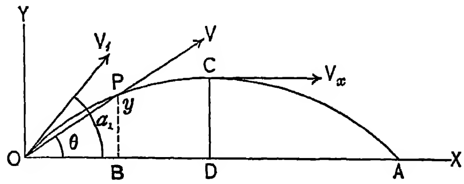
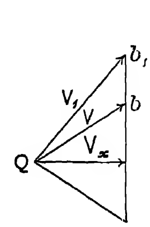


FIG. (*b*)



EQUATION OF THE PATH.—Let the uniform acceleration f be vertical and act downwards. (In the case of gravity $f = g$.) Take the origin O at the initial position of the point as shown in Fig. (*a*), and let the initial velocity v_1 make the angle α_1 with the horizontal.

Let the co-ordinates of any point P of the path or *trajectory* be $OB = x$ and $BP = y$.

Let the angle $POB = \theta$.

If in Fig. (*b*) we lay off from Q the line representative $Qb_1 = v_1$ of v_1, the line representative $Qb = v$ of v, etc., we see that the hodograph b_1b, etc., is a straight line. (See example (2), page 80.)

It is at once evident that the horizontal component v_x of the velocity at every point is constant and equal to

$$v_x = v_1 \cos \alpha_1. \qquad (1)$$

In any time t, then, the horizontal distance described is

$$x = v_1 t \cos \alpha_1. \qquad (2)$$

The vertical component of the initial velocity v_1 is $v_1 \sin \alpha_1$ upwards. But the uniform acceleration f is downwards. Hence the vertical velocity at the end of the time t is, from equation (3), page 92,

$$v_y = v_1 \sin \alpha_1 - ft. \qquad (3)$$

The *mean vertical velocity* during the time t is, then,

$$\frac{v_1 \sin \alpha_1 + v_y}{2} = \frac{2v_1 \sin \alpha_1 - ft}{2} = v_1 \sin \alpha_1 - \tfrac{1}{2}ft.$$

The vertical distance passed over in the time t is, then,

$$y = v_1 t \sin \alpha_1 - \tfrac{1}{2}ft^2. \qquad (4)$$

If we combine (2) and (4) by eliminating t, we have for the equation of the path

$$y = x \tan \alpha_1 - \frac{fx^2}{2v_1^2 \cos^2 \alpha_1}. \qquad (5)$$

This is the equation of a *parabola*.

The time of reaching the highest point C is the time of describing the vertical distance DC. Denote this time by T_v. Since at this point the vertical velocity is zero, we have, by making $v_y = 0$ in (3),

$$T_v = \frac{v_1 \sin \alpha_1}{f}. \qquad (6)$$

If we substitute this for t in (2) and (4), we have for the co-ordinates of the vertex C of the parabola

$$OD = x_0 = \frac{v_1^2 \sin \alpha_1 \cos \alpha_1}{f} = \frac{v_1^2 \sin^2 \alpha_1}{2f}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$DC = y_0 = \frac{v_1^2 \sin^2 \alpha_1}{2f} = \frac{x_0^2 f}{2v_1^2 \cos^2 \alpha_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The parameter of the parabola is then

$$\frac{x_0^2}{2y_0} = \frac{v_1^2 \cos^2 \alpha_1}{f}.$$

The directrix is parallel to OD at a distance above the vertex C equal to one half the parameter, or

$$\frac{v_1^2 \cos^2 \alpha_1}{2f},$$

or at a distance of $\frac{v_1^2}{2f}$ above O. That is, *the distance of the directrix above O is the height due to the velocity* v_1.

If we transfer the origin to the vertex C, then the horizontal velocity at C is $v_1 \cos \alpha_1$, the horizontal distance described in any time t is $x = v_1 t \cos \alpha_1$. The mean vertical velocity is $\frac{1}{2}$ ft., and the vertical distance is $y = \frac{1}{2}$ ft.². Eliminating t, we have

$$x^2 = \frac{2v_1^2 \cos^2 \alpha_1}{f} y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which is the equation of the path for origin at the vertex C.

VELOCITY AT ANY POINT OF THE PATH.—The magnitude of the velocity at any point P is the resultant of the vertical and horizontal components, or, from (1) and (3),

$$v^2 = v_x^2 + v_y^2 = v_1^2 - 2v_1 ft \sin \alpha_1 + f^2 t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Inserting the value of y from (4),

$$v^2 = v_1^2 - 2fy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If the acceleration is due to gravity, we replace f by g and have

$$\frac{v^2}{2g} = \frac{v_1^2}{2g} - y.$$

But we have just seen that $\frac{v_1^2}{2g}$ is the distance of the directrix above O. Therefore $\frac{v_1^2}{2g} - y$ is the distance of the directrix above any point P, and $\frac{v^2}{2g}$ is the height due to the velocity v (page 101). Hence for a body acted upon by gravity *the speed at any point is the same as that acquired by a body falling from the directrix to that point.*

To find the *direction* of the velocity v at any point P, the magnitude of which is given by (10) or (11), let α be the angle which it makes with the horizontal. Then we have directly from the hodograph (Fig. (*b*), page 103), and from equations (3) and (1),

$$v \sin \alpha = v_y = v_1 \sin \alpha_1 - ft,$$

$$v \cos \alpha = v_x = v_1 \cos \alpha_1.$$

Hence, from equation (2),

$$\tan \alpha = \tan \alpha_1 - \frac{ft}{v_1 \cos \alpha_1} = \tan \alpha_1 - \frac{ft^2}{x}, \quad \ldots \ldots \quad (12)$$

or

$$\tan \alpha = \tan \alpha_1 - \frac{fx}{v_1^2 \cos^2 \alpha_1}. \quad \ldots \ldots \quad (13)$$

TIME OF FLIGHT AND RANGE.—If in (4) we make $y = 0$, we have for the time T_h in which the body reaches the line OX, or the time of flight in a horizontal direction,

$$T_h = \frac{2v_1 \sin \alpha_1}{f}. \quad \ldots \ldots \quad (14)$$

Inserting this value of t in (2), we have for the horizontal range $OA = R_h$

$$R_h = \frac{2v_1^2 \sin \alpha_1 \cos \alpha_1}{f} = \frac{v_1^2 \sin 2\alpha_1}{f}. \quad \ldots \ldots \quad (15)$$

This, we see from (7), is twice the distance OD.

We see from (15) that the range is greatest when $\sin 2\alpha_1$ is greatest, or when $2\alpha_1 = 90°$ or $\alpha_1 = 45°$. Therefore the range, neglecting resistance of the air, is greatest for an angle of elevation of 45°, and is equal to $\frac{v_1^2}{f}$.

DISPLACEMENT IN ANY GIVEN DIRECTION AND THE CORRESPONDING TIME.—Let θ be the angle which any displacement $OP = R$ makes with the horizontal. Then we have $BP = y = x \tan \theta$. Substituting this value of y in (5), we have for the abscissa of the point P

$$x = \frac{2v_1^2 \cos \alpha_1 \sin (\alpha_1 - \theta)}{f \cos \theta} = \frac{v_1^2[\sin (2\alpha_1 - \theta) - \sin \theta]}{f \cos \theta}, \quad \ldots \ldots \quad (16)$$

and therefore, since $R = \frac{x}{\cos \theta}$,

$$R = \frac{2v_1^2 \cos \alpha_1 \sin (\alpha_1 - \theta)}{f \cos^2 \theta} = \frac{v_1^2[\sin (2\alpha_1 - \theta) - \sin \theta]}{f \cos^2 \theta}. \quad \ldots \quad (17)$$

If in (17) we make $\theta = 0$, we have the horizontal range given by (15).

If we divide (16) by the horizontal component of the velocity, $v_1 \cos \alpha_1$, we have for the *time of flight*

$$t = \frac{2v_1 \sin (\alpha_1 - \theta)}{f \cos \theta}; \quad \ldots \ldots \quad (18)$$

This reduces to (14) for $\theta = 0$.

ANGLE OF ELEVATION FOR GREATEST RANGE IN ANY DIRECTION.—The range R given (by (17) is greatest when $\sin (2\alpha_1 - \theta)$ is greatest, or when $2\alpha_1 - \theta = 90°$, or $\alpha_1 = \frac{1}{2}(90° + \theta)$. The direction for greatest range makes, therefore, with the vertical an angle of $90° - \alpha_1 = \frac{1}{2}(90° - \theta)$, that is, *it bisects the angle between the vertical and the range.*)

ELEVATION NECESSARY TO HIT A GIVEN POINT.—To determine the direction of the velocity v_1 in order that the path may pass through a given point given by x and y, we

substitute for $\frac{1}{\cos^2 \alpha_1}$ in equation (5) the equivalent expression $1 + \tan^2 \alpha_1$ and obtain at once

$$\tan \alpha_1 = \frac{v_1^2}{fx} \pm \sqrt{\left(\frac{v_1^2}{fx}\right)^2 - \left(1 + \frac{2v_1^2 y}{fx^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We have also, from (16),

$$\alpha_1 = \frac{\theta}{2} + \frac{1}{2} \sin^{-1}\left(\frac{fx \cos \theta}{v_1^2} + \sin \theta\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

or, since $R \cos \theta = x$,

$$\alpha_1 = \frac{\theta}{2} + \frac{1}{2} \sin^{-1}\left(\frac{fR \cos^2 \theta}{v_1^2} + \sin \theta\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

We see from (19) that α_1 has two values. If α_1' is an angle such that $\sin (2\alpha_1' - \theta) = \sin (2\alpha_1 - \theta)$, then $2\alpha_1' - \theta = 180° - (2\alpha_1 - \theta)$, or $\alpha_1' = 90° - (\alpha_1 - \theta)$, and either α_1' or α_1 will satisfy equation (16).

With a given acceleration and initial velocity there are, then, two directions of the initial velocity, α_1 and $90° - (\alpha_1 - \theta)$, and therefore two paths by which the projectile may hit the same point.

If, in (19), we put $\frac{v_1^2}{fx} = 1 + \frac{2v_1^2 y}{fx^2}$, we have

$$v_1^2 = f(y + \sqrt{x^2 + y^2}) \quad \text{and} \quad \tan \alpha_1 = \frac{v_1^2}{fx}.$$

Smaller values of v_1 make $\tan \alpha_1$ imaginary. Larger values of v_1 give two values for $\tan \alpha_1$. In the first case the point cannot be hit. In the second case it can be hit during either the rise or fall of the projectile.

ENVELOPE OF ALL TRAJECTORIES.—Equation (5) gives the equation of the trajectory corresponding to the angle of elevation α_1. If we substitute $1 + \tan^2 \alpha_1$ for $\frac{1}{\cos^2 \alpha_1}$, equation (5) becomes

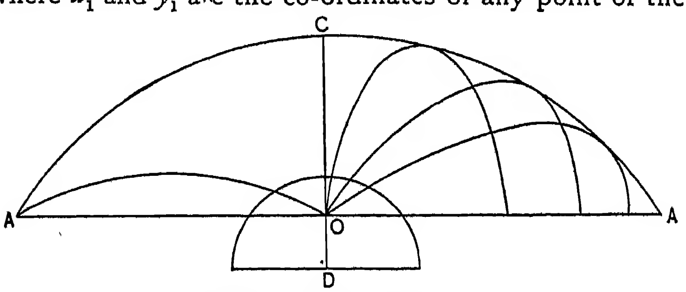
$$y = x \tan \alpha_1 - \frac{fx^2(1 + \tan^2 \alpha_1)}{2v_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where x and y are the co-ordinates of any point of the path.

For another angle of elevation, α_1', and the same initial speed, v_1, we have

$$y_1 = x_1 \tan \alpha_1' - \frac{fx_1^2(1 + \tan^2 \alpha_1')}{2v_1^2},$$

where x_1 and y_1 are the co-ordinates of any point of the new trajectory.



If we make $x = x_1$ and $y = y_1$ and equate these two equations, we have for the point of intersection of the two trajectories

$$\frac{fx}{2v_1^2}(\tan \alpha_1 + \tan \alpha_1') = 1.$$

As the angles α_1' and α_1 approach equality, this expression approaches the limit

$$\frac{fx}{v_1^2} \tan \alpha_1 = 1, \quad \text{or} \quad \tan \alpha_1 = \frac{v_1^2}{fx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Equation (23) then gives the value of tan α_1 when the two trajectories starting from the same point O with the same speed v_1 have angles of elevation at O whose difference is indefinitely small.

Substituting this value of tan α_1 in (22) we obtain

$$y = \frac{v_1^2}{2f} - \frac{fx^2}{2v_1^2}. \qquad (24)$$

Equation (24) is then the equation of a curve which passes through all the points in which every two trajectories, starting from O at angles of elevation whose difference is indefinitely small, cut each other. It is therefore the equation of the *envelope* or curve which touches all the trajectories or parabolas described from O with the same speed v_1.

Equation (24) is the equation of a parabola ACA whose axis OC is vertical, whose focus is at O, and whose vertex C is in the common directrix of the trajectories.

With the given initial speed v_1 the projectile can reach any point inside this envelope by two angles of elevation and two trajectories, as already proved. It can reach any point in the envelope by only one elevation and path. It cannot reach any point outside this envelope with any elevation and the given speed v_1.

The point, therefore, where this envelope cuts any range gives the *maximum range* in that direction for given v_1.

Thus the maximum horizontal range is found from (24) by making $y = 0$ to be $\frac{v_1^2}{f}$. This is the same as given by (15) when we make $\alpha_{1'} = 45°$, or the same as given by (17) when we make $\alpha_1 = 45°$ and $\theta = 0$.

Questions of maximum range may thus be readily solved by the equation of the envelope.

From (2) we have $\cos \alpha_1 = \frac{x}{v_1 t}$, and from (4) $\sin \alpha_1 = \frac{y + \frac{1}{2}ft^2}{v_1 t}$.

Since $\cos^2 \alpha_1 + \sin^2 \alpha_1 = 1$, we have

$$x^2 + (y + \tfrac{1}{2}ft^2)^2 = v_1^2 t^2. \qquad (25)$$

This is the equation of a circle whose radius is $v_1 t$ and whose centre is situated vertically below O at a distance $OD = \frac{1}{2}ft^2$.

The circumference of this circle is reached in the same time by a projectile starting from O with the velocity v_1 in any direction.

In all equations the resistance of the air is neglected. Hence the following examples have little practical value except as illustrating the application of the equations.

Examples. –($g = 32.16$ *ft.-per-sec. per sec*)

(1) *If the angle of elevation is 30°, find the velocity of projection in order to hit a point at a distance of 2500 ft. on an ascent of 1 in 40.*

ANS. 311.5 ft. per sec.

(2) *Find the direction and magnitude of the velocity of projection in order that a projectile may reach its maximum height at a point whose horizontal and vertical distances from the starting-point are x_0 and y_0.*

ANS. From equations (7) and (8). $\tan \alpha_1 = \frac{2y_0}{x_0}$, $v_1 = \sqrt{\frac{(4y_0^2 + x_0^2)g}{2y_0}}$.

(3) *A gun is fired horizontally at a height of 144.72 ft. above the surface of a lake, and the initial speed of the ball is 1000 ft. per sec. Find after what time and at what horizontal distance the ball strikes the lake and with what velocity.*

ANS. Time 3 sec.; distance 3000 ft.; velocity 1004.64 ft. per sec. at an angle below the horizontal whose tangent is 0.09648.

(4) *A ball is projected with a velocity of 100 ft. per sec. at an angle of 75° to the horizon. Find the horizontal range; the range on a line 30° to the horizon; and what other directions of the initial velocity would give the same ranges.*

ANS. Horizontal range 155.5 ft.; range on 30° line 207.3 $(\sqrt{3} - 1)$; 15° and 45°.

(5) *Find the angle of elevation in order to hit a point at a distance of 1000 ft. and an elevation of 500 ft. with a velocity of projection of 310.8 ft. per sec.*

ANS. 39° 17′ or 80° 43′.

(6) *A body is projected with a velocity of 30 ft. per sec. inclined 60° to the horizon. Find the velocity after 20 seconds.*

ANS. 617.3 ft. per sec. inclined 148° 36′.6 to the direction of the initial velocity.

(7) *A projectile is fired at an angle of 30° at a target distant 1200 meters horizontally.* ($g = 9.81$ *meters-per-sec. per sec.*) *Find* (a) *the initial velocity;* (b) *the time of flight;* (c) *the highest point of the trajectory;* (d) *the velocity of striking.*

ANS. (*a*) 116 6 meters per sec.; (*b*) about 12 seconds; (*c*) 173.21 meters; (*d*) same as the initial velocity making angle 30° below horizontal.

(8) *A projectile is fired with an initial velocity of 150 meters per sec. from a point 100 meters below a target which is distant horizontally 1525 meters.* ($g = 9.81$ *meters-per-sec. per sec.* *Find* (a) *the angle of elevation;* (b) *the velocity of striking;* (c) *the time of flight.*

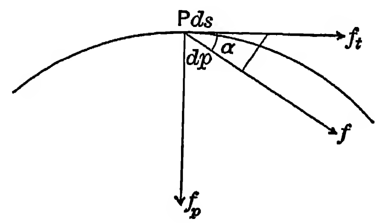
ANS. (*a*) 68° 28′ 50″ or 25° 16′; (*b*) 143.3 meters per sec.; (*c*) 27.715 sec. or 11.24 sec.

CHAPTER III.

MOTION UNDER VARIABLE ACCELERATION IN GENERAL. CENTRAL ACCELERATION. CENTRAL ACCELERATION INVERSELY AS THE SQUARE OF THE DISTANCE. PLANETARY MOTION.

Motion under Variable Acceleration in General.—Let P be a point moving in any path, and f its acceleration. The acceleration f can be resolved into the tangential acceleration f and the central acceleration f_p.

Let α be the angle between f and f_t. Then we have



$$f \cos \alpha = f_t.$$

Let ds be the element of the path at P, and dp the projection of this element upon f. Then we have

$$dp = ds \,.\, \cos \alpha.$$

Multiplying these two equations we have

$$fdp = f_t ds.$$

That is, the acceleration f multiplied by the elementary displacement in the direction of f is equal to the tangential acceleration f_t multiplied by the elementary displacement in the direction of f_t.

Since this holds at every point of the path, we have for the entire path

$$\Sigma f \,.\, dp = \Sigma f_t ds. \quad \ldots \ldots \ldots \quad (1)$$

Let v_1 and v_2 be the velocities at two consecutive points P_1 and P_2 of the path. Then for the indefinitely small time τ

$$f_t = \frac{v_2 - v_1}{\tau}.$$

The average velocity for this time is $\frac{v_2 + v_1}{2}$, and the distance ds described is

$$ds = \frac{v_2 + v_1}{2} \tau.$$

We have then

$$f_t ds = \frac{v_2^2 - v_1^2}{2}.$$

For the next two consective points, P_2 and P_3, we have then

$$f_t ds = \frac{v_3^2 - v_2^2}{2}.$$

For the next two, P_3 and P_4,

$$f_t ds = \frac{v_4^2 - v_3^2}{2}.$$

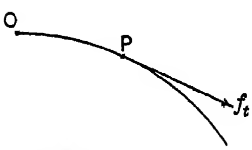
Adding all these, we have, if v_1 is the initial and v the final **velocity,**

$$\Sigma f_t ds = \frac{v^2 - v_1^2}{2}.$$

Equation (1) then becomes

$$\Sigma f . dp = \Sigma f_t ds = \frac{v^2 - v_1^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

Equation (I) is general, as is seen by its derivation, whatever the path and whatever the acceleration f in magnitude, and however it may change in direction. It will therefore in all cases give us the relation between v and f or f_t. In performing the summations in (I) we should take f or f_t positive when acting *away* from the origin, and negative when acting *towards* the origin.



ILLUSTRATIONS.—Thus for a point moving in any path with uniform rate of change of speed we have f_t constant in magnitude, and positive if away from the origin O, as shown in the figure. Hence

$$\Sigma f_t ds = f_t \Sigma ds = f_t(s - s_1),$$

where s is the final and s_1 the initial distance of the point *measured along the path* from any point O of the path assumed as origin. We have then, from (I),

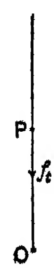
$$v^2 = v_1^2 + 2f_t(s - s_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which holds no matter whether the distance from O is increasing or decreasing. If increasing, s is greater than s_1. If decreasing, s is less than s_1. This is equation (7), page 92.

If f_t acts towards the origin, we should take f_t negative and obtain

$$v^2 = v_1^2 - 2f_t(s - s_1),$$

and this again holds whether the distance from O is increasing or decreasing.



Again, for a point moving with uniform acceleration in a straight line, we have for acceleration towards the origin $f = f_t$ negative. Hence

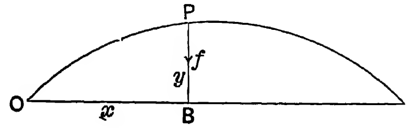
$$- \Sigma f_t ds = -f_t \Sigma ds = - f_t(s - s_1)$$

and, from (I),

$$v^2 = v_1^2 - 2f_t(s - s_1).$$

If $f_t = g$, this is the case of a body under the action of the earth's attraction, either projected upwards or falling, origin at surface of earth (page 100). If projected upwards, s is greater than s_1. If falling, s is less than s_1.

Again, if the acceleration f is uniform and does not coincide with the velocity, we have for origin at O, as shown in the figure, f negative if *downwards*. Hence



$$-\Sigma f \,.\, dp = -f\Sigma dp = -fy,$$

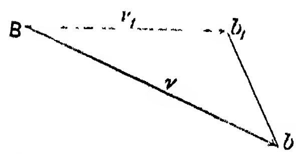
and, from (I),

$$v^2 = v_1^2 - 2fy,$$

which is equation (11) (page 104). We see, then, that equation (I) is general and applies to all cases.

Central Acceleration.—If the acceleration of a moving point is always directed towards or away from a fixed point or CENTRE OF ACCELERATION, the acceleration is said to be CENTRAL.

The velocity $v = Bb$ of the moving point at any instant is the resultant of the velocity $v_1 = Bb_1$ at the preceding instant, and of the change of velocity $b_1 b$ (page 75).



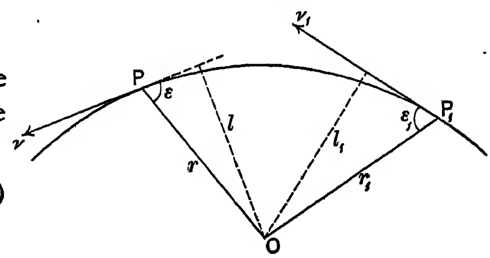
But if $b_1 b$ is always directed towards or away from a fixed point, its moment relative to that point is zero. Since the moment of the resultant is equal to the algebraic sum of the moments of the components (page 89), and since in this case the moment of one of the two components $b_1 b$ is always zero, it follows that the moment of any velocity v relative to the centre of acceleration is for central acceleration *always constant*.

Conversely, if the moment of the velocity of a moving point relative to any fixed point is constant, the acceleration must be central.

But the moment of the velocity is twice the areal velocity of the radius vector (page 89). Hence *in all cases of central acceleration the radius vector describes equal areas in equal times.*

If v_1 is the known initial velocity at any given instant and l_1 is its lever-arm, and if at any other instant the velocity is v and lever-arm l, we have for constant moment

$$vl = v_1 l_1.$$



If r_1 is the initial radius vector and ϵ_1 is the angle of v_1 with r_1, we have $l_1 = r_1 \sin \epsilon_1$. In the same way $l = r \sin \epsilon$. Hence

$$vl = vr \sin \epsilon = v_1 r_1 \sin \epsilon_1. \quad . \quad . \quad (3)$$

If ω_1 is the initial angular velocity, $r_1\omega_1 = v_1 \sin \epsilon_1$. In the same way $r\omega = v \sin \epsilon$. Hence

$$r^2\omega = r_1^2\omega_1, \quad \text{or} \quad \frac{\omega}{\omega_1} = \frac{r^2}{r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, *in all cases of central acceleration the angular velocity is inversely as the square of the radius vector.*

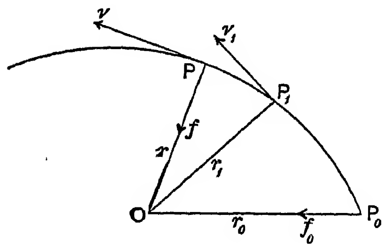
Cases of Central Acceleration.—Two of the most important cases of central acceleration are:

1st. Central acceleration varying inversely as the square of the distance from the centre of acceleration.

2d. Central acceleration varying directly as the distance from the centre of acceleration.

We shall discuss in this chapter the first case.

Central Acceleration Inversely as the Square of the Distance.—Let r_1 be the initial distance from the centre of acceleration O, the velocity at the point P_1 being v_1, and r the distance to any point P at which the velocity is v.



Let f_0 be the known acceleration at a given distance r_0, and f the acceleration at any distance r. Then

$$f : f_0 :: r_0^2 : r^2, \quad \text{or} \quad f = \frac{f_0 r_0^2}{r^2}.$$

This gives the magnitude of the central acceleration. If f is towards the centre O, it is negative and we have

$$f = -\frac{f_0 r_0^2}{r^2}.$$

If f is away from the centre O, it is positive.

(1) ACCELERATION TOWARDS THE CENTRE.—From (I), page 110, we have

$$\Sigma f.\,dp = \frac{v^2 - v_1^2}{2}, \quad \text{where} \quad f = -\frac{f_0 r_0^2}{r^2}.$$

Let P_1 and P_2 be two consecutive points at distances r_1 and r_2 from O. Then, if the points are consecutive, $dp = r_2 - r_1$, and $r^2 = r_1 r_2$. Hence

$$fdp = -\frac{f_0 r_0^2}{r_1 r_2}(r_2 - r_1) = f_0 r_0^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

For the next two consecutive points, P_2 and P_3, we have, in the same way,

$$fdp = f_0 r_0^2\left(\frac{1}{r_3} - \frac{1}{r_2}\right).$$

For the next two consecutive points, P_3 and P_4,

$$fdp = f_0 r_0^2\left(\frac{1}{r_4} - \frac{1}{r_3}\right),$$

and so on.

Summing up, if r_1 is the initial and r the final distance, we have

$$\Sigma f.\,dp = f_0 r_0^2\left(\frac{1}{r} - \frac{1}{r_1}\right),$$

and hence, from (I),

$$v^2 = v_1^2 - 2f_0 r_0^2\left(\frac{1}{r_1} - \frac{1}{r}\right). \quad \ldots \ldots \ldots \quad (1)$$

If the distance is increasing, r is greater than r_1. If decreasing, r is less than r_1. Equation (1) holds in both cases if the acceleration is *towards the centre*.

(2) ACCELERATION AWAY FROM CENTRE.—In this case we have

$$f = +\frac{f_0 r_0^2}{r^2},$$

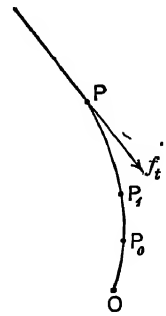
and proceeding just as before, we obtain

$$v^2 = v_1^2 + 2f_0 r_0^2\left(\frac{1}{r_1} - \frac{1}{r}\right). \quad \ldots \ldots \ldots \quad (2)$$

This also holds whether r is increasing or decreasing. We see that in order to obtain (2) we have simply to change the sign of f_0 in (1).

COR. It is evident that the same equations hold for motion in any path, if we take the pole O in the path and *measure all distances along the path*, if the *tangential acceleration f_t is inversely as the square of the distance.*

We have, in such case,



for f_t towards O, $$v^2 = v_1^2 - 2f_0 s_0^2\left(\frac{1}{s_1} - \frac{1}{s}\right); \quad \ldots \quad (3)$$

for f_t away from O, $$v^2 = v_1^2 + 2f_0 s_0^2\left(\frac{1}{s_1} - \frac{1}{s}\right), \quad \ldots \quad (4)$$

where f_0 is the tangential acceleration at a distance s_0, s_1 is the initial distance from O to P_1, s the final distance from O to P, all distances measured *along the path.*

Central Acceleration Inversely as the Square of the Distance—Motion Rectilinear.—If the central acceleration coincides with the direction of the velocity, the motion is rectilinear. In this case equations (3) and (4) hold.

This is the case of a body falling freely under the action of gravity at a very great distance from the earth.

In this case let $s_0 = r_0$ be the radius of the earth. The known acceleration at this distance is $f_0 = g$. We have then, from (3),

$$v^2 = v_1^2 - 2gr_0^2\left(\frac{1}{s_1} - \frac{1}{s}\right). \quad \ldots \quad (5)$$

If the body falls from rest from a distance s_1, we have from (5), making $v_1 = 0$,

$$v^2 = 2gr_0^2\left(\frac{1}{s} - \frac{1}{s_1}\right).$$

If the distance s_1 is infinite, we have

$$v = \sqrt{\frac{2gr_0^2}{s}}$$

for the velocity acquired by a body falling from an infinite distance.

If the body is projected upwards, we have from (5), making $v = 0$,

$$v_1^2 = 2gr_0^2\left(\frac{1}{s_1} - \frac{1}{s}\right).$$

If the distance s is infinite, we have

$$v_1 = \sqrt{\frac{2gr_0^2}{s_1}}$$

for the velocity of projection which would carry the body to an infinite distance.

If we take $g = 32\frac{1}{6}$ ft.-per-sec. per sec., the mean radius of the earth $r_0 = 3960$ miles, we have, making $s = r_0$ and $s_1 = r_0$, for the velocity acquired in falling to the earth from an infinite distance, or for the velocity necessary to project a body from the earth's surface to an infinite distance, *in vacuo,*

$$v = v_1 = \sqrt{2gr_0} = 6.95 \text{ miles per sec.}$$

We can put equation (5) in the form

$$v^2 = v_1^2 + 2gr_0^2\left(\frac{s_1 - s}{ss_1}\right).$$

If the fall takes place near the earth's surface, ss_1 will be practically equal to r_0^2 and we have

$$v^2 = v_1^2 + 2g(s_1 - s).$$

This is the same as equation (7), page 92, when applied to a falling body (page 101).

Examples.—(1) *A body falls to the earth from a point 1000 miles above the surface. Find the speed on reaching the surface, neglecting air-resistance, and taking the earth's radius 4000 miles and $g = 32.16$.*

ANS. $v = 3.12$ miles per sec.

(2) *At what point on a line joining the centres of the earth and moon would the acceleration of a body be zero? (Take f at the surface of the moon 5.5 ft.-per-sec. per sec.; radius of moon 1080 miles; distance between centres of earth and moon 240000 miles; $g = 32.16$ ft.-per-sec. per sec.; radius of earth 4000 miles.)*

ANS. Let x_1 = distance from earth's centre, x_2 from moon's centre, R radius of earth, r radius of moon, f acceleration at moon's surface, f_1 acceleration at x_1 towards earth, f_2 acceleration at x_2 towards moon.

Then $\qquad \dfrac{f_1}{g} = \dfrac{R^2}{x_1^2}$, or $f_1 = \dfrac{gR^2}{x_1^2}$. Also $\dfrac{f_2}{f} = \dfrac{r^2}{x_2^2}$, or $f_2 = \dfrac{fr^2}{x_2^2}$.

If f_1 and f_2 are equal

$$\frac{gR^2}{x_1^2} = \frac{fr^2}{x_2^2}.$$

Also $x_1 + x_2 = 240000$. Hence substituting numerical values $x_1 = 215893$ miles.

Central Acceleration Inversely as the Square of the Distance—Motion in a Curve.—If the central acceleration does not coincide with the velocity, we have motion in a curve.

(*a*) HODOGRAPH.—Since the acceleration is central we have, from equation (4), page 111,

$$r^2\omega = r_1^2\omega_1, \quad \text{or} \quad \omega = \frac{r_1^2\omega_1}{r^2},$$

where r is the radius vector for any point whose angular velocity is ω, and r_1, ω_1 are the known radius vector and angular velocity at some given point.

FIG. (*a*).

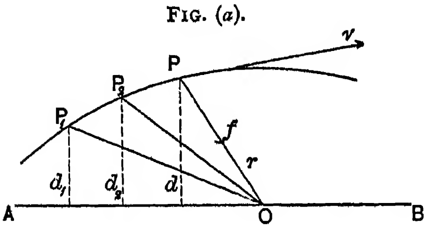
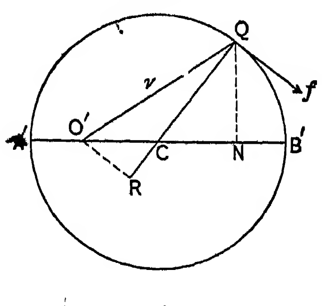


FIG. (*b*).



We have also, by assumption (page 112),

$$f = \frac{f_0 r_0^2}{r^2},$$

where f is the central acceleration at any point whose radius vector is r, and f_0 is the known central acceleration at a given point whose radius vector is r_0.

Let P, Fig. (*a*), be a point which has the velocity v and central acceleration directed always to the point O, the radius vector being $OP = r$.

Take O', Fig. (*b*), as the pole of the hodograph (page 79) and draw $O'Q$ parallel and equal to v. Then the tangent to the hodograph at Q is the direction of the acceleration f at P and is parallel to $OP = r$ (page 79).

Since the angular velocity of r, Fig. (*a*), is ω, the angular velocity of the tangent at Q, Fig. (*b*), is also ω.

Let C, Fig. (*b*), be the centre of curvature of the hodograph, so that CQ is perpen-

dicular to the tangent at Q, and CQ is the radius of the hodograph. Then, since the acceleration f of P, Fig. (a), is the velocity of Q, Fig. (b), we have $f = CQ \cdot \omega$ or $CQ = \frac{f}{\omega}$.

But $\omega = \frac{r_1^2\omega_1}{r^2}$, and $f = \frac{f_0 r_0^2}{r^2}$. Hence

$$CQ = \frac{f_0 r_0^2}{r_1^2\omega_1} = \text{a constant quantity.}$$

The radius of curvature of the hodograph is therefore constant, *and the hodograph is a circle.*

(b) PATH.—Draw $O'R$, Fig. (b), at right angles to CQ and therefore parallel to r. $O'R$ is the component of the velocity v along the radius vector. Draw QN perpendicular to $O'C$. Then QN is the component of v at right angles to the line $O'C$, or the diameter $A'B'$ of the hodograph.

But by similar triangles

$$\frac{O'R}{O'C} = \frac{QN}{CQ}, \quad \text{or} \quad \frac{O'R}{QN} = \frac{O'C}{CQ}.$$

But $O'C$ is a constant quantity by construction, and CQ, as we have just proved, is also constant. Hence

$$\frac{O'R}{QN} = \text{a constant quantity} = c.$$

That is, the ratio of the velocity along the radius vector to the velocity at right angles to the diameter $A'B'$ of the hodograph is a constant ratio c.

If, then, r_1 and r_2 are the initial and final values of r for an indefinitely short time τ, and d_1, d_2 are the corresponding distances of P from any line AB, Fig. (a), parallel to $A'B'$, we have for the velocity along the radius vector

$$O'R = \frac{r_1 - r_2}{\tau},$$

and for the velocity at right angles to AB

$$QN = \frac{d_1 - d_2}{\tau}.$$

Hence

$$\frac{O'R}{QN} = \frac{r_1 - r_2}{d_1 - d_2} = c, \quad \text{or} \quad r_1 - r_2 = c(d_1 - d_2).$$

Since this holds wherever we take the line AB, let us take the initial distance $d_1 = \frac{r_1}{c}$ or $c = \frac{r_1}{d_1}$. Then we have

$$\frac{r_2}{d_2} = \frac{r_1}{d_1} = c.$$

That is, the ratio $\frac{r}{d} = c$ of the distance r of the point P from a fixed point O to its distance d from a fixed line AB is constant.

This is the property of a conic section for a focus at O and directrix AB.

When, therefore, a point has a central acceleration inversely as the square of the radius vector, it must move in a conic section with the centre of acceleration at a focus, and, as we have seen (page 111) in all cases of central acceleration, the radius vector describes equal areas in equal times.

Conversely, if the radius vector describes equal areas in equal times, the acceleration must be central; and if the path is a conic section and the centre of acceleration is at a *focus*, the acceleration is inversely as the square of the radius vector.

If $c = 1$ or $r = d$, then the path is a parabola, $O'R = QN$, and the pole O' is on the circumference of the hodograph. The parabola becomes a straight line when the directrix passes through the focus.

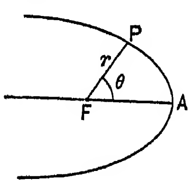
If c is less than unity, or r is less than d, then the path is an ellipse, $O'R$ is less than QN, and the pole O' is inside the hodograph. If d is infinity, the path is a circle and the pole O' coincides with the centre of the hodograph.

If c is greater than unity, or r is greater than d, then the path is an hyperbola, $O'R$ is, greater than QN, and the pole O' is outside the hodograph.

General Equation of a Conic Section.—The general polar equation of a conic section is by Analytical Geometry,

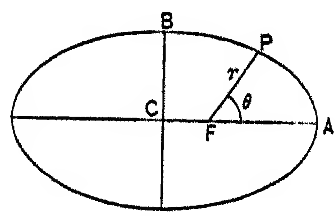
$$r = \frac{A(1 - e^2)}{1 + e\cos\theta}, \qquad (1)$$

where r is the radius vector of any point making the angle θ with an axis, and A and e are constants.



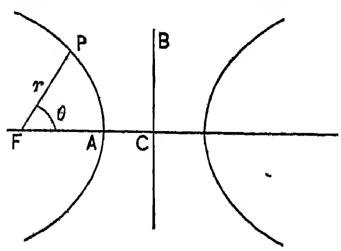
If $e = 1$, we have a parabola, r is the radius vector from the focus F, as shown in the figure, θ is the angle AFP of the radius vector with the axis measured round from the apex A. A straight-line path is a special case of the parabola when the directrix passes through the focus.

If $e < 1$, we have an ellipse, r is the radius vector from a focus F, and θ is the angle AFP of the radius vector *with the major axis CA* measured round from *the nearer vertex A*, as shown in the figure.



In this case A in equation (1) *is the semi-major axis CA*, and e is the eccentricity, or $e = \frac{CF}{CA}$. The circle is a special case when $e = 0$.

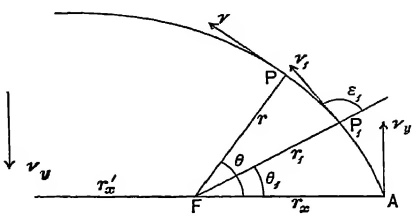
If $e > 1$, we have an hyperbola, r is the radius vector from a focus F, and θ is the angle AFP of the radius vector *with the major axis AC* measured round from the *nearer vertex A*, as shown in the figure.



In this case A in equation (1) is the semi-major axis CA, and e is the eccentricity, or $e = \frac{CF}{CA}$.

Central Acceleration Inversely as the Square of the Distance—Equation of the Path.—We have for the path in general a conic section, as proved on page 115, and for the general polar equation of a conic section, as just explained,

$$r = \frac{A(1 - e^2)}{1 + e\cos\theta}. \qquad (1)$$



Let the notation be as in the figure. Thus r_1, v_1 and θ_1 are the given radius vector, velocity and angle for the initial position P_1, v_1 making the given angle ϵ_1 with r_1. For any position P we have r, v and θ. For the apex A we have $\theta = 0$, r_x and v_y. For $\theta = 180$ we have r'_x and v'_y. Let f_0 be the known central acceleration at a known distance r_0.

For acceleration *towards the centre* we have, as already proved, page 112,

$$v^2 = v_1^2 - 2f_0r_0^2\left(\frac{1}{r_1} - \frac{1}{r}\right),$$

or, solving for r,

$$r = \frac{2f_0r_0^2}{v^2 - v_1^2 + \dfrac{2f_0r_0^2}{r_1}}. \qquad (2)$$

If in (1) we make θ θ_1, v becomes v_1, r becomes r_1, and we have

$$A(1 - e^2) = r_1(1 + e\cos\theta_1). \qquad (3)$$

Substituting in (1) we have

$$r = \frac{r_1(1 + e\cos\theta_1)}{1 + e\cos\theta}. \qquad (4)$$

If we make $\theta = 0$, r becomes r_x, v becomes v_y, and we have, from (2) and (4),

$$r_x = \frac{2f_0r_0^2}{v_y^2 - v_1^2 + \dfrac{2f_0r_0^2}{r_1}} = \frac{r_1(1 + e\cos\theta_1)}{1 + e}. \qquad (5)$$

If we make $\theta = 180°$, r becomes r'_x, v becomes v'_y, and we have, from (2) and (4),

$$r'_x = \frac{2f_0r_0^2}{v_y'^2 - v_1^2 + \dfrac{2f_0r_0^2}{r_1}} = \frac{r_1(1 + e\cos\theta_1)}{1 - e}. \qquad (6)$$

We have also, by the principle of equal areas (page 111),

$$v'_yr'_x = v_yr_x = v_1r_1\sin\epsilon_1. \qquad (7)$$

From (7) and (5) we obtain

$$v_y = \frac{v_1\sin\epsilon_1(1 + e)}{1 + e\cos\theta_1}. \qquad (8)$$

From (7) and (6) we obtain

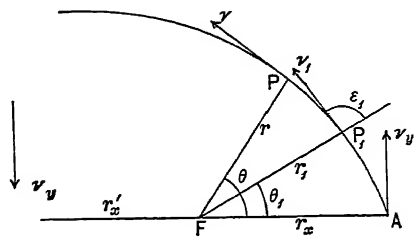
$$v'_y = \frac{v_1\sin\epsilon_1(1 - e)}{1 + e\cos\theta_1}. \qquad (9)$$

Substituting (8) in (5), and (9) in (6), we have, after reduction,

$$2f_0r_0^2(1 + e)(1 + e\cos\theta_1) = r_1v_1^2\sin^2\epsilon_1(1 + e)^2 + (1 + e\cos\theta_1)^2(2f_0r_0^2 - r_1v_1^2),$$

$$2f_0r_0^2(1 - e)(1 + e\cos\theta_1) = r_1v_1^2\sin^2\epsilon_1(1 - e)^2 + (1 + e\cos\theta_1)^2(2f_0r_0^2 - r_1v_1^2).$$

Let the notation be as in the figure. Thus r_1, v_1 and θ_1 are the given radius vector, velocity and angle for the initial position P_1, v_1 making the given angle ϵ_1 with r_1. For any position P we have r, v and θ. For the apex A we have $\theta = 0$, r_x and v_y. For $\theta = 180$ we have r'_x and v'_y. Let f_0 be the known central acceleration at a known distance r_0.



For acceleration *towards the centre* we have, as already proved, page 112,

$$v^2 = v_1^2 - 2f_0r_0^2\left(\frac{1}{r_1} - \frac{1}{r}\right),$$

or, solving for r,

$$r = \frac{2f_0r_0^2}{v^2 - v_1^2 + \frac{2f_0r_0^2}{r_1}}. \qquad (2)$$

If in (1) we make $\theta = \theta_1$, v becomes v_1, r becomes r_1, and we have

$$A(1 - e^2) = r_1(1 + e\cos\theta_1). \qquad (3)$$

Substituting in (1) we have

$$r = \frac{r_1(1 + e\cos\theta_1)}{1 + e\cos\theta}. \qquad (4)$$

If we make $\theta = 0$, r becomes r_x, v becomes v_y, and we have, from (2) and (4),

$$r_x = \frac{2f_0r_0^2}{v_y^2 - v_1^2 + \frac{2f_0r_0^2}{r_1}} = \frac{r_1(1 + e\cos\theta_1)}{1 + e}. \qquad (5)$$

If we make $\theta = 180°$, r becomes r'_x, v becomes v'_y, and we have, from (2) and (4),

$$r'_x = \frac{2f_0r_0^2}{v'^2_y - v_1^2 + \frac{2f_0r_0^2}{r_1}} = \frac{r_1(1 + e\cos\theta_1)}{1 - e}. \qquad (6)$$

We have also, by the principle of equal areas (page 111),

$$v'_y r'_x = v_y r_x = v_1 r_1 \sin\epsilon_1. \qquad (7)$$

From (7) and (5) we obtain

$$v_y = \frac{v_1\sin\epsilon_1(1 + e)}{1 + e\cos\theta_1}. \qquad (8)$$

From (7) and (6) we obtain

$$v'_y = \frac{v_1\sin\epsilon_1(1 - e)}{1 + e\cos\theta_1}. \qquad (9)$$

Substituting (8) in (5), and (9) in (6), we have, after reduction,

$$2f_0r_0^2(1 + e)(1 + e\cos\theta_1) = r_1v_1^2\sin^2\epsilon_1(1 + e)^2 + (1 + e\cos\theta_1)^2(2f_0r_0^2 - r_1v_1^2),$$

$$2f_0r_0^2(1 - e)(1 + e\cos\theta_1) = r_1v_1^2\sin^2\epsilon_1(1 - e)^2 + (1 + e\cos\theta_1)^2(2f_0r_0^2 - r_1v_1^2).$$

From these two equations we obtain

$$1 + e\cos\theta_1 = \frac{r_1 v_1^2 \sin^2 \epsilon_1}{f_0 r_0^2}, \quad . \quad . \quad . \quad . \quad (10)$$

$$e = \sqrt{1 + \frac{r_1^2 v_1^2 \sin^2 \epsilon_1 \left(v_1^2 - \frac{2f_0 r_0^2}{r_1}\right)}{f_0^2 r_0^4}}, \quad . \quad . \quad . \quad . \quad (11)$$

Substituting (10) and (11 in (4) we have for the general equation of the path, in terms of known quantities,

$$r = \frac{\frac{r_1^2 v_1^2 \sin^2 \epsilon_1}{f_0 r_0^2}}{1 + \cos\theta \sqrt{1 + \frac{r_1^2 v_1^2 \sin^2 \epsilon_1 \left(v_1^2 - \frac{2f_0 r_0^2}{r_1}\right)}{f_0^2 r_0^4}}}, \quad . \quad . \quad . \quad . \quad (12)$$

where f_0 is the known central acceleration at a known distance r_0, v_1 is the given initial velocity at a given distance r_1, making the angle ϵ_1 with r_1, and r is the radius vector for any angle θ. The quantity under the radical is the value of e.

The path will be a parabola when $e = 1$, or when

$$v_1^2 = \frac{2f_0 r_0^2}{r_1}, \quad \text{or} \quad v_1 = \sqrt{\frac{2f_0 r_0^2}{r_1}}.$$

The path will be an ellipse when $e < 1$, or when

$$v_1^2 < \frac{2f_0 r_0^2}{r_1}, \quad \text{or} \quad v_1 < \sqrt{\frac{2f_0 r_0^2}{r_1}}.$$

The path will be an hyberbola when $e > 1$, or when

$$v_1^2 > \frac{2f_0 r_0^2}{r_1}, \quad \text{or} \quad v_1 > \sqrt{\frac{2f_0 r_0^2}{r_1}}.$$

We see, then, that *the form of the path depends solely upon the magnitude of the initial velocity and not upon its direction, that is simply upon the initial speed.*

From (2), page 117, making $v = 0$ and r infinity, we have

$$v_1 = \sqrt{\frac{2f_0 r_0^2}{r_1}}$$

for the velocity of projection which would carry the point from r_1 to rest at an infinite distance, or which the point would acquire in moving from rest at an infinite distance to r_1.

The path is an ellipse, parabola or hyperbola according as v_1 is less than, equal to or greater than this.

For acceleration away from the centre we should change the sign of f_0 in equation (2), as pointed out page 112. The value of e, equation (11), in such case is always greater than unity. Hence for acceleration away from the centre the path is always an hyperbola.

Path a Parabola.—If the path is a parabola, $e = 1$, $v_1^2 = \frac{2f_0r_0^2}{r_1}$ and equation (12) becomes

$$r = \frac{2r_1 \sin^2 \epsilon_1}{1 + \cos \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We have also, from (10), for the angle θ_1 of the initial radius vector r_1 with the axis

$$\cos \theta_1 = 2 \sin^2 \epsilon_1 - 1. \quad . \quad . \quad . \quad . \quad (14)$$

The path is then completely determined. The equation referred to the axis and vertex as origin is

$$y^2 = 4r_1 \sin^2 \epsilon_1 . x. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Fig. (a).

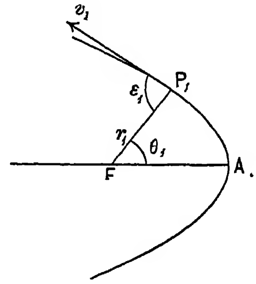
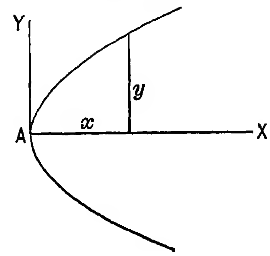


Fig. (b).



Path an Ellipse or Hyperbola.—If the path is an ellipse or hyperbola, the eccentricity in either case is given by (11), viz.,

$$e = \sqrt{1 + \frac{r_1v_1^2 \sin^2 \epsilon_1(r_1v_1^2 - 2f_0r_0^2)}{f_0^2r_0^4}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From (16), (3) and (10) we have for the semi-major axis of the ellipse or the semi-transverse axis of the hyperbola

$$A = \pm \frac{f_0r_0^2r_1}{2f_0r_0^2 - r_1v_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where the (+) sign is taken for the ellipse and the (−) sign for the hyperbola. We see that this is independent of the angle ϵ_1 or the direction of the initial velocity v_1, and is the same for v_1 and r_1 constant in magnitude no matter what the direction.

The semi-conjugate axis is given by $B = A\sqrt{1 - e^2}$ for the ellipse or $B = A\sqrt{e^2 - 1}$ for the hyperbola. From (16) and (17) we have, then,

$$B = \frac{r_1v_1 \sin \epsilon_1}{\sqrt{\pm(2f_0r_0^2 - r_1v_1^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where the (+) sign is taken for the ellipse and the (−) sign for the hyperbola.

We have, from (10), for the angle θ_1 of the initial radius vector r_1 with the axis

$$\cos \theta_1 = \frac{r_1v_1^2 \sin^2 \epsilon_1 - f_0r_0^2}{\sqrt{f_0^2r_0^4 + r_1v_1^2 \sin^2 \epsilon_1(r_1v_1^2 - 2f_0r_0^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The elliptic or hyperbolic path is thus determined.

The equation of the path referred to the centre and axes is

$$A^2y^2 \pm B^2x^2 = A^2B^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the (+) sign is taken for the ellipse and the (−) sign for the hyperbola.

If the path is a circle, we have $e = 0$, $\epsilon_1 = 90^\circ$, $f_0 = f$, $r_1 = r$, $v_1 r_1 = vr$, and hence, from (16),

$$f = \frac{v_1^2}{r_1} = \frac{v^2}{r},$$

as should be (p. 79).

Planetary Motion—Kepler's Laws.—By long and laborious comparison of the observations which TYCHO BRAHE had made, through many years, of the planets, especially of *Mars*, KEPLER discovered the three laws of planetary motion which are known as KEPLER'S LAWS. He gave these laws simply as the expression of facts which seemed established by the observations.

The three laws are as follows:

I. *The planets describe ellipses, the sun occupying one of the foci.*

II. *The radius vector of each planet describes equal areas in equal times.* This is known as the law of equal areas.

III. *The squares of the periodic times of the planets are proportional to the cubes of their mean distances from the sun.*

The second law, as we have seen (page 111), is a necessary consequence of *all* central acceleration.

The first law, as we have just seen, follows if the central acceleration is inversely as the square of the radius vector.

The third law is also a direct consequence of such central acceleration, as we shall see in the next article.

Verification by Application to the Moon.—If ω_1 and ω_2 are the mean angular velocities of two planets and T_1, T_2 their periodic times, then

$$\omega_1 = \frac{2\pi}{T_1}, \qquad \omega_2 = \frac{2\pi}{T_2}.$$

We have also for the mean central accelerations, if r_1, r_2 are the mean radius vectors,

$$f_1 = r_1\omega_1^2 = \frac{4\pi^2 r_1}{T_1^2}, \qquad f_2 = r_2\omega_2^2 = \frac{4\pi^2 r_2}{T_2^2}.$$

Hence

$$\frac{f_1}{f_2} = \frac{T_2^2 r_1}{T_1^2 r_2}.$$

But if the accelerations are inversely as the squares of the radius vectors, we have also

$$\frac{f_1}{f_2} = \frac{r_2^2}{r_1^2}.$$

Hence we obtain

$$\frac{T_2^2}{T_1^2} = \frac{r_2^3}{r_1^3},$$

This is Kepler's third law, and, as we see, it is a direct consequence of central acceleration, varying inversely as the square of the radius vector.

Assuming Kepler's third law, Newton was led directly to this conclusion. He tested it by application to the moon, as follows:

The moon moves in a sensibly circular orbit, the centre being at the centre of the earth.

Let the mean radius of the earth be r_0, the acceleration at the surface be g, the distance from centre of earth to centre of moon be r, and acceleration of moon be f. Then, by the hypothesis of inverse squares, we have

$$f : g :: r_0^2 : r^2, \quad \text{or} \quad f = \frac{r_0^2}{r^2} g.$$

Inserting $r_0 = 4000$ miles, $r = 240000$ miles, and $g = 32.2$ ft.-per-sec. per sec., we have by hypothesis for the moon's central acceleration

$$f = \frac{(4000)^2}{(240000)^2} \times 32.2 = 0.0089 \text{ ft.-per-sec. per sec.}$$

But the speed of the moon in its orbit is $v = 3375$ ft. per sec.

Its radial acceleration is then in reality (page 79)

$$f = \frac{v^2}{r} = \frac{(3375)^2}{240000 \times 5280} = 0.0089 \text{ ft.-per-sec. per sec.}$$

Newton's hypothesis of inverse squares is thus verified in the case of the moon.

As we have seen (page 116), the path in general for central acceleration varying inversely as the square of the distance is either an ellipse, a parabola or an hyperbola Instances of all these paths are found in the solar system. Thus the planets and satellites move in elliptic orbits, while comets have paths elliptic, parabolic or hyperbolic.

Velocity of a Planet at any Point of its Orbit.—We have from equation (2), page 112,

$$v^2 = v_1^2 - 2f_0 r_0^2 \left(\frac{1}{r_1} - \frac{1}{r}\right),$$

which gives the velocity for any distance r when v_1 and r_1 are given.

From equation (1), page 116, and equation (12), page 118, we have

$$A(1 - e^2) = \frac{r_1^2 v_1^2 \sin^2 \epsilon_1}{f_0 r_0^2}, \quad \text{or} \quad f_0 r_0^2 = \frac{r_1^2 v_1^2 \sin^2 \epsilon_1}{A(1 - e^2)}. \quad . \quad . \quad . \quad . \quad (21)$$

From page 111, $vl = rv \sin \epsilon$, or $v^2 = \dfrac{r_1^2 v_1^2 \sin^2 \epsilon_1}{l^2}$, where l is the lever-arm of v.

From Analytical Geometry, for an ellipse $l^2 = \dfrac{A^2(1 - e^2)r}{2A - r}$. Hence we have

$$v^2 = \frac{r_1^2 v_1^2 \sin^2 \epsilon_1 (2A - r)}{A^2(1 - e^2)r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or, from (21),

$$v^2 = \frac{f_0 r_0^2}{Ar}(2A - r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Equation (23) gives the velocity for any distance r if the semi-major axis A is known.

COR. 1.—We see that the velocity is greatest where r is least, or at perihelion, and least where r is greatest, or at aphelion.

COR. 2.—If a point moves in a circle of radius r with velocity v', its radial acceleration is $\dfrac{v'^2}{r}$ (page 79).

If this acceleration is equal to the acceleration of the planet, we have

$$\frac{v'^2}{r} = \frac{f_0 r_0^2}{r^2},$$

or, from (21),

$$\frac{v'^2}{r} = \frac{r_1^2 v_1^2 \sin^2 \epsilon_1}{A(1 - e^2) r^2}.$$

Therefore, from (22),

$$v^2 : v'^2 :: 2A - r : A.$$

That is, *the square of the speed in the ellipse is to the square of the speed in the circle as the distance of the planet from the unoccupied focus is to the semi-major axis.*

COR. 3.—If r_1 is the perihelion distance and r_2 the aphelion distance, we have, from (23),

$$\text{for} \quad r = r_1, \quad v^2 = \frac{f_0 r_0^2 r_2}{A r_1};$$

$$\text{for} \quad r = r_2, \quad v^2 = \frac{f_0 r_0^2 r_1}{A r_2};$$

while for $r = A$ we have

$$v^2 = \frac{f_0 r_0^2}{A}.$$

That is, *the speed at the extremity of the minor axis is a mean proportional between the speeds at perihelion and aphelion.*

Periodic Time.—The moment $v_1 r_1 \sin \epsilon_1$ is equal to twice the areal velocity of the radius vector (page 89), and this areal velocity, as we have seen (page 111), is constant. Twice the area of an ellipse is $2\pi A^2 \sqrt{1 - e^2}$. Hence the periodic time is

$$T = \frac{2\pi A^2 \sqrt{1 - e^2}}{v_1 r_1 \sin \epsilon_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

or, substituting the values for A and (e), equations (17) and (16),

$$T = \frac{2\pi f_0 r_0^2 r_1 \sqrt{r_1}}{(2f_0 r_0^2 - r_1 v_1^2)\sqrt{2f_0 r_0^2 - r_1 v_1^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From (25) we see that the periodic time is independent of the angle ϵ_1 or the direction of the initial velocity v_1.

We can write (24) in the form

$$\frac{4\pi^2 A^3}{T^2} = \frac{v_1^2 r_1^2 \sin^2 \epsilon_1}{A(1 - e^2)},$$

or, from (21),

$$\frac{4\pi^2 A^3}{T^2} = f_0 r_0^2 = \frac{v_1^2 r_1^2 \sin^2 \epsilon_1}{A(1 - e^2)}.$$

But, by Kepler's third law, we have for two different planets of periodic times T and T_1

$$\frac{T^2}{T_1^2} = \frac{A^3}{A_1^3}, \quad \text{or} \quad \frac{A^3}{T^2} = \frac{A_1^3}{T_1^3}.$$

Hence

$$f_0 r_0^2 = \frac{v_1^2 r_1^2 \sin^2 \epsilon_1}{A(1 - e^2)}$$

is a constant quantity for all the planets.

Now f_0 is the acceleration at a distance r_0, and, since the acceleration f at any distance r is

$$f = \frac{f_0 r_0^2}{r},$$

we see that $f_0 r_0^2$ is the magnitude of the acceleration at the distance unity, or $r = 1$.

Hence it follows, from Kepler's third law, *that for all the planets the acceleration would be the same at the same distance from the sun.**

Value of f_0 for Planetary Motion.—In all our equations for central acceleration we see that it is necessary to know the acceleration f_0 at some known distance r_0.

It will be proved hereafter (page 206) that if M is the mass of the sun and m the mass of a planet, the value of f_0 at a distance r_0 *equal to the mean radius of the earth* is given by

$$f_0 = \frac{M + m}{m_0} g, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where m_0 is the mass of the earth and g the mean acceleration of a body at the earth's surface.

If the two bodies are the earth and a small body of mass m', then $f_0 = \frac{m_0 + m'}{m_0} g$, or, since m' is insignificant with respect to m_0, $f_0 = g$. If, in the preceding article, we had used the value of f_0 given by (1), we should have obtained

$$\frac{M + m}{m_0} g r_0^2 = \frac{4\pi^2 A^3}{T^2} \quad \text{and} \quad \frac{M + m_1}{m_0} g r_0^2 = \frac{4\pi^2 A_1^3}{T_1^2}.$$

Hence

$$\frac{T^2}{T_1^2} = \frac{M + m_1}{M + m} \cdot \frac{A^3}{A_1^3}.$$

We see, then, that Kepler's third law is not strictly exact. The value of $f_0 r_0^2$, or the acceleration at units distance, is not strictly constant. The more accurate expression is that the squares of the periodic times are directly as the cubes of the semi-major axes and *inversely as the sum of the masses of sun and planet.*

The error from this source is insignificant, the mass of Jupiter, the largest of the planets, being less than a thousandth part of that of the sun.

* "Of all the laws," says Sir John Herschel, "to which induction from pure observation has ever conducted man, this third law of Kepler may justly be regarded as the most remarkable, and the most pregnant with important consequences. When we contemplate the constituents of the planetary system from the point of view which this relation affords us, it is no longer mere analogy which strikes us, no longer a general resemblance among them as individuals independent of each other, and circulating about the sun each according to its own peculiar nature, and connected with it by its own peculiar tie. The resemblance is now perceived to be a true *family* likeness; they are bound up in one chain; interwoven in one web of mutual relation and harmonious agreement; subjected to one pervading influence, which extends from the centre to the farthest limit of that great system, of which all of them, the earth included, must henceforth be regarded as members." —*Outlines of Astronomy.*

The motion of translation of the planets is not affected by their rotation on their axes, and we may treat them as material points at their centres of mass, so far as translation is concerned.

The centre of mass of the sun is not strictly a fixed point, but both sun and planet move in orbits about a common centre of mass. The sun is also affected by the other planets, and the planets by each other.

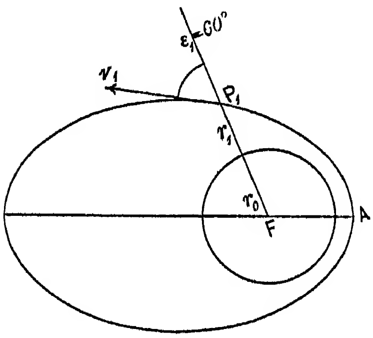
The attraction of the planets for each other sensibly modifies their orbits.

Kepler's laws are thus approximate. If we had but two bodies, one fixed and the other free to move, then Kepler's first two laws would be accurate, and the third would approach aecuracy as the mass of the moving body becomes insignificant with respect to the mass of the other.

Examples.—(1) *Find the speed and periodic time of a body moving in a circle at a distance from the earth's centre of n times the earth's radius, the central acceleration being inversely as the square of the distance.*

ANS. $v = \sqrt{\frac{r_0 g}{n}}$. $T = 2\pi\sqrt{\frac{n^3 r_0}{g}}$.

(2) *A body at a distance of r_1 from the centre of the earth is projected in a direction which makes an angle of $\epsilon_1 = 60°$ with r_1 with a speed v_1, which is to the speed acquired by falling from an infinite distance as 1 to. $\sqrt{3}$. Find the path, the major axis, eccentricity and periodic time.*



ANS. We have $\epsilon_1 = 60°$, $\sin\epsilon_1 = \sqrt{\frac{3}{4}}$, $v_1 = \frac{1}{\sqrt{3}}\cdot\sqrt{\frac{2g r_0^2}{r_1}}$, r_0 g.

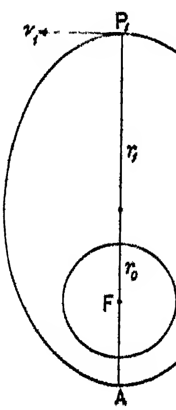
The path is an ellipse. From (17), $2A = \frac{3}{2}r_1$. From (16), $e = \sqrt{\frac{1}{3}}$.

From (25) $T = \frac{3\pi r_1}{4r_0}\sqrt{\frac{3r_1}{g}}$, where r_0 is the radius of the earth. From (12) we have, by making $\theta = 0$, the perihelion distance FA equal to $\frac{3}{4}r_1\left(1 - \sqrt{\frac{1}{3}}\right)$. From (19) we have for the angle $AFP_1 = \theta_1$

$$\cos\theta_1 = -\frac{1}{2}\sqrt{3}.$$

The centre of the earth is at the focus F.

(3) *In the preceding example let $\epsilon_1 = 90°$, so that the body is projected in a direction at right angles to r.*



ANS. From (17), $2A = \frac{3}{2}r_1$, as before.

From (16), $e = \frac{1}{3}$.

From (25), $T = \frac{3\pi r_1}{4r_0}\sqrt{\frac{3r_1}{g}}$, as before.

From (12) we have, by making $\theta = 0$, the perihelion distance FA equal to $\frac{1}{2}r_1$.

From (19) we have for the angle AFP_1

$$\cos\theta_1 = -1, \quad \text{or} \quad \theta_1 = 180°.$$

CHAPTER IV.

CENTRAL ACCELERATION DIRECTLY AS THE DISTANCE. HARMONIC MOTION.

Harmonic Motion.—The motion of a point moving in any path, the acceleration being always directed towards a fixed point and varying *directly* as the distance from that point, is called HARMONIC motion.

If the path is a straight line, the motion is SIMPLE HARMONIC; if the path is a curve, the motion is COMPOUND HARMONIC.

The vibrations of such bodies as a tuning-fork or a piano-wire are approximate examples of such motion, hence the term "*harmonic.*" The vibrations of an elastic body, such as a spring or the air, are also examples of such motion.

It is also (page 207) the motion of a body under the action of gravitation, within a homogeneous sphere.

The motion of the piston of a steam-engine when moved by a crank and connecting-rod approximates the same motion if the rotation of the crank is uniform, the approximation being closer the longer the connecting-rod.

Harmonic Motion—Velocity.—Let f_0 be the known acceleration at a given distance r_0, and f the acceleration at any distance r. Then

$$f : f_0 :: r : r_0, \quad \text{or} \quad f = \frac{r}{r_0} f_0.$$

This gives the magnitude of the central acceleration.

If f is towards the centre of acceleration, it is negative and we have

$$f = -\frac{r}{r_0} f_0.$$

From (I), page 110, we have

$$\Sigma f \,.\, dp = \frac{v^2 - v_1^2}{2}, \quad \text{where} \quad f = -\frac{r}{r_0} f_0.$$

Let P_1 and P_2 be two consecutive points at distances r_1 and r_2 from the centre of acceleration. Then if the points are consecutive, $dp = r_2 - r_1$, and $r = \frac{r_1 + r_2}{2}$. Hence

$$f \,.\, dp = -\frac{r}{r_0} f_0 (r_2 - r_1) = \frac{f_0}{2r_0}(r_1^2 - r_2^2).$$

For the next two consecutive points, P_2 and P_3, we have in the same way

$$f \,.\, dp = \frac{f_0}{2r_0}(r_2^2 - r_3^2).$$

For the next two consecutive points, P_3 and P_4,

$$f \cdot dp = \frac{f_0}{2r^0}(r_3^2 - r_4^2),$$

and so on.

Summing up, if r_1 is the initial and r the final distance, we have

$$\Sigma f \cdot dp = \frac{f_0}{2r_0}(r_1^2 - r^2) = \frac{v^2 - v_1}{2}$$

Hence

$$v^2 = v_1^2 + \frac{f_0}{r_0}(r_1^2 - r^2). \quad \ldots \ldots \ldots \ldots \quad (1)$$

If the distance is increasing, r is greater than r_1; if decreasing, r is less than r_1. Equation (1) holds in both cases.

COR. It is evident that the same equation holds for motion in any path if we take the centre of acceleration at some fixed point of the path and measure all distances *along the path, if the tangential acceleration f_t is directly as the distance.*

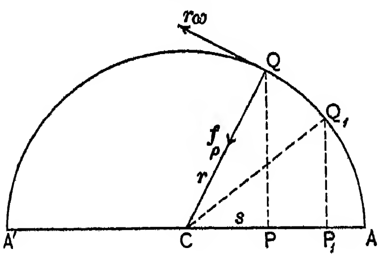
We have, in such case,

$$v^2 = v_1^2 + \frac{f_0}{s_0}(s_1^2 - s^2), \quad \ldots \ldots \ldots \ldots \quad (2)$$

where f_0 is the tangential acceleration at a distance s_0, s_1 is the initial and s the final distance from the fixed point of the path, all distances measured *along the path.*

Simple Harmonic Motion.—If the central acceleration varies directly as the distance from the centre of acceleration and coincides with the direction of the velocity, the motion is rectilinear and we have simple harmonic motion.

In this case equation (2) still holds.



Let a point Q move with uniform speed $r\omega$ in a circle of radius $CQ = r$ with uniform angular speed ω. Then the radial acceleration f_ρ is towards the centre C and equal to

$$f_\rho = r\omega^2.$$

The projection of f_ρ upon a diameter CA is $r\omega^2 \cdot \cos QCA$. But $r \cos QCA$ is the distance $CP = s$, if P is the projection of Q upon the diameter. The acceleration of P along the diameter towards C is then $s\omega^2$, or directly proportional to the distance s. The motion of P is then simple harmonic.

Hence, *if a point Q move in a circle with uniform speed, its projection P upon a diameter moves with simple harmonic motion along the diameter.*

If v_1 is the initial velocity at the initial position P_1, so that $CP_1 = s_1$, equation (2) becomes

$$v^2 = v_1^2 + \frac{f_0}{s_0}(s_1^2 - s^2). \quad \ldots \ldots \ldots \ldots \quad (3)$$

The point P starts from rest at A at the distance $CA = r$. If then we make $v = 0$ in (3) and $s = r$, we have

$$v_1^2 = \frac{f_0}{s_0}(r^2 - s_1^2),$$

and substituting this in (3),

$$v^2 = \frac{f_0}{s_0}(r^2 - s^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We see from (4) that the velocity increases as the distance s decreases till P arrives at C, where the velocity is a maximum and equal to

$$r\omega = r\sqrt{\frac{f_0}{s_0}}; \qquad \text{hence} \qquad \omega = \sqrt{\frac{f_0}{s_0}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then the velocity decreases and finally becomes zero when P arrives at A' at the distance $-r$ on the other side of C.

From A to A' is called a VIBRATION, from A to A' and back to A is called an OSCILLATION. The radius r is called the RANGE or AMPLITUDE of an oscillation.

PERIODIC TIME.—The time from A back to A, or the time of an oscillation, is called the PERIODIC TIME. Since the uniform speed is $r\omega = r\sqrt{\frac{f_0}{s_0}}$, the periodic time is

$$T = \frac{2\pi r}{r\sqrt{\frac{f_0}{s_0}}} = 2\pi\sqrt{\frac{s_0}{f_0}} = \frac{2\pi}{\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If f is the acceleration at any distance s, we have for harmonic motion

$$\frac{s_0}{f_0} = \frac{s}{f} = \frac{1}{\omega^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{s}{f}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The periodic time depends, then, only upon the constant ratio $\frac{s}{f} = \frac{1}{\omega^2}$, and is independent of the range r or amplitude of oscillation. For this reason the oscillations are said to be ISOCHRONOUS, or made in equal times, no matter what the range or amplitude.

COR.—Since the motion of a body under the action of gravity in a homogeneous sphere is harmonic (page 207), if we put g for f_0, and r_0, the radius of the earth, for s_0, we have from (3), for a body falling under the action of gravity in a well or shaft assuming the earth to be a homogeneous sphere and neglecting the resistance of the air,

$$v^2 = v_1^2 + \frac{g}{r_0}(s_1^2 - s^2).$$

This can be written

$$v^2 = v_1^2 + \frac{g}{r_0}(s_1 + s)(s_1 - s).$$

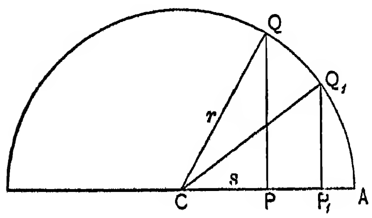
If the fall takes place near the surface, for a short distance compared to r_0, we have $s_1 + s$ practically equal to $2r_0$, and hence

$$v^2 = v_1^2 + 2g(s_1 - s),$$

which is the same as for uniform acceleration g (page 101).

We obtained the same result (page 101) for a body external to the earth. The equations of page 101 hold good, then, in all practical cases, whether the fall takes place above the earth or within the earth, for small fall near the surface, neglecting air resistance.

Epoch—Phase.—If P_1 is the initial position or the position of P at zero of time, the time of passing from A to P_1 is called the EPOCH. The epoch may also be defined with reference to the auxiliary circle as the angle ACQ_1 in radians. This is the epoch in angular measure. The epoch in angular measure is, then, the angle described on the auxiliary circle in the interval of time defined as the epoch.



The epoch locates the initial position of P.

The fraction of the periodic time in passing from A to any position P is called the PHASE. Measured on the circle is the ratio of the angle ACQ radians to 2π radians.

The phase locates the position of P at any instant.

It therefore varies with the time or with the position of P. The phase at the initial position P_1 multiplied by 2π gives, then, the epoch in angular measure, and multiplied by the periodic time gives the epoch in time.

Examples.—(1) *A point whose motion is simple harmonic has velocities 20 and 25 m. per sec. at distances 10 and 8 m. from the centre of acceleration. Find the period and acceleration at units distance.*

ANS. We have $400 = \frac{f_0}{s_0}(r^2 - 100)$ and $625 = \frac{f_0}{s_0}(r^2 - 64)$. Therefore

$$\sqrt{\frac{f_0}{s_0}} = \frac{15}{6} \text{ and period } = \frac{2\pi}{\sqrt{\frac{f_0}{s_0}}} = \frac{4\pi}{5} \text{ sec.}$$

We have also $\frac{f}{s} = \frac{f_0}{s_0}$, or $f = \frac{f_0 s}{s_0}$. Making $s = 1, f = \pm\frac{225}{36} = \pm 6.25$ ft.-per-sec. per sec.

(2) *The period of a simple harmonic motion is 20 sec. and the maximum velocity is 10 ft. per sec. Find the velocity at a distance of $\frac{60}{\pi}$ ft. from the centre.*

ANS. We have $\frac{2\pi}{\sqrt{\frac{f_0}{s_0}}} = 20$ sec. Therefore $\frac{f_0}{s_0} = \frac{\pi^2}{100}$. When $s = 0$, $v = 10$ and $100 = \frac{\pi^2}{100}r^2$, or $r = \frac{100}{\pi}$ ft.

Hence

$$v^2 = \frac{\pi^2}{100}\left(\frac{100^2}{\pi^2} - \frac{60^2}{\pi^2}\right), \quad \text{or} \quad v = 8 \text{ ft. per sec.}$$

(3) *Find the mean speed of a point in simple harmonic motion during the time of moving from one to the other extremity of its range, its maximum speed being 5 ft. per sec.*

ANS. The distance is $2r$. The time $\dfrac{\pi}{\sqrt{\dfrac{f_0}{s_0}}}$. The mean speed $\dfrac{2r\sqrt{f_0}}{\pi\sqrt{s_0}}$. When $s = 0$, we have

$$25 = \frac{f_0}{s_0}r^2, \quad \text{or} \quad r\sqrt{\frac{f_0}{s_0}} = 5.$$

Therefore the mean speed is $\dfrac{10}{\pi}$ ft. per sec.

(4) *If T is the period and r the amplitude of a simple harmonic motion, v the velocity and s the distance from the centre at any instant, show that*

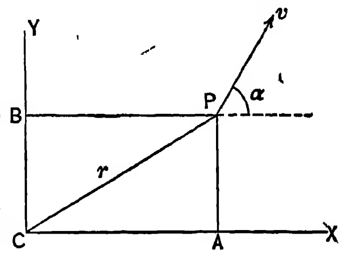
$$r = \left(\frac{T^2v^2}{4\pi^2} + s^2\right)^{\frac{1}{2}}.$$

(5) *A point has simple harmonic motion whose period is 4 min. 12 sec. Find the time during which its phase changes from $\frac{1}{12}$ to $\frac{1}{8}$ of a period.*

ANS. 21 sec.

Compound Harmonic Motion.—If the central acceleration varies directly as the distance from the centre of acceleration and does not coincide with the direction of the velocity, we have motion in a curve and the motion is COMPOUND HARMONIC.

Any Compound Harmonic Motion may be Resolved into Two Simple Harmonic Motions at Right Angles.—Let C be the centre of acceleration, and P the position of the moving point at any instant. Let the velocity v of P make an angle α with the axis of X, and let the motion of P be harmonic so that the acceleration of P is $\dfrac{f_0}{r_0}r$, where f_0 is the acceleration at a known distance r_0, and r is the distance CP.



The velocity v may be resolved into $v \cos \alpha$ and $v \sin \alpha$ in the directions CX and CY, and the acceleration may be resolved into $\dfrac{f_0}{r_0} r \cos PCA$ or $\dfrac{f_0}{r_0}\overline{CA}$, and $\dfrac{f_0}{r_0} r \cos PCB$ or $\dfrac{f_0}{r_0}\overline{CB}$, in the same directions.

The component accelerations are therefore directly as the distances CA and CB, and the component velocities are in the directions of CA and CB. The compound harmonic motion of P, whatever the direction of the velocity v, is therefore the resultant of two simple harmonic motions in the lines CA and CB at right angles.

If, then, any compound harmonic motion is resolved into two components at right angles, the component motions are rectilinear harmonic.

Conversely, *the resultant of two rectilinear harmonic motions at right angles is a compound harmonic motion.*

Composition of Simple Rectilinear Harmonic Motions in Different Lines.—Let the point Q move in a circle AQA' of radius $r = CA = CQ$ with a constant angular velocity ω. Then the motion of the projection P in the line AA' is simple harmonic (page 126).

Let the point Q_1 move in the circle CBQ_1 of radius $r_1 = CB = CQ_1$, with constant angular velocity ω_1. Then the motion of the projection P_1 in the line CB is simple harmonic. Let the angle BCA between the planes of the circles be α.

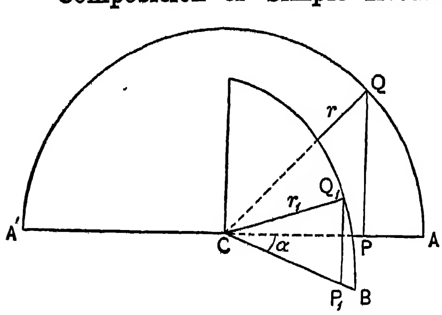


FIG. 1.

Let the time count from the instant when Q_1 is at B, so that the *epoch* of P_1 is zero (page 128). At this instant let the epoch of P be ϵ. Then ϵ is the difference of epoch, or, in angular measure, the angle of Q above or below A at the beginning of the time. In any time t, Q_1 will have moved from B through the angle $\omega_1 t$ measured from CB, and Q through the angle $\omega t \pm \epsilon$ measured from CA.

By the preceding article we can resolve the harmonic motion of P_1 into a simple rectilinear harmonic motion at right angles to CA, and another along CA.

The displacement of P_1 from C for any time t is $r_1 \cos(\omega_1 t)$, and this displacement may be resolved into $r_1 \cos\alpha \cos(\omega_1 t)$ along CA, and $r_1 \sin\alpha \cos(\omega_1 t)$ perpendicular to CA. The displacement of P from C in the same time t is $r \cos(\omega t \pm \epsilon)$.

If a point undergoes these displacements simultaneously, its resultant displacement along CA will be

$$x = r \cos(\omega t \pm \epsilon) + r_1 \cos\alpha \cos(\omega_1 t), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and perpendicular to CA

$$y = r_1 \sin\alpha \cos(\omega_1 t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of the curve in which the point moves, referred to rectangular co-ordinates with C for the origin, will then be obtained by combining (1) and (2) so as to eliminate t. Such combination (page 129) gives always compound harmonic motion about C, the radius vector from C passing over equal areas in equal times (page 111).

Equations (1) and (2) enable us, then, to find the curve resulting from the combination of any two simple rectilinear harmonic motions inclined at any angle α.

If the component motions are at right angles, $\alpha = 90°$. If the amplitudes are equal, $r = r_1$. If the periods are equal, $\omega = \omega_1$, the difference of epoch is constant, and, since the epoch equals the product of the phase at zero of time by 2π radians (page 128), when the periods are equal the difference of phase is constant. When, then, the periods are equal and $\epsilon = 0$, or the epochs are equal, the phases are also equal at any instant.

Two Component Simple Harmonic Motions in Different Lines with the Same Period.—In this case $\omega = \omega_1$ and ϵ is constant, or the difference of epochs is constant and difference of phase at any instant is constant.

We have then, from (1) and (2),

$$x = r \cos(\omega t + \epsilon) + r_1 \cos\alpha \cos(\omega t), \qquad y = r \sin\alpha (\cos \omega t).$$

Combining these two equations by eliminating ωt, we have

$$(r_1^2 \sin^2\alpha)x^2 - 2r_1 \sin\alpha(r\cos\epsilon + r_1\cos\alpha)xy + (r^2 + 2rr_1\cos\epsilon\cos\alpha + r_1^2\cos^2\alpha)y^2 = r^2 r_1^2 \sin^2\alpha \sin^2\epsilon. \quad (3)$$

This is the equation of an ellipse referred to its centre and rectangular axes.

Hence if a point has two component simple harmonic motions in any directions, of any amplitudes, and any difference of epoch, *if the periods of the two components are the same*, the resultant motion of the point will be harmonic in an ellipse, the centre of acceleration at the centre of the ellipse. The areal velocity of the radius vector about the centre is constant (page 111).

Such motion is called *elliptic harmonic motion.* Elliptic harmonic motion, then, is compound harmonic motion when the periods of the components are the same.

Equation (3) gives all cases of compound harmonic motion for equal periods of the components.

It will be instructive to derive from it special cases.

(*a*) TWO COMPONENT SIMPLE RECTILINEAR MOTIONS IN DIFFERENT LINES WITH THE SAME PERIOD AND PHASE.—In this case we make in (3) $\epsilon = 0$, and therefore the phases are equal, and we have at once

$$x = \frac{r + r_1 \cos \alpha}{r_1 \sin \alpha} y.$$

This is the equation of a straight line passing through the centre C. The resultant motion is therefore central harmonic in a straight line, or simple rectilinear harmonic.

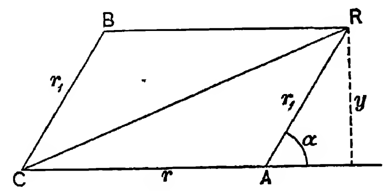


FIG. 2.

If CA and CB are the amplitudes r and r_1 inclined at the angle α, the resultant motion has the amplitude CR, in direction and magnitude the diagonal of the parallelogram whose adjacent sides are r and r_1, inclined at the angle α.

Conversely, a simple rectilinear harmonic motion whose amplitude is CR may be resolved, by completing the parallelogram, into two others in any two directions, *of the same period, epoch and phase*.

If $\alpha = 90°$, we have $y = \frac{r_1}{r} x$. Therefore the projection of a simple rectilinear harmonic motion on any straight line is also a simple rectilinear harmonic motion of the same period, epoch and phase.

If the component motions are more than two, they may be compounded two and two, and therefore *any number* of component simple rectilinear harmonic motions in any directions, of the same period, epoch and phase, give a single resultant rectilinear harmonic motion of determinate direction and amplitude, which may be resolved into two components in any two directions, of the same period, epoch and phase.

(*b*) TWO COMPONENT SIMPLE RECTILINEAR MOTIONS IN THE SAME LINE WITH THE SAME PERIOD AND DIFFERENT EPOCHS AND PHASES.—In this case we make in (3) $\alpha = 0$, and obtain at once

$$(r^2 + 2rr_1 \cos \epsilon + r_1^2) y^2 = 0.$$

But since for $\alpha = 0$, $y = 0$, (see Fig. 1,) we have

$$r^2 + 2rr_1 \cos \epsilon + r_1^2 = \text{constant}.$$

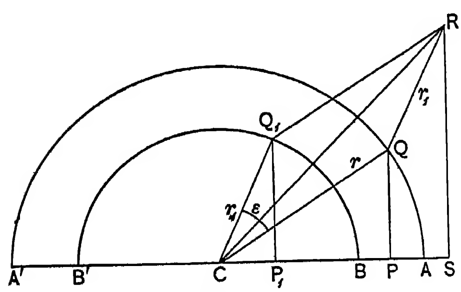


FIG. 3.

In Fig. 3 the points P and P_1 move in the line AA' with simple harmonic motion and the diagonal $CR = \sqrt{r^2 + 2rr_1 \cos \epsilon + r_1^2}$, where ϵ is the constant difference of epoch and phase.

Since ϵ is constant and CR is constant, its inclination to CQ or CQ_1 is constant. At any instant the resultant displacement is $CP_1 + CP = CS$, and the motion of S is therefore the resultant motion and is simple rectilinear harmonic, with the amplitude CR, the diagonal of the parallelogram on r and r_1. The epoch and phase are intermediate between the epochs and phases of the components.

If the epochs and phases are the same, $\epsilon = 0$ and the amplitude of the resultant motion

is $r+r_1$, or the sum of those of the components. If the difference of epoch or phase is $\epsilon=\pi$ radians, the amplitude is $r-r_1$, or the difference of those of the components.

By taking CQ_1 and CQ of proper lengths we can make QCP and Q_1CQ what we please without changing CR. *Therefore any simple harmonic motion may be resolved into two others in the same line, with any required difference of phase and one of them having any desired epoch, the periods being the same.*

Three or more component simple harmonic motions in the same line and with the same period may be compounded two and two, and the resultant will be rectilinear harmonic with the same period.

If the periods are different, the angle $Q_1CQ=\epsilon$ will vary and CR will vary. When $\epsilon=0$, CR will have its maximum value $r+r_1$. When the difference of epoch ϵ is π radians, CR has its minimum value $r-r_1$. The angular velocity of CR is also variable. The direction of CR will oscillate back and forth about CQ, the maximum inclination being $\sin^{-1}\frac{r_1}{r}$. The resultant motion is therefore not simple harmonic, but a more complex motion. It is, as it were, simple harmonic with periodically increasing and decreasing amplitude, and periodical acceleration and retardation of phase or epoch.

(*c*) TWO COMPONENT SIMPLE RECTILINEAR HARMONIC MOTIONS AT RIGHT ANGLES WITH THE SAME PERIOD AND DIFFERENT PHASES OR EPOCHS.—The general equation for this case is given by (3). If the directions are at right angles, we have $\alpha=90°$. Suppose in addition the amplitudes equal, so that $r=r_1$, and the difference of epoch $\epsilon=90°$. We have then, from (3),

$$x^2+y^2=r^2.$$

Since the motion is central harmonic, according to page 129, the areal velocity of the radius vector is constant; and since the radius is constant, the speed in the circle is constant. We have already seen, page 126, that the projection of the motion of a point moving with uniform speed in a circle, upon a diameter, gives rectilinear harmonic motion. The projection upon two diameters at right angles gives, then, two component rectilinear harmonic motions of the same period, with a difference of epoch of 90°, or of phase of $\frac{1}{4}$, since, when one component has its greatest displacement, the other is at the centre with displacement zero.

It follows also that two component simple harmonic motions at right angles, with the same period and equal amplitudes, differing in epoch by 90° or in phase by one quarter of a period, will give, as a resultant, uniform motion in a circle whose radius is the common amplitude of the components.

If the amplitudes are not equal, but α and ϵ still 90°, and periods the same, we have, from (3),

$$r_1^2x^2+r^2y^2=r^2r_1^2,$$

which is the equation of an ellipse referred to its centre and axes.

The resultant motion is therefore harmonic in an ellipse, whose semi-diameters are r and r_1, the centre at the centre of the ellipse.

The same result is evidently obtained by projecting the circle in the preceding case upon a plane, so as to obtain the required amplitude r_1, r remaining unchanged.

(*d*) THREE OR MORE COMPONENT SIMPLE RECTILINEAR HARMONIC MOTIONS IN DIFFERENT LINES WITH THE SAME PERIOD BUT DIFFERENT PHASES OR EPOCHS.—We have seen from equation (3) that the resultant of two simple rectilinear component harmonic

motions in any two directions, of the same period and different epoch or phase, is elliptic harmonic motion.

We have also seen from (*a*) that any simple rectilinear harmonic motion may be resolved into two others of the same period and phase or epoch in any two given directions. Any number of given simple rectilinear harmonic motions, then, of the same period and different phases or epochs may each be resolved into its own pair in any two given directions. These pairs constitute a number of simple rectilinear harmonic motions in two given lines, all of the same period and different phases or epochs.

According to (*b*), all in one line may be compounded into one resultant, and all in the other line into another resultant, these two resultants having the same period and different phases or epochs. The resultant of these two is, according to equation (3), elliptic harmonic motion.

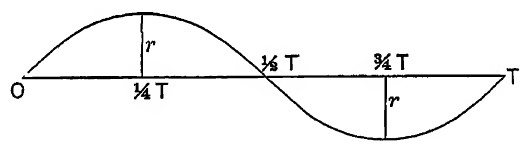
Hence the resultant of *any number* of component simple rectilinear harmonic motions *of the same period*, whatever their amplitudes, directions, phases or epochs, is elliptic harmonic motion, the centre of the ellipse being then centre of acceleration, and the radius vector describing equal areas in equal times.

In special cases this becomes, as we have seen, uniform circular motion or simple rectilinear harmonic motion.

Since the above holds whatever the inclination of the two resultants, elliptic harmonic motion may be considered as the resultant of two component simple harmonic motions of the same period and different epochs or phases at right angles.

Graphic Representation.—We may exhibit graphically simple or compound rectilinear harmonic motion by a curve in which the abscissas represent intervals of time, and the ordinates the corresponding distance of the moving point from its mean position.

In the case of a single harmonic motion we have (page 130) $x = r \cos (\omega t \pm \epsilon)$. If the distance x is to be zero when $t = 0$, we must have the epoch $\epsilon = \frac{\pi}{2}$ radians, or one fourth of the periodic time. This gives $x = r \sin \omega t$.



Since $\omega = \frac{2\pi}{T}$, where T is the periodic time, we have for $t = 0$, $x = 0$; for $t = \frac{1}{4}T$, $x = r$; for $t = \frac{1}{2}T$, $x = 0$; for $t = \frac{3}{4}T$, $x = -r$; for $t = T$, $x = 0$.

The curve is the curve of sines, or the curve which would be described by the point P (page 126) if, while Q maintained its uniform circular motion, the circle itself were to move with uniform speed in a direction perpendicular to CA.

It is the simplest possible form assumed by a vibrating string, when it is assumed that at each instant the motion of every particle of the string is simple harmonic.

If the rectilinear harmonic motion is compound, we have (page 130) in general

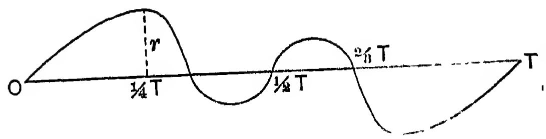
$$x = r \cos (\omega t \pm \epsilon) + r_1 \cos (\omega_1 t \pm \epsilon_1).$$

If the displacement of one of the motions is zero when $t = 0$, we have $\epsilon = \frac{\pi}{2}$; if $\epsilon_1 = \epsilon + n\pi$, we have

$$x = r \sin \omega t + r_1 \sin (\omega_1 t + n\pi).$$

If the period of one motion is twice that of the other, for instance, we have $\omega_1 = 2\omega$, and

$$x = r \sin \omega t + r_1 \sin (2\omega t + n\pi).$$



If the difference of phase is zero, $n = 0$; and if the amplitudes are equal also, we have

$$x = r \sin \omega t + r \sin (2\omega t).$$

This gives a curve as shown in the figure.

Periods Unequal.—We have in general

$$x = r \cos (\omega t + \epsilon), \quad y = r_1 \cos (\omega_1 t + \epsilon_1)$$

for the two component rectilinear harmonic motions at right angles. The elimination of t in any case gives the curve of resultant compound harmonic motion.

If the periods of the components are as 1 to 2, and ϵ is the *difference of the epochs*, we have for equal amplitudes

$$x = r \cos (2\omega t + \epsilon), \quad y = r \cos \omega t.$$

Eliminating t,

$$x = r \left\{ \left(\frac{2y^2}{r^2} - 1\right) \cos \epsilon + 2\frac{y}{r}\sqrt{1 - \frac{y^2}{r^2}} \cdot \sin \epsilon \right\},$$

which is the general equation of the curve for any value of ϵ.

Thus for $\epsilon = 0$, or equal epochs,

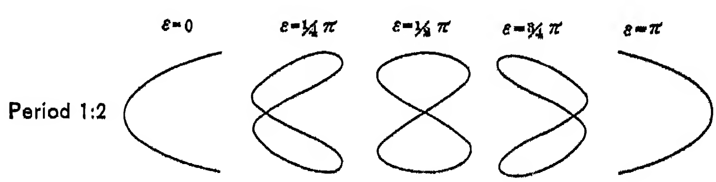
$$\frac{x}{r} = \frac{2y^2}{r^2} - 1, \quad \text{or} \quad y^2 = \frac{r}{2}(x + r),$$

which is the equation of a parabola. For $\epsilon = \frac{\pi}{2}$,

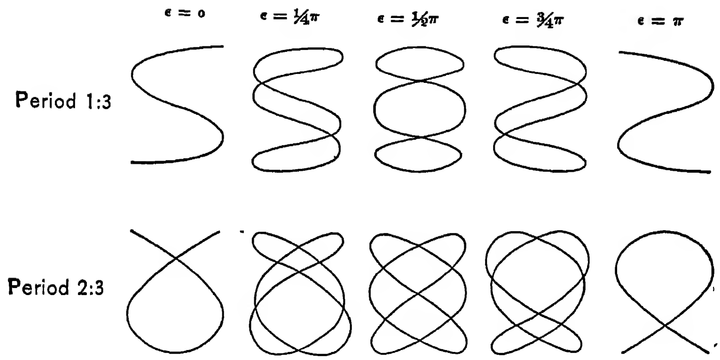
$$\frac{x}{r} = 2\frac{y}{r}\sqrt{1 - \frac{y^2}{r^2}}, \quad \text{or} \quad r^2x^2 = 4y^2(r^2 - y^2),$$

which is also the equation of a parabola.

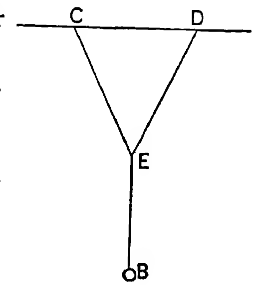
If we make ϵ in succession, 0, 1, 2, etc., eighths of a circumference, we obtain a series of curves as shown.



In the same way we can find the curve for any ratio of periods and difference of epoch. Thus if the periods are as 1 to 3 or 2 to 3, and we make ϵ in succession 0, 1, 2, etc., eighths of a circumference, we obtain the following series of curves:



Blackburn's Pendulum.—The motion of a pendulum which swings through a small arc is, as we shall see hereafter (page 138), simple harmonic, and the projection of the bob on a horizontal plane moves with simple rectilinear harmonic motion.



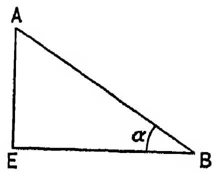
Curves similar to those just given are therefore traced by *Blackburn's pendulum.* This consists of two pendulums, *CED* and *EB*, arranged so as to swing in two planes at right angles.

Any difference of period may be made by adjusting the lengths of the pendulums, and they may be started with any difference of epoch. If the bob *B* is made to trace its path on a horizontal plane, we have, approximately, the compound harmonic curve.

CHAPTER V.

CONSTRAINED MOTION OF A POINT.

Motion on an Inclined Plane—Uniform Acceleration.—Let a point have a uniform acceleration f in the direction AE, and let the point be constrained to move in the straight line AB which makes the angle α with the horizon.



The component of the acceleration in the direction of the motion is then $f \sin \alpha$.

The motion along AB is then rectilinear motion under uniform acceleration $f \sin \alpha$, and equations (2) to (7), page 92, apply directly if we substitute $f \sin \alpha$ in place of f.

If v_1 is the initial velocity at A, and v is the velocity at B we have from (7), page 92,

$$v^2 - v_1^2 = 2fl \sin \alpha,$$

where l is the length of the inclined plane AB. But $l \sin \alpha = AE$.

The speed, therefore, gained in moving from A to B *is equal to that which would be gained in falling through AE with the uniform acceleration f.*

The time in falling from A to E is, from (2), page 91, $t' = \dfrac{v - v_1}{f}$, and in passing from A to B, $t = \dfrac{v - v_1}{f \sin \alpha}$. Hence

$$\frac{t}{t'} = \frac{l}{AE},$$

or the times are proportional to the distances l and AE.

The distance passed through along AB is, from (5), page 92,

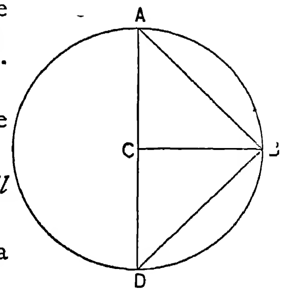
$$l = v_1 t + \frac{1}{2} f \sin \alpha \,.\, t^2,$$

where v_1 is the initial velocity.

If the point starts from rest, we have for the distance along AB

$$l = \frac{1}{2} f \sin \alpha \,.\, t^2.$$

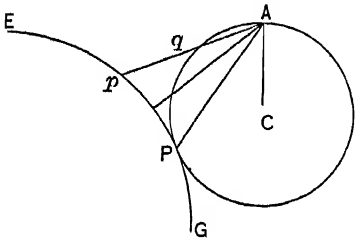
Let AD be the vertical diameter of a circle, and $AB = l$ any chord. Join DB. Then we have $AB = AD \cos DAB = AD \sin ABC$. If $AB = l$, we have also $AB = \frac{1}{2}ft^2 \sin ABC$. Therefore $AD = \frac{1}{2}ft^2$, or $t = \sqrt{\frac{2AD}{f}}$.



This is independent of the position of the chord AB, and therefore *it is the same for any chord* through A or D.

Hence for uniform acceleration f, *the time of descent down all chords through the highest and lowest points of a circle are equal.*

This property enables us to find the line of swiftest descent to a given curve from any point in the same vertical plane.



Thus if EG is the curve and A the point, draw AC parallel to the direction of f, and with centre in AC describe a circle passing through A and tangent to the curve EG at P.

Then AP is the line of swiftest descent from A to the curve EG. For any other point p in EG, Ap cuts the circle in some point q, and since the time from A to q is equal to that from A to P, the time from A to p is greater.

Examples.—(g = 32.16 ft.-per-sec. per sec. Friction, etc., disregarded.)

(1) *Find the position of a point on the circumference of a vertical circle, in order that the time of rectilinear descent from it to the centre may be the same as the time of descent to the lowest point. Acceleration due to gravity.*

ANS. 30° from the top.

(2) *The straight line down which a particle will slide, under the action of gravity, in the shortest time from a given point to a given circle in the same vertical plane, is the line joining the point to the upper or lower extremity of the veritcal diameter, according as the point is within or without the circle*

(3) *Find the line of quickest descent from the focus to a parabola whose axis is vertical and vertex upwards, and show that its length is equal to that of the latus-rectum. Acceleration vertical and uniform.*

(4) *Find the straight line of swiftest descent from the focus of a parabola to the curve when the axis is horizontal. Acceleration vertical and uniform.*

(5) *The times in which heavy particles slide from rest down inclined planes of equal height are proportional to their lengths.*

(6) *Show that if a circle be drawn touching a horizontal straight line in a point P and a given curve in a point Q below P, PQ is the line of swiftest descent to the curve, under constant vertical acceleration.*

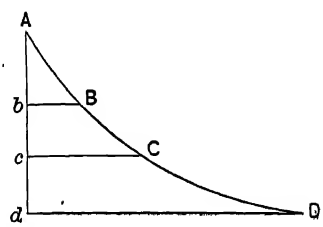
(7) *Find the straight line of quickest descent from a given point to a given straight line, the point and the line being in the same vertical plane. Acceleration constant and vertical.*

ANS. From P, the given point, draw a horizontal line meeting the given line in C. Lay off along the given line CD equal to PC. PD is the required line of swiftest descent.

(8) *A given point P is in the same plane with a given vertical circle and outside it, the highest point Q of the circle being below P. Find the line of slowest descent from P to the circle. Acceleration constant and vertical.*

ANS. Join PQ and produce it to meet the circumference in R. PR is the line required.

Motion in a Curved Path—Uniform Acceleration.—Let $ABCD$ be any curved path, and Ad the direction of the acceleration f. Any very small portion of the curve, AB, may be considered as a straight line. We have then, as on page 136, the change of speed in moving from A to B, the same as in moving from A to b with the constant acceleration f. So, also, in moving from B to C the change of speed is the same as in moving from b to c with the constant acceleration f.



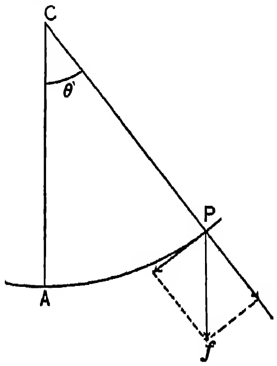
Hence the change of speed in traversing any portion of the

path AD is the same as in traversing with constant acceleration f the projection Ad of the path on a line in the direction of the acceleration.

If, then, v_1 is the initial speed at A, and v is the speed at any point D, we have

$$v^2 - v_1^2 = 2f.\,Ad.$$

Motion in a Circle—Uniform Acceleration.—This is the case of the *simple pendulum*, which consists of a heavy particle attached to a fixed point by a massless inextensible string.



Let C be the point of suspension and CA the radius, and let the acceleration f be uniform and vertical. For any position of the point P the angle $ACP = \theta$, and the acceleration may be resolved into a tangential component $f \sin \theta$ and into a normal component $f \cos \theta$.

The normal component has no effect upon the motion in the curve at P.

If the angle θ is very small, the arc will not differ materially from the sine, and we have $\sin \theta = \frac{\text{arc } AP}{l}$, where l is the length o the radius CA.

The tangential acceleration at the point P is then $f_t = \frac{f \times \text{arc } AP}{l}$. It is therefore directly proportional to the distance of P from A, measured along the path.

The motion of P is thus *harmonic* in the path, and the periodic time is then (page 127)

$$T = 2\pi\sqrt{\frac{\text{arc } AP}{\frac{f \times \text{arc } AP}{l}}} = 2\pi\sqrt{\frac{l}{f}},$$

or for the simple pendulum the time of a *vibration* is $t = \pi\sqrt{\frac{l}{g}}$.

The periodic time is therefore *for small displacements* independent of the amplitude, and therefore for small arcs the oscillations are *isochronous*.

The time of vibration is half the periodic time, or the time between the instants at which the pendulum reaches opposite ends of its swing. Thus the seconds pendulum makes a complete oscillation in 2 seconds.

If θ is not very small the time is different, but the variation is practically very slight.

COR.—If the velocity of P at any instant is not wholly in the plane PCA, it may be resolved into two components, one in the plane PCA and the other perpendicular to it, and both tangential to a spherical surface. Hence, in the case in which θ is small, P's motion may be resolved into two simple harmonic motions of the same period; and its motion is therefore (page 133) elliptic harmonic motion, the period being the common period of the components. The ellipse described will depend upon the amplitude and epoch of the components and therefore upon the magnitude and direction of the initial velocity of P.

If θ is not very small, and the component motions are of different amplitudes, the periods will have different values, and the point P describes curves similar to those given on page 135.

If the component motions are equal in amplitude and therefore in period and differ in phase by one quarter period, the point P moves (page 132) in a circle about the foot of the perpendicular on CA as a centre. This is the case of the conical pendulum.

Examples.—(1) *Find the time of oscillation of a pendulum 10 ft. long at a place at which $g = 32$ ft.-per-sec. per sec.*

ANS 1.75 sec.

(2) *Find the length of the seconds pendulum at a place at which $g = 31.9$.*

ANS. 3.232 ft.

(3) *Find the length of the pendulum which makes 24 beats in 1 min. when $g = 32.2$.*

ANS. 20.39 ft.

(4) *A seconds pendulum was lengthened 1 per cent. How much does it lose per day?*

ANS. 7 min. 8.8 sec.

(5) *The length of the seconds pendulnm being 99.414 cm, find the value of g.*

ANS. 981.17 cm.-per-sec. per sec.

(6) *A pendulum 37.8 inches long makes 182 beats in 3 min. Find the value of g.*

ANS. 31.78 ft.-per-sec. per sec.

(7) *If two pendulums at the same place make 25 and 30 oscillations respectively in 1 sec., what are their relative lengths?*

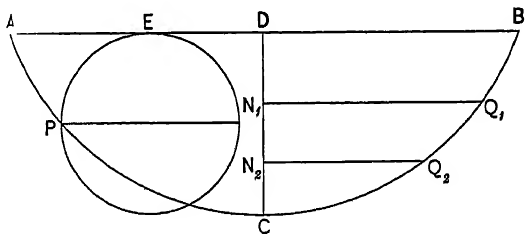
ANS. 1.44 to 1.

(8) *A pendulum which beats seconds at one place is carried to another where it gains 2 sec. per day. Compare the value of g at the two places.*

ANS. As 0.999953 to 1.

(9) *A pendulum which beats seconds at the sea-level is carried to the top of a mountain, where it loses 40.1 sec. per day. Assuming the value of g to be inversely proportional to the distance from the centre of the earth, and the sea-level to be 4000 miles from that point, find the height of the mountain.*

ANS. 1 86 miles.



Motion in a Cycloid—Uniform Acceleration.—A cycloid is the curve traced by a point in the circumference of a circle which rolls along a straight line.

If the circle EP rolls along the line AB, the point P being originally at A, the path of P is the cycloid ACB.

If C is the position of P when the diameter of the circle through P is perpendicular to AB, the line CD perpendicular to AB is the axis and C is the vertex of the cycloid.

Let the uniform acceleration f be always parallel to DC and vertical.

Let the moving point Q have at Q_1 a speed zero. Its speed at Q_2 is then (page 136)

$$v^2 = 2f \,.\, N_1N_2.$$

Let t be the time in which the point would with the same acceleration and with initial speed zero move from D to C. Then $CD = \frac{1}{2}ft^2$. Hence

$$v^2 = \frac{4}{t^2} \,.\, N_1N_2 \,.\, CD = \frac{4}{t^2} CD(CN_1 - CN_2).$$

Now by a property of the cycloid

$$4CD \,.\, CN_1 = CQ_1^2 \quad \text{and} \quad 4CD \,.\, CN_2 = CQ_2^2.$$

Hence

$$v^2 = \frac{1}{t^2}(CQ_1^2 - CQ_2^2).$$

Now $t^2 = \dfrac{2CD}{f}$ is a constant. Hence the motion of Q in the cycloid is harmonic (page 126), where $\dfrac{1}{t^2} = \dfrac{f_0}{s_0}$, f_0 being the tangential acceleration of Q at the distance s_0 measured along the curve. If T is the time of a complete oscillation, we have

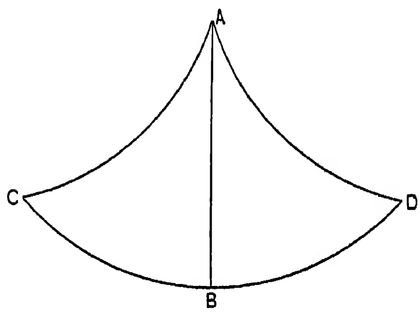
$$T = 2\pi\sqrt{\frac{s_0}{f_0}} = 2\pi t = 2\pi\sqrt{\frac{2CD}{f}}.$$

If t' is the time occupied in moving from Q_1 to C,

$$t' = \frac{\pi}{2}\sqrt{\frac{2CD}{f}},$$

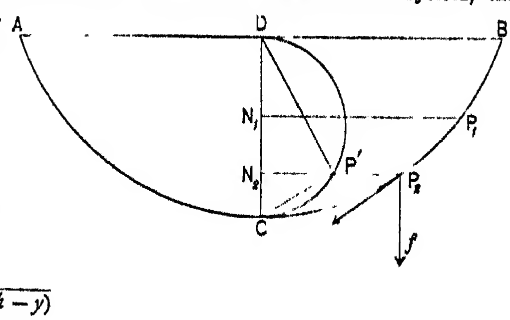
or the time of a pendulnm whose length is $2CD$, or 4 times the radius of the generating circle.

As this involves only constant quantities, the time is the same whatever be the position of the starting-point Q_1, or the oscillations are isochronous. Hence the cycloid is called a *tautochrone*.



This result is rendered of practical importance by one of the properties of the cycloid, viz., that if a flexible and inextensible string AB is fixed at the end A and wrapped tightly round the semi-cycloid AC, the end B of the string as it unwinds will describe another semi-cycloid. If, then, AC and AD are fixed semi-cycloids, symmetrical with reference to the vertical AB, and AB is a simple pendulum, B will describe a cycloid, and its oscillations will be isochronous whatever their extent.

[**Application of the Calculus.—To Determine the Motion of a Point Constrained to Move in a Cycloid, the Acceleration being Constant, in the Direction of the Axis and towards the Vertex.**]—By the application of the general formulas of page 92 we can deduce the results already obtained.



Let the axis $CD = 2r$, where r is the radius of the generating circle $DP'C$. Let the acceleration f act downward. Let $CN = y$, $NP = x$ and the length of arc $CP = s$. Let the initial position be P_1 at the height $CN_1 = h$ above C, and the speed at P_1 be $v_1 = 0$.

We have

$$v = \sqrt{2f(h-y)}$$

for the speed at any point given by $CN = y$. When $y = 0$, we have, at the lowest point C, $v = \sqrt{2fh}$, which is the same as that due to the vertical height h.

By the property of the cycloid we have

$$s = \text{arc } CP = 2\sqrt{DC \times CN} = 2\sqrt{2ry} = 2 \text{ chord } CP'.$$

Hence

$$ds = \pm\, dy\sqrt{\frac{2r}{y}}.$$

We have then

$$dt = -\sqrt{\frac{r}{f}}\,\frac{dy}{\sqrt{hy - y^2}}.$$

Integrating, since for $t = 0, y = h$, we have

$$t = \sqrt{\frac{r}{f}} \cdot \left(\pi - \text{versin}^{-1}\frac{2y}{h}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

For the time of descent to the lowest point where $y = 0$, or for the time of one quarter of a complete oscillation,

$$t = \pi\sqrt{\frac{r}{f}} = \frac{\pi}{2}\sqrt{\frac{4r}{f}}.$$

The periodic time is then

$$T = 2\pi\sqrt{\frac{4r}{f}},$$

or the same as a simple pendulum (page 138) whose length is 4 times the radius of the generating circle $DP'C$.

The time is independent of h and is the same no matter what the position from which the point begins to descend. The oscillations are therefore *isochronous* and hence the cycloid is called the *tautochrone*.

The reason of this remarkable property is easily seen by considering the tangential acceleration.

In the cycloid the chord CP' is always parallel to the tangent TP. The tangential acceleration or tangential component of f is then

$$\frac{d^2s}{dt^2} = f \sin TPf = f \sin CDP' = f\frac{CP'}{CD} = f\frac{s}{4r}.$$

The tangential acceleration is therefore directly proportional to the distance from the vertex measured along the path, and the motion of P is *simple harmonic* (page 125).

The periodic time is then (page 127)

$$T = 2\pi\sqrt{\frac{4r}{f}}.$$

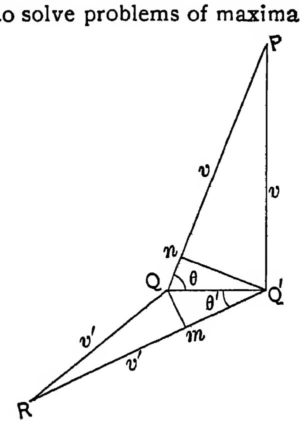
If in (1) we make $y = \frac{h}{2}$, we have $t = \frac{\pi}{2}\sqrt{\frac{r}{f}}$, or half the time from P_1 to C. The time, therefore, in descending through half the vertical space to C is half the time of passing from P_1 to C.

[**To Find a Curve such that a Point Moving on it under the Action of Gravity will Pass from any one Given Position to any Other in Less Time than by any Other Curve through the Same Two Points.**]—This is the celebrated problem of the "curve of swiftest descent" first propounded by Bernoulli The following is founded upon his original solution.

If the time of descent through the entire curve is a minimum, that through any portion of the curve is a minimum.

It is also obvious that *between any two contiguous equal values of a continuously varying quantity, a maximum or minimum must lie.*

This principle, though simple, is of very great power, and often enables us to solve problems of maxima and minima, such as require not merely the processes of the Differential Calculus but those of the Calculus of Variations. The present case is a good example.



Let, then, PQ, QR and PQ', $Q'R$ be two pairs of indefinitely small sides of a polygon such that the time of descending through either pair, starting from P, may be *equal*. Let QQ' be horizontal and indefinitely small compared with PQ and QR. The curve of swiftest descent must lie *between* these paths, and must possess any property which they have in common. Hence if we draw Qm, $Q'n$ perpendicular to RQ', PQ, and let v be the speed down PQ or PQ' (supposed uniform) and v' that down QR or $Q'R$, we have for the time from P to R by either path

$$\frac{PQ}{v} + \frac{QR}{v'} = \frac{PQ'}{v} + \frac{Q'R}{v'}, \quad \text{or} \quad \frac{PQ - PQ'}{v} = \frac{Q'R - QR}{v'},$$

or

$$\frac{Qn}{v} = \frac{Q'm}{v'}.$$

Now if θ be the inclination of PQ to the horizontal, and θ' that of QR, we have $Qn = QQ' \cos\theta$, $Q'm = QQ' \cos\theta'$. Hence

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'}$$

This is true for any two consecutive elements of the required curve, and therefore throughout the curve we have, at any point, v proportional to the cosine of the angle which the tangent to the curve at that point makes with the horizontal. But v^2 is proportional to the vertical distance h fallen through.

Hence the curve required is such that the cosine of the angle it makes with the horizontal line through the point of departure varies as the square root of the distance from that line.

Now in the figure of page 140 we have, from the property of a cycloid,

$$\cos CP'N = \cos TP'N = \cos CDP' = \frac{DP'}{DC} = \sqrt{\frac{DN}{DC}}.$$

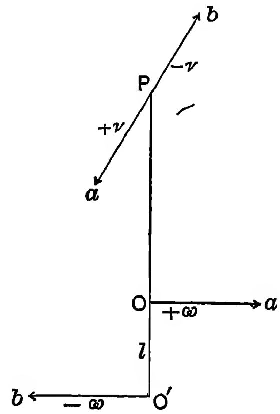
The curve required is therefore the cycloid. The cycloid has received on account of this property the name of *Brachistochrone*.

KINEMATICS OF A RIGID BODY.

CHAPTER I.

ANGULAR VELOCITY AND ACCELERATION COUPLES ANGULAR AND LINEAR VELOCITY AND ACCELERATION COMBINED.

Angular Velocity Couple.—Two simultaneous, equal, parallel and opposite angular velocities, not in the same straight line, we call an ANGULAR VELOCITY COUPLE.



Thus, if the point P has an angular velocity $+\ \omega$ about an axis Oa, so that $Oa = +\ \omega$ is its line representative, and at the same time has an angular velocity $-\ \omega$ about an axis $O'b$, so that $O'b = -\ \omega$ is its line representative, then, since the line representatives are equal, parallel and opposite, and do not coincide, they constitute a couple.

Moment of Angular Velocity Couple.—We have seen (page 90) that the moment $OP \times \omega$ of the angular velocity $Oa = +\ \omega$ relative to P, gives the linear velocity $Pa = +\ v$ at right angles to the plane of Oa and OP, due to angular velocity about the axis Oa. The direction of $+\ v$ is such that when we look along Oa in its direction, $+\ v$ is seen as clockwise rotation, or towards the reader in the preceding figure.

In the same way $O'P \times \omega$ gives the linear velocity $Pb = -\ v$ of P in a direction away from the reader, at right angles to the plane of the couple.

The resultant linear velocity of P is then

$$v = (O'P - OP)\ \omega = O'O \times \omega,$$

where $O'O$ is the perpendicular distance between the line representatives Oa, $O'b$ of the couple

This resultant velocity is at right angles to the plane of the couple and in such a direction that when we look along its line representative, rotation as indicated by the arrows of the couple is seen clockwise.

With this convention, if the distance $O'O = l$, we have

$$v = l\omega$$

This result holds *no matter where the point P may be taken in the plane of the couple.*

If, then, P is a point of a rigid body, every point of this body in the plane of the couple must have the same velocity v in the same direction. That is, the body has a velocity of translation.

Hence, *if a rigid body is acted upon by an angular velocity couple, the result is a velocity of translation for every point of the body.*

We denote velocity of translation always by $\overline{v}$.

Angular Acceleration Couple.—Two simultaneous, equal parallel and opposite angular accelerations not in the same line we call an ANGULAR ACCELERATION COUPLE. We have then, in precisely the same way and with the same convention as to direction, the acceleration for every point of the body

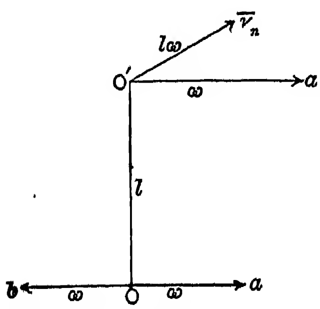
$$f = l\alpha,$$

where α is the angular acceleration and f is the linear acceleration of translation of the body.

Hence, *if a rigid body is acted upon by an angular acceleration couple, the result is a linear acceleration of translation for every point of the body.*

We denote acceleration of translation always by $\overline{f}$.

Angular and Linear Velocity Combined.—Let a rigid body have an angular velocity ω about an axis $O'a$, so that $O'a = \omega$ is the line representative. Let O be any point of the body. If at this point we apply two equal and opposite angular velocities $Oa = \omega$ and $Ob = \omega$, both parallel to $O'a = \omega$, the previous motion of the body is evidently not affected.



We see, then, that the single angular velocity $O'a = \omega$ about an axis $O'a$ can be reduced to the same angular velocity $Oa = \omega$ about a parallel axis Oa through any point O, and an angular velocity couple $O'a$ and Ob.

Let l be the distance between the parallel axes. Then, as we have just seen, the couple causes a velocity of translation $\overline{v}_n = l\omega$ at right angles to the plane of the couple, so that looking along the line representative of $\overline{v}_n$ in its direction, the arrows of the couple indicate clockwise rotation. In the figure $\overline{v}_n$ is at right angles to the plane of the couple and away from the reader.

Hence

(*a*) A single angular velocity ω of a rigid body about a given axis, can be resolved into an equal angular velocity about a parallel axis through any point O of the body at a distance l, and a normal velocity of translation $\overline{v}_n = l\omega$ of this axis in a direction at right angles to the plane of the two axes.

(*b*) Conversely, the resultant of an angular velocity ω of a rigid body about a given axis and a simultaneous velocity of translation $\overline{v}_n$ normal to that axis, is a single equal angular velocity about a parallel axis at a distance $l = \frac{\overline{v}_n}{\omega}$, the plane of the two axes being perpendicular to $\overline{v}_n$.

(*c*) If, then, a rigid body has any number of angular velocities, each one about a different axis through a different point, then by (*a*) we can reduce each one to an equal angular velocity about an axis through any point O we please, and a normal velocity of translation of this axis.

All the angular velocities at this point O can then be reduced to a single resultant angular velocity ω about a resultant axis by the polygon of angular velocities (page 68), and all the normal velocities of translation can be reduced to a single resultant velocity of translation $\overline{v}$, not necessarily normal to the resultant axis, by the polygon of linear velocities (page 66).

The motion of a rigid body in general can then be reduced at any instant to an angular

velocity ω about an axis through any point O we please, and a velocity of translation $\bar{v}$ of this axis. This velocity $\bar{v}$ of translation is not necessarily normal to the axis.

The angular velocity ω has the same magnitude and direction no matter what point is taken, but the velocity of translation $\bar{v}$ varies in magnitude and direction with the position of this point.

(d) This velocity of translation $\bar{v}$ is not necessarily normal to the axis and can in general be resolved into a component $\bar{v}_a$ along the axis through O, and a component $\bar{v}_n$ normal to this axis.

But by (b) we can reduce ω and $\bar{v}_n$ to the same angular velocity ω about a parallel axis at a distance $l = \frac{\bar{v}_n}{\omega}$.

Instantaneous Axis of Rotation.—This axis is called the INSTANTANEOUS AXIS OF ROTATION because it is the axis *without translation* about which at a given instant angular velocity takes place.

Hence, in general, the motion of a rigid body, at any instant, can be reduced to an angular velocity ω about an axis through any point of the body, a velocity of translation v_a along this axis, and a normal velocity v_n of translation of this axis. Or to an angular velocity ω about a parallel instantaneous axis at a distance $l = \frac{v_n}{\omega}$ and a velocity of translation $\bar{v}_a$ along this axis.

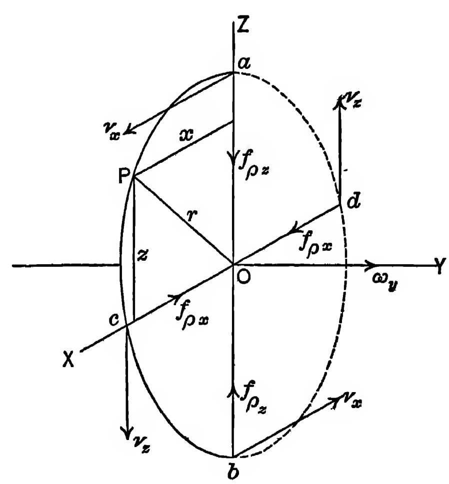
Spin. Screw-Spin.—Angular velocity of a rigid body about any axis we call a SPIN about that axis. Angular velocity of a rigid body about any axis together with velocity of translation along that axis we call a SCREW-SPIN. The velocity of translation $\bar{v}_a$ along the axis we call the VELOCITY OF ADVANCE.

Hence the motion of a rigid body at any instant can be reduced in general to a *spin or a screw-spin about an instantaneous axis*, or to a *spin or screw-spin about a parallel axis through any point together with normal velocity of translation* $\bar{v}_n$ *of that axis.*

Spontaneous Axis of Rotation.—The axis of rotation through the centre of mass of a rigid body at any instant is called the SPONTANEOUS axis of rotation. If it has normal velocity of translation $\bar{v}_n$, the instantaneous axis of rotation is parallel to it at the distance $l = \frac{\bar{v}_n}{\omega}$, the plane of these two axes being perpendicular to $\bar{v}_n$ (page 144).

The velocity of any point is that due to angular velocity about the instantaneous axis of rotation, or angular velocity about the parallel *translating* spontaneous axis (page 144).

If the spontaneous axis of rotation has no normal velocity of translation v_n, the spontaneous and instantaneous axes coincide.



Examples.—(1) *A vertical circle of radius $r = 2$ ft. rotates about a fixed horizontal axis through the centre of mass at right angles to the plane of the circle with angular velocity of 3 radians per sec. Find the velocity and central acceleration of the top, bottom, forward and rear points, and of any point in general.*

ANS. Take co-ordinate axes as shown in the figure. Then $r = 2$ ft., $\omega_y = +3$ radians per sec. Since the spontaneous axis OY is without translation, the instantaneous axis coincides with it.

We have then at the top point a the velocity $v_x = + r\omega_y = + 6$ ft. per sec. At the bottom point

b the velocity $v_x = -r\omega_y = -6$ ft. per sec. At the forward point c the velocity $v_z = -r\omega_y = -6$ ft per sec. At the rear point d the velocity $v_z = +r\omega_y = +6$ ft. per sec.

Also, at the top point a the central acceleration $f_{\rho z} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. At the bottom point b the central acceleration $f_{\rho z} = +r\omega_y^2 = +18$ ft.-per-sec. per sec. At the forward point c the central acceleration $f_{\rho x} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. At the rear point d the central acceleration $f_{\rho x} = +18$ ft.-per-sec. per sec.

Let $+x$, $+y$, $+z$ be the co-ordinates of any point P. Then we have the component velocities of that point given by

$$v_x = z\omega_y, \quad v_y = 0, \quad v_z = -x\omega_y \quad \ldots \ldots \ldots \quad (1)$$

The resultant velocity is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \omega_y\sqrt{z^2 + x^2} = r\omega_y, \quad \ldots \ldots \quad (2)$$

and its direction cosines are

$$\cos\alpha = \frac{v_x}{v} = +\frac{z}{r}, \quad \cos\beta = 0, \quad \cos\gamma = -\frac{x}{r}. \quad \ldots \ldots \quad (3)$$

The central acceleration is $f_\rho = r\omega_y^2$ towards the centre. The radius r has the direction cosines $\cos\alpha = \frac{x}{r}$, $\cos\beta = 0$, $\cos\gamma = \frac{z}{r}$. The component central accelerations are then

$$f_{\rho x} = -f_\rho\frac{x}{r} = -x\omega_y^2, \quad f_{\rho y} = 0, \quad f_{\rho z} = -f_\rho\frac{z}{r} = -z\omega_y^2. \quad \ldots \ldots \quad (4)$$

The resultant central acceleration is

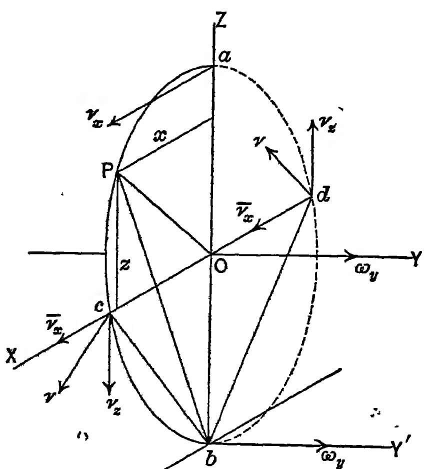
$$f_\rho = \sqrt{f_{\rho x}^2 + f_{\rho y}^2 + f_{\rho z}^2} = \omega_y^2\sqrt{x^2 + z^2} = r\omega_y^2, \quad \ldots \ldots \quad (5)$$

and its direction cosines are

$$\cos\alpha = \frac{f_{\rho x}}{f_\rho} = -\frac{x}{r}, \quad \cos\beta = 0, \quad \cos\gamma = \frac{f_{\rho z}}{f_\rho} = -\frac{z}{r}. \quad \ldots \ldots \quad (6)$$

(2) *A vertical circle of radius $r = 2$ ft. rolls on a horizontal straight line. The centre moves parallel to that line with a velocity of 6 ft. per sec. Find the angular velocity; the velocity and central acceleration of the top, bottom, forward and rear points, and of any point in general.*

ANS. Take co-ordinate axes as shown in the figure. Then $r = 2$ ft., $\bar{v}_x = +6$ ft. per sec. The spontaneous axis is OY.



Since the circle rolls, the instantaneous axis bY' passes through the bottom point b parallel to the spontaneous axis.

We have then for velocity of translation

$$\bar{v}_x = r\omega_y, \text{ or } \omega_y = \frac{\bar{v}_x}{r} = +3 \text{ radians per sec.}$$

The velocity of any point is that due to translation and angular velocity ω about OY, or to angular velocity ω only about bY'.

We have, then, at the top point a the velocity $v_x = \bar{v}_x + r\omega_y = 2r\omega_y = +12$ ft. per sec.

At the bottom point b the velocity is $v_x = \bar{v}_x - r\omega_y = 0$. At the forward point c we have the component velocities $v_x = \bar{v}_x = r\omega_y = +6$ ft. per sec., and $v_z = -r\omega_y = -6$ ft. per sec. The resultant velocity is then $v = 6\sqrt{2}$ ft. per sec., making an angle of 45° below OX, as shown in the figure.

At the rear point d we have the component velocities $v_x = \bar{v}_x = r\omega_y = +6$ ft. per sec., and $v_z = +r\omega_y = +6$ ft. per sec. The resultant velocity is then $v = 6\sqrt{2}$ ft. per sec., making an angle of 45° above OX, as shown in the figure.

The central acceleration of any point is that due to angular velocity about OY *considered as without translation* (page 145).

We have then, just as in the preceding example, at the top point a the central acceleration $f_{\rho z} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. At the bottom point b, $f_{\rho z} = +r\omega_y^2 = +18$ ft.-per-sec. per sec. At the forward point c, $f_{\rho x} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. At the rear point d, $f_{\rho x} = +r\omega_y^2 = +18$ ft.-per-sec. per sec.

Let $+x$, $+z$ be the co-ordinates of any point P for the origin O. Then we have for the component velocities of that point

$$v_x = \overline{v}_x + z\omega_y = r\omega_y + z\omega_y, \qquad v_y = 0, \qquad v_z = -x\omega_y. \quad . \quad . \quad . \quad (1)$$

Equations (1) give the component velocities for any point in general.

The resultant velocity of any point is then

$$v = \sqrt{v_x^2 + v_z^2} = \omega_y \sqrt{(z+r)^2 + x^2} = r'\omega_y, \quad . \quad . \quad . \quad (2)$$

where r' is the radius vector of the point relative to b.

The direction cosines of v are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = 0, \qquad \cos\gamma = \frac{v_z}{v}. \quad . \quad . \quad . \quad (3)$$

These equations reduce to (1), (2), (3) of the preceding example if $\overline{v}_x = 0$.

The radius r for the point P has the direction cosines $\cos\alpha = \frac{x}{r}$, $\cos\gamma = \frac{z}{r}$. The central acceleration is $f_\rho = r\omega_y^2$ towards O. The component central accelerations for any point are then, just as in the preceding example,

$$f_{\rho x} = -x\omega_y^2, \qquad f_{\rho y} = 0, \qquad f_{\rho z} = -z\omega_y^2. \quad . \quad . \quad . \quad (4)$$

The resultant is

$$f_\rho = \sqrt{f_x^2 + f_z^2} = r\omega_y^2, \quad . \quad . \quad . \quad (5)$$

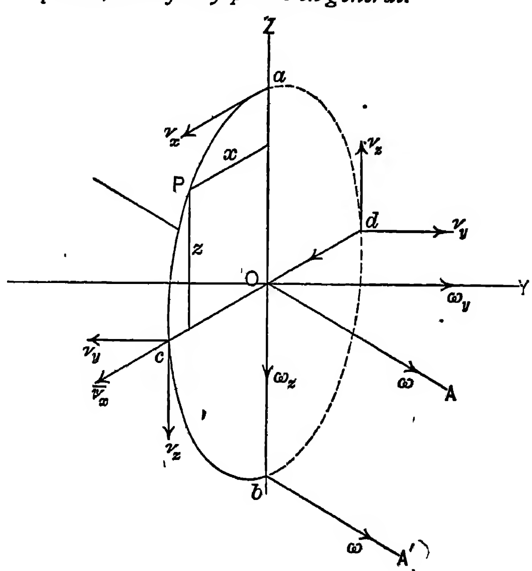
and its direction cosines are

$$\cos\alpha = \frac{f_{\rho x}}{f_\rho} = -\frac{x}{r}, \qquad \cos\gamma = \frac{f_{\rho z}}{f_\rho} = -\frac{z}{r}. \quad . \quad . \quad . \quad (6)$$

These equations are the same as (4), (5), (6) of the preceding example.

(3) *Let a vertical circle of radius $r = 2$ ft. roll on a horizontal plane. The centre moves with a velocity of 6 ft. per sec. At the same time let the plane of the circle rotate about the vertical diameter with an angular velocity of 2 radians per sec. downwards. Find the angular velocity about the horizontal axis; the velocity and central acceleration of the top, bottom, forward and rear points, and of any points in general.*

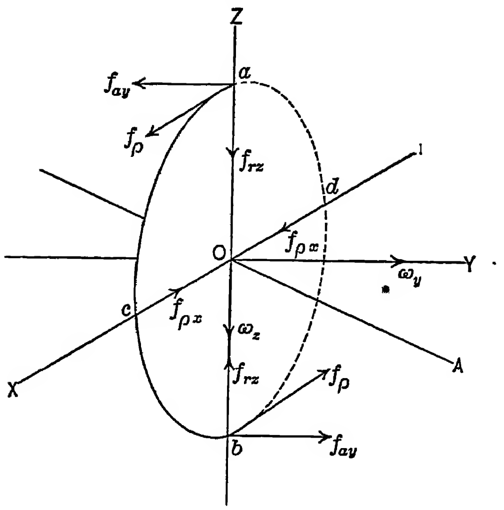
ANS. Take the co-ordinate axes as shown in the figure. We have then $r = 2$ ft., $\omega_z = -2$ radians per sec., $v_x = +6$ ft. per sec. Since the circle rolls, we have $r\omega_y = \overline{v}_x$ or $\omega_y = \frac{\overline{v}_x}{r} = +3$ radians per sec., and the instantaneous axis passes through the bottom point b. The velocity of any point is that due to translation and angular velocity ω_y about OY and ω_z about OZ, or to translation and angular velocity ω about the spontaneous axis OA, or to angular velocity ω only about the instantaneous axis bA' parallel to OA.



We have then at the top point a the velocity $v_x = \overline{v}_x + r\omega_y = 2r\omega_y = +12$ ft. per sec. At the bottom point b the velocity is $v_x = \overline{v}_x - r\omega_y = 0$. At the forward point c we have the component velocities $v_x = \overline{v}_x = r\omega_y = +6$ ft. per sec., $v_y = r\omega_z = -4$ ft. per sec., $v_z = -r\omega_y = -6$ ft per sec. The resultant velocity is then $v = \sqrt{v_x^2 + v_y^2 + v^2} = 2\sqrt{22}$ ft. per sec, and its direction cosines are given by

$$\cos\alpha = \frac{v_x}{v} = +\frac{3}{\sqrt{22}}, \qquad \cos\beta = \frac{v_y}{v} = -\frac{2}{\sqrt{22}}, \qquad \cos\gamma = \frac{v_z}{v} = -\frac{3}{\sqrt{22}}.$$

At the rear point d we have the component velocities $v_x = \bar{v}_x = r\omega_y = +6$ ft. per sec., $v_y = -r\omega_z$ $= +4$ ft. per sec., $v_z = +r\omega_y = +6$ ft. per sec. The resultant velocity is as before $v = 2\sqrt{22}$ ft. per sec., and its direction cosines are



$$\cos\alpha = \frac{v_x}{v} = +\frac{3}{\sqrt{22}}, \quad \cos\beta = \frac{v_y}{v} = +\frac{2}{\sqrt{22}}$$

$$\cos\gamma = \frac{v_z}{v} = +\frac{3}{\sqrt{22}}.$$

The central acceleration of any point is that due to angular velocity about OA *considered as without translation* (page 145), or to angular velocity ω_y about OY and ω_z about OZ without translation.

We have then at the top point a the *deflecting* acceleration (page 80) $f_{rz} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. and the *deviating* acceleration (page 80) $f_{ay} = r\omega_y\omega_z = -12$ ft.-per-sec. per sec. The resultant central acceleration is then $f_\rho = \sqrt{f_{rz}^2 + f_{ay}^2} = 6\sqrt{13}$ ft.-per-sec. per sec., and its direction cosines are

$$\cos\alpha = 0, \quad \cos\beta = \frac{f_{ax}}{f_\rho} = -\frac{2}{3\sqrt{13}}, \quad \cos\gamma = \frac{f_{rz}}{f_\rho} = -\frac{3}{\sqrt{13}}.$$

At the bottom point b the *deflecting* acceleration (page 80) is $f_{rz} = +r\omega_y^2 = +18$ ft.-per.-sec. per sec., and the *deviating* acceleration is $f_{ax} = -r\omega_y\omega_z = +12$ ft.-per-sec. per sec. The resultant central acceleration is then $f_\rho = \sqrt{f_{rz}^2 + f_{ax}^2} = 6\sqrt{13}$ ft.-per-sec. per sec., and its direction cosines are

$$\cos\alpha = 0, \quad \cos\beta = \frac{f_{ax}}{f_\rho} = +\frac{2}{3\sqrt{13}}, \quad \cos\gamma = \frac{f_{rz}}{f_\rho} = +\frac{3}{\sqrt{13}}.$$

At the forward point c the central acceleration is $f_{\rho x} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. At the rear point d the central acceleration is $f_{\rho x} = +r\omega_y^2 = +18$ ft.-per-sec. per sec.

Let $+x$, $+z$, be the co-ordinates of any point P. Then we have the component velocities of that point:

$$v_x = \bar{v}_x + z\omega_y = r\omega_y + z\omega_y, \qquad v_y = x\omega_z, \qquad v_z = -x\omega_y. \quad \ldots \ldots \quad (1)$$

The resultant velocity of any point is then

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{[(z+r)^2 + x^2]\omega_y^2 + x^2\omega_z^2} = \sqrt{r'^2\omega_y^2 + x^2\omega_z^2}, \quad \ldots \ldots \quad (2)$$

where r' is the radius vector of the point relative to b.

The direction cosines of v are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v}, \qquad \cos\gamma = \frac{v_z}{v}. \quad \ldots \ldots \quad (3)$$

These equations reduce to equations (1), (2), (3) of the preceding example if $\omega_z = 0$.

The radius r for the point P has the direction cosines $\cos\alpha = \frac{x}{r}$, $\cos\gamma = \frac{z}{r}$. The deflecting acceleration is $f_r = r\omega_y^2$ towards O for rotation about OY, and $x\omega_z^2$ towards OZ for rotation about OZ. The component deflecting accelerations are then

$$f_{rx} = -x\omega_y^2 - x\omega_z^2, \qquad f_{ry} = 0, \qquad f_{rz} = -z\omega_y^2.$$

The deviating acceleration is $f_{ay} = z\omega_y\omega_z$. The components of the central acceleration are then

$$f_{\rho x} = -x\omega_y^2 - x\omega_z^2, \qquad f_{\rho y} = z\omega_y\omega_z, \qquad f_{\rho z} = -z\omega_y^2. \quad \ldots \ldots \quad (4)$$

The resultant is

$$f_\rho = \sqrt{f_{\rho x}^2 + f_{\rho y}^2 + f_{\rho z}^2} = \sqrt{(x^2 + z^2)\omega_y^4 + z^2\omega_y^2\omega_z^2} = \sqrt{r^2\omega_y^4 + z^2\omega_y\omega_z^2}. \quad \ldots \ldots \quad (5)$$

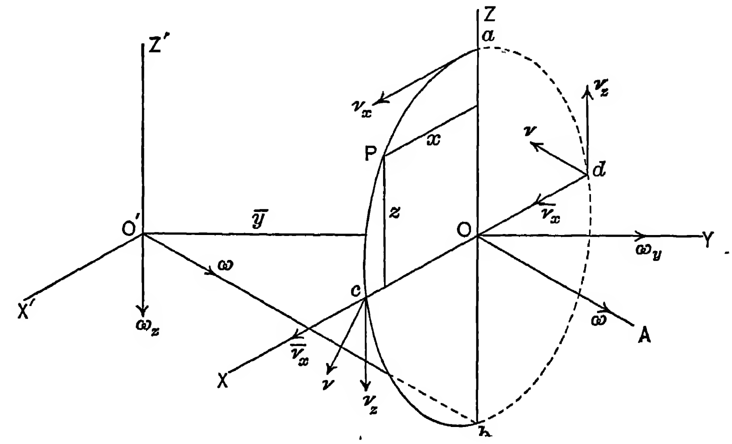
The direction cosines of f_ρ are

$$\cos\alpha = \frac{f_{\rho x}}{f_\rho}, \qquad \cos\beta = \frac{f_{\rho y}}{f_\rho}, \qquad \cos\gamma = \frac{f_{\rho z}}{f_\rho}. \quad \ldots \ldots \quad (6)$$

These equations reduce to (4), (5) (6) of the preceding example if $\omega_z = 0$.

(4) *Let a vertical circle of radius $r = 2$ ft. roll on a horizontal plane with an angular velocity of 3 radians per sec., while its centre describes a horizontal circle of radius 3 ft. with angular velocity about a fixed vertical axis. Find the angular velocity about the fixed axis; the velocity and central acceleration of the top, bottom, forward and rear points, and of any point in general.*

ANS. Take the co-ordinate axes as shown in the figure. Let $O'Z'$ be the fixed axis. Then $r = 2$ ft., $\bar{x} = 3$ ft., $\omega_y = +3$ radians per sec. Since the circle rolls, the bottom point b is on the instantaneous axis. The instantaneous axis must also pass through the point O'. Hence $O'b$ is the instantaneous axis, and OA parallel to it is the spontaneous axis. We have then



$$r\omega_y = -\bar{y}\omega_z,$$

or

$$\omega_z = -\frac{r\omega_y}{\bar{y}} = -2 \text{ radians per sec.}$$

ω_z is therefore negative, as shown in the figure. The velocity of translation of the centre is given by

$$\bar{v}_x = r\omega_y = +6 \text{ ft. per sec.}$$

The velocity at any point is that due to translation and angular velocity ω about the spontaneous axis OA, or to angular velocity ω only about the instantaneous axis $O'b$.

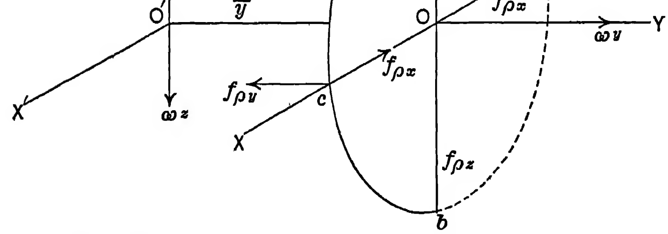
We have then at the top point a the velocity $v_x = \bar{v}_x + r\omega_y = +2r y = +12$ ft. per sec. At the bottom point b the velocity is $v_x = \bar{v}_x - r\omega_y = 0$. At the forward point c we have the component velocities $v_x = \bar{v}_x = r\omega_y = +6$ ft. per sec. $v_y = r\omega_z = -4$ ft. per sec., $v_z = -r\omega_y = -6$ ft. per sec. The resultant velocity is then $v = \sqrt{v_x^2 + v_y^2 + v_z^2} = 2\sqrt{22}$ ft. per sec., and its direction cosines are given by

$$\cos\alpha = \frac{v_x}{v} = +\frac{3}{\sqrt{22}}, \qquad \cos\beta = \frac{v_y}{v} = -\frac{2}{\sqrt{22}}, \qquad \cos\gamma = \frac{v_z}{v} = -\frac{3}{\sqrt{22}}.$$

At the rear point d we have the component velocities $v_x = \bar{v}_x = r\omega_y = +6$ ft. per sec., $v_y = -r\omega_z = +4$ ft. per sec., $v_z = +r\omega_y = +6$ ft per sec. The resultant velocity is, as before, $v = 2\sqrt{22}$ ft. per sec., and its direction cosines are

$$\cos\alpha = \frac{v_x}{v} = +\frac{3}{\sqrt{22}}, \qquad \cos\beta = \frac{v_y}{v} = +\frac{2}{\sqrt{22}}, \qquad \cos\gamma = \frac{v_z}{v} = +\frac{3}{\sqrt{22}}.$$

The central acceleration of any point is that due to angular velocity about OA *considered as without translation* (page 145) or to anglar velocity ω_y about OY and ω_z about OZ, without translation. In either case O and hence every point has acceleration $\bar{x}\omega_z^2$ towards $O'Z'$, owing to rotation about OZ'.



We have then at the top point a, the *deflecting* acceleration (page 80) $f_{rz} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. due to rotation about OY, the deviating acceleration (page 80) $f_{ay} = r\omega_y\omega_z = -12$ ft.-per-sec. per sec. due to rotation of the plane about OZ, and the deflecting acceleration $f_{ry} = -\bar{x}\omega_z^2 = -12$ ft.-per-sec. per sec. due to rotation about $O'Z'$. The component central accelerations are then $f_{\rho z} = -r\omega_y^2 = -18$ ft.-per-sec. per sec. and $f_{\rho y} = f_{ry} + f_{ay} = r\omega_y\omega_z - \bar{y}\omega_z^2 = -24$ ft.-per-sec. per sec. The resultant is then $f_\rho = \sqrt{f_{\rho z}^2 + f_{\rho y}^2} = 30$ ft.-per-sec. per sec., and its direction cosines are

$$\cos\alpha = 0, \qquad \cos\beta = \frac{f_{\rho y}}{f_\rho} = -0.8, \qquad \cos\gamma = \frac{f_{\rho z}}{f_\rho} = -0.6.$$

At the bottom point b the deflecting acceleration due to rotation about OY is $f_{rz} = +r\omega_y^2$. The deviating acceleration due to rotation of the plane about OZ is $f_{ay} = -r\omega_y\omega_z$. The deflecting acceleration due to rotation about $O'Z'$ is $f_{ry} = -x\omega_z^2$. The component central accelerations are then $f_{pz} = r\omega_y^2 = +18$ ft.-per-sec. per sec., $f_{py} = f_{ay} + f_{ry} = -r\omega_y\omega_z - \bar{y}\omega_z^2 = 0$. The resultant central acceleration is then f_{pz}.

At the forward point c the deflecting acceleration due to rotation about OY is $f_{rx} = -r\omega_y^2$, and due to rotation about $O'Z$, $f_{ry} = -\bar{y}\omega_z^2$. The deflecting acceleration due to rotation of the plane about OZ is $f_{rx} = -r\omega_z^2$. The component central accelerations are then $f_{px} = -r\omega_y^2 - r\omega_z^2 = -26$ ft.-per-sec. per sec., $f_{py} = -\bar{y}\omega_z^2 = -12$ ft.-per-sec. per sec. The resultant acceleration is then $f_p = \sqrt{f_{py}^2 + f_{px}^2} = 2\sqrt{205}$ ft.-per-sec. per sec., and its direction cosines are

$$\cos\alpha = \frac{f_{px}}{f_p} = -\frac{13}{\sqrt{205}}, \qquad \cos\beta = \frac{f_{py}}{f_p} = -\frac{6}{\sqrt{205}}, \qquad \cos\gamma = 0.$$

At the rear point d we have, in the same way, $f_{px} = +r\omega_y^2 + r\omega_z^2$, $f_{py} = -\bar{y}\omega_z^2$, or $f_{px} = +26$ ft.-per-sec. per sec., $f_{py} = -12$ ft.-per-sec. per sec., $f_p = 2\sqrt{205}$ ft.-per-sec. per sec., and the direction cosines are

$$\cos\alpha = +\frac{13}{\sqrt{205}}, \qquad \cos\beta = -\frac{6}{\sqrt{205}}, \qquad \cos\gamma = 0.$$

Let $+x$, $+y$, $+z$ be the co-ordinates of any point P. Then we have the component velocities

$$v_x = \bar{v}_x + z\omega_y = r\omega_y + z\omega_y, \qquad v_y = x\omega_z, \qquad v_z = -x\omega_y. \quad . \quad . \quad . \quad . \quad (1)$$

The resultant velocity is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{[(z+r)^2 + x^2]\omega_y^2 + x^2\omega_z^2} = \sqrt{r'^2\omega_y^2 + x^2\omega_z^2}, \quad . \quad . \quad . \quad (2)$$

where r' is the radius vector of the point from b.

The direction cosines of v are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v}, \qquad \cos\gamma = \frac{v_z}{v}. \quad . \quad . \quad . \quad . \quad (3)$$

These equations are the same as in the preceding example. For the component deviating accelerations due to rotation of the plane about OZ we have

$$f_{ax} = 0, \qquad f_{ay} = z\omega_y\omega_z, \qquad f_{az} = 0.$$

For the component deflecting accelerations due to rotation about OY, OZ, and $O'Z'$ we have

$$f_{rx} = -x\omega_y^2 - x\omega_z^2, \qquad f_{ry} = -\bar{x}\omega_z^2, \qquad f_{rz} = -z\omega_y^2.$$

Hence

$$f_{px} = -x\omega_y^2 - x\omega_z^2, \qquad f_{py} = z\omega_y\omega_z - \bar{x}\omega_z^2, \qquad f_{pz} = -z\omega_y^2. \quad . \quad . \quad . \quad . \quad (4)$$

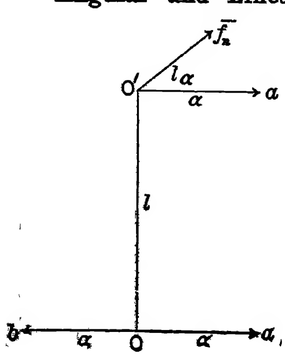
The resultant is

$$f_p = \sqrt{f_{px}^2 + f_{py}^2 + f_{pz}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and its direction cosines are

$$\cos\alpha = \frac{f_{px}}{f_p}, \qquad \cos\beta = \frac{f_{py}}{f_p}, \qquad \cos\gamma = \frac{f_{pz}}{f_p}. \quad . \quad . \quad . \quad . \quad (6)$$

These equations reduce to (4), (5), (6) of the preceding example if $\bar{x} = 0$.



Angular and Linear Acceleration Combined.—Let a rigid body have an angular acceleration α about an axis $O'a$ so that $O'a = \alpha$ is the line representative. Let O be any point of the body. If at this point we apply two equal and opposite angular accelerations $Oa = \alpha$ and $Ob = \alpha$, both parallel and equal to $O'a = \alpha$, the motion of the body is evidently not affected.

We see, then, that the single angular acceleration $O'a = \alpha$ about an axis $O'a$ can be reduced to the same angular acceleration $Oa = \alpha$ about a parallel axis Oa through any point O, and an angular acceleration couple $O'a$ and Ob.

Let l be the distance between the parallel axes. Then

(page 144) the couple causes acceleration of translation $\bar{f}_n = l\alpha$ at right angles to the plane of the couple, so that looking along the line representative of $\bar{f}_n$ in its direction, the arrows of the couple indicate clockwise rotation. In the figure $\bar{f}_n$ is at right angles to the plane of the couple and away from the reader. Hence

(*a*) A single angular acceleration α of a rigid body about a given axis can be resolved into an equal angular acceleration about a parallel axis through any point of the body at a distance l, and a normal acceleration of translation $\bar{f}_n = l\alpha$ of this axis in a direction at right angles to the plane of the two axes.

(*b*) Conversely, the resultant of an angular acceleration α of a rigid body about a given axis and a simultaneous acceleration of translation $\bar{f}_n$ normal to that axis, is a single equal angular acceleration about a parallel axis at a distance $l = \frac{\bar{f}_n}{\alpha}$, the plane of the two axes being perpendicular to $\bar{f}_n$.

(*c*) If, then, a rigid body has any number of angular accelerations, each one about a different axis through a different point, then by (*a*) we can reduce each one to an equal angular acceleration about an axis through any point we please, and a normal acceleration of translation of this axis.

All the angular accelerations at this point can then be reduced to a single resultant angular acceleration α about a resultant axis by the polygon of angular accelerations (page 83), and all the normal accelerations of translation can be reduced to a single resultant acceleration of translation $\bar{f}$ not necessarily normal to the resultant axis, by the polygon of linear accelerations (page 76).

The change of motion of a rigid body in general can then be reduced, at any instant, to an angular acceleration α about an axis through any point we please, and an acceleration of translation $\bar{f}$ of this axis. This acceleration $\bar{f}$ of translation is not necessarily normal to the axis.

The angular acceleration α has the same magnitude and direction no matter what point is taken, but the acceleration of translation $\bar{f}$ varies in magnitude and direction with the position of this point.

(*d*) This acceleration of translation $\bar{f}$ is not necessarily normal to the axis and can in general be resolved into a component $\bar{f}_a$ along the axis of α and a component $\bar{f}_n$ normal to this axis.

But by (*b*) we can reduce α and $\bar{f}_n$ to the same angular acceleration α about a parallel axis at a distance $l = \frac{\bar{f}_n}{\alpha}$.

INSTANTANEOUS AXIS OF ACCELERATION.—This axis is the INSTANTANEOUS AXIS OF ACCELERATION because it is the axis *without acceleration* about which at a given instant angular acceleration takes place.

Hence, in general, the change of motion of a rigid body at any instant can be reduced to an angular acceleration α about an axis through any point of the body, an acceleration of translation $\bar{f}_a$ along this axis, and a normal acceleration $\bar{f}_n$ of translation of this axis. Or to an angular acceleration α about a parallel instantaneous axis at a distance $l = \frac{\bar{f}_n}{\alpha}$ and an acceleration of translation $\bar{f}_a$ along this axis.

Twist—Screw-Twist.—Angular acceleration of a rigid body about any axis we call a TWIST. Angular acceleration about any axis together with acceleration of translation along that axis we call a SCREW-TWIST.

Hence the change of motion of a rigid body at any instant can be reduced in general to a *twist* or *screw-twist about an instantaneous axis;* or to a *twist* or *screw-twist about a parallel axis through any point together with a normal acceleration of translation* $\bar{f}_n$ *of that axis.*

SPONTANEOUS AXIS OF ACCELERATION.—The axis of acceleration through the centre of mass of a rigid body at any instant is called the spontaneous axis of acceleration. If it has normal acceleration of translation $\bar{f}_n$, the instantaneous axis of translation is parallel to it at the distance $l = \frac{\bar{f}_n}{\alpha}$, the plane of these two axes being perpendicular to $\bar{f}_n$ (page 151).

The acceleration of any point is that due to angular acceleration about the instantaneous axis of acceleration, or angular acceleration about the parallel translating spontaneous axis (page 151).

If the spontaneous axis of acceleration has no normal acceleration of translation $\bar{f}_n$, the spontaneous and instantaneous axes coincide.

CHAPTER II.

ROTATION AND TRANSLATION—ANALYTIC RELATIONS.

Components of Motion.—We have seen (page 144) that the *motion* of a rigid body at any instant can be reduced to a resultant angular velocity ω about an axis through any point of the body and a resultant velocity of translation $\bar{v}$ of this axis or of the body.

The motion of the body at any instant is then known, if we know at that instant the velocity of translation of the axis through some given point of the body and the angular velocity of the body about this axis.

We always take the given point *at the centre of mass* of the body, and denote the components of the velocity of translation $\bar{v}$ of the axis through it, along the co-ordinate axes, by $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$, and the components of the angular velocity w by ω_x, ω_y, ω_z.

The motion of the body at any instant is then known when these six quantities are known:

$$\bar{v}_x,\ \bar{v}_y,\ \bar{v}_z,\ \omega_x,\ \omega_y,\ \omega_z.$$

These six quantities are therefore called the COMPONENTS OF MOTION of the body.

Components of Change of Motion.—We have seen (page 151) that the *change* of motion of a rigid body at any instant can be reduced to a resultant angular acceleration α about an axis through any point of the body and a resultant acceleration of translation $\bar{f}$ of this axis.

The change of motion of the body at any instant is then known, if we know at that instant the acceleration of translation of the axis through some given point of the body and the angular acceleration of the body about this axis.

We always take the given point *at the centre of mass* of the body, and denote the components of the acceleration of translation $\bar{f}$ of the axis through it, along the co-ordinate axes, by $\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z$, and the components of the angular acceleration α by α_x, α_y, α_z.

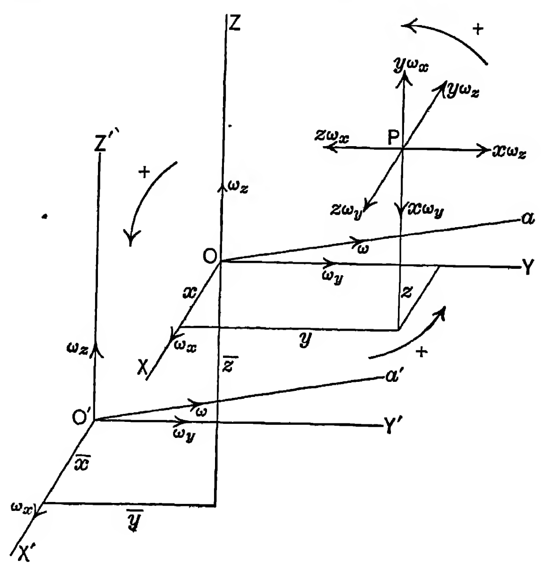
The change of motion of the body at any instant is then known when these six quantities are known:

$$\bar{f}_x,\ \bar{f}_y,\ \bar{f}_z,\ \alpha_x,\ \alpha_y,\ \alpha_z.$$

These six quantities are therefore called the COMPONENTS OF CHANGE OF MOTION of the body.

Motion of a Point of a Rigid Body—General Analytic Equations.—Let O be the centre of mass of a rigid body. Take any co-ordinate axes X, Y, Z through the centre of mass O as origin, and let the co-ordinates of any point P of the body relative to O be x, y, z.

Let the point P have the angular velocity ω about any axis Oa through the centre of mass



O, and let the components of ω along the co-ordinate axes be ω_x, ω_y, ω_z. Let the axes X, Y, Z be *fixed in the body and move with it.*

ROTATION—CENTRE OF MASS FIXED.—Let the centre of mass O be fixed. Then $z\omega_y$ is the velocity of P parallel to OX in a positive direction due to angular velocity ω_y about OY, and $y\omega_z$ is the velocity parallel to OX in a negative direction due to angular velocity ω_z about OZ. In the same way we have, parallel to OY, $x\omega_z$ positive and $z\omega_x$ negative, and parallel to OZ, $y\omega_x$ positive and $x\omega_y$ negative.

If, then, v_x, v_y, v_z are the component velocities of the velocity v of the point P due to rotation about the axis Oa through the fixed point O, we have *for rotation only* about an axis Oa through the fixed centre of mass

$$v_x = z\omega_y - y\omega_z, \qquad v_y = x\omega_z - z\omega_x, \qquad v_z = y\omega_x - x\omega_y. \quad . \quad . \quad . \quad . \quad (1)$$

ROTATION—ANY POINT FIXED.—Let any point O' of the body be fixed, and take the co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$, *parallel to* OX, OY, OZ. Let the point P have the angular velocity ω about an axis $O'a'$ through the fixed point O', and let the components of ω along the co-ordinate axes X', Y', Z' be ω_x, ω_y, ω_z. Let the co-ordinates of the centre of mass O relative to O' be $\bar{x}$, $\bar{y}$, $\bar{z}$. Then the co-ordinates of any point P will be $\bar{x} + x$, $\bar{y} + y$, $\bar{z} + z$, where x, y, z are the co-ordinates of P relative to the centre of mass O.

We have then, from (1), for the component velocities of the velocity v of the point P *due to rotation only* about an axis $O'a'$ through any fixed point O' of the body

$$v_x = (\bar{z} + z)\omega_y - (\bar{y} + y)\omega_z, \quad v_y = (\bar{x} + x)\omega_z - (\bar{z} + z)\omega_x, \quad v_z = (\bar{y} + y)\omega_x - (\bar{x} + x)\omega_y. \quad (2)$$

If, in these equations, we make $x = 0$, $y = 0$, $z = 0$, we have for the component velocities $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ of the centre of mass O due to rotation about the axis $O'a'$ through the fixed point O'

$$\bar{v}_x = \bar{z}\omega_y - \bar{y}\omega_z, \qquad \bar{v}_y = \bar{x}\omega_z - \bar{z}\omega_x, \qquad \bar{v}_z = \bar{y}\omega_x - \bar{x}\omega_y. \quad . \quad . \quad . \quad . \quad (3)$$

ROTATION ABOUT TRANSLATING AXIS THROUGH CENTRE OF MASS.—Let the axis Oa through the centre of mass O have the velocity of translation $\bar{v}$ in any direction, and let $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ be the components of $\bar{v}$. The point P will have, then, component velocities of translation in addition to the component velocities given by equations (1) for rotation only about an axis through the fixed centre of mass. Hence, for rotation and translation combined, we have the component velocities of P:

$$v_x = \bar{v}_x + z\omega_y - y\omega_z, \qquad v_y = \bar{v}_y + x\omega_z - z\omega_x, \qquad v_z = \bar{v}_z + y\omega_x - x\omega_y. \quad . \quad . \quad (4)$$

As we have seen, page 145, the motion of a body in general can be reduced to angular velocity about an axis through the centre of mass and a velocity of translation of the body Equations (4), then, are general and include all cases.

Thus if $\bar{v}_x = 0$, $\bar{v}_y = 0$, $\bar{v}_z = 0$, we have rotation only about an axis through the fixed centre of mass, as given by equations (1); if in addition we put $\bar{x} + x$, $\bar{y} + y$, $\bar{z} + z$ for x, y, z, we have equations (2) for rotation only about an axis through any fixed point. If we put $\bar{x}$, $\bar{y}$, $\bar{z}$ for x, y, z, we have equations (3) for the component velocities of the centre of mass due to rotation about an axis through any fixed point

RESULTANT VELOCITY.—In all cases the resultant velocity of P is given by

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Its line representative passes through P, and its direction cosines are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v}, \qquad \cos\gamma = \frac{v_z}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

RESULTANT ANGULAR VELOCITY.—The resultant angular velocity is

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Its line representative passes through the centre of mass O, and its direction cosines are

$$\cos\alpha = \frac{\omega_x}{\omega}, \qquad \cos\beta = \frac{\omega_y}{\omega}, \qquad \cos\gamma = \frac{\omega_z}{\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

MOMENT OF VELOCITY.—Let the moment of the velocity of P about the axis $O'a'$ through any point O' be M'_v. Then for the component moments M'_{vx}, M'_{vy}, M'_{vz} about the axes X', Y', Z' of the component velocities v_x, v_y, v_z of the point P due to translation and rotation we have

$$\left.\begin{array}{ll} \text{about} & O'X' \ldots\ldots M'_{vx} = v_z(\bar{y}+y) - v_y(\bar{z}+z), \\ \text{``} & O'Y' \ldots\ldots M'_{vy} = v_x(\bar{z}+z) - v_z(\bar{x}+x), \\ \text{``} & O'Z' \ldots\ldots M'_{vz} = v_y(\bar{x}+x) - v_x(\bar{y}+y), \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where v_x, v_y, v_z are given by equations (4). Substituting these values, we have for rotation and translation

$$\left.\begin{array}{l} M'_{vx} = \bar{v}_z(\bar{y}+y) - \bar{v}_y(\bar{z}+z) + y(\bar{y}+y)\omega_x - x(\bar{y}+y)\omega_y - x(\bar{z}+z)\omega_z + z(\bar{z}+z)\omega_x, \\ M_{vy} = \bar{v}_x(\bar{z}+z) - \bar{v}_z(\bar{x}+x) + z(\bar{z}+z)\omega_y - y(\bar{z}+z)\omega_z - y(\bar{x}+x)\omega_x + x(\bar{x}+x)\omega_y, \\ M'_{vz} = \bar{v}_y(\bar{x}+x) - \bar{v}_x(\bar{y}+y) + x(\bar{x}+x)\omega_z - z(\bar{x}+x)\omega_x - z(\bar{y}+y)\omega_y + y(\bar{y}+y)\omega_z. \end{array}\right\} (10)$$

For rotation only $\bar{v}_x = 0$, $\bar{v}_y = 0$, $\bar{v}_z = 0$. For axis through centre of mass $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$.

VELOCITY ALONG THE AXIS OF ROTATION.—For the velocity of translation $\bar{v}_a$ along the axis of rotation we have

$$\bar{v}_a = \bar{v}_x \cos\alpha + \bar{v}_y \cos\beta + \bar{v}_z \cos\gamma,$$

or, from (8),

$$\bar{v}_a = \frac{\bar{v}_x\omega_x + \bar{v}_y\omega_y + \bar{v}_z\omega_z}{\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is called the *axial velocity* or *velocity of advance*. It is the velocity with which the body moves along the axis of rotation.

VELOCITY NORMAL TO THE AXIS OF ROTATION.—The components, along the axes, of the axial velocity are

$$\bar{v}_a \cos\alpha, \qquad \bar{v}_a \cos\beta, \qquad \bar{v}_a \cos\gamma.$$

If we subtract these from the components $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ of the velocity $\bar{v}$ of translation, we have the components of the velocity of translation $\bar{v}_n$ normal to the axis of rotation:

$$\left.\begin{array}{l} \bar{v}_{nx} = \bar{v}_x - \bar{v}_a \cos\alpha = \bar{v}_x - \dfrac{\bar{v}_a\omega_x}{\omega}, \\[2ex] \bar{v}_{ny} = \bar{v}_y - \bar{v}_a \cos\beta = \bar{v}_y - \dfrac{\bar{v}_a\omega_y}{\omega}, \\[2ex] \bar{v}_{nz} = \bar{v}_z - \bar{v}_a \cos\gamma = \bar{v}_z - \dfrac{\bar{v}_a\omega_z}{\omega}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Equations (7), (8) and (11) give the *screw-spin* (page 145) about the axis of rotation through the centre of mass. Equations (7) and (8) give the angular velocity ω about the axis of rotation and the direction of this axis, and equations (11) give the velocity $\overline{v}_a$ along this axis. Equations (12) give the components of the velocity of translation normal to this axis.

INSTANTANEOUS AXIS OF ROTATION.—Let x, y, z be the co-ordinates of any point of the instantaneous axis of rotation. Since the normal velocity of any point of this axis is zero, we have, from equations (4) (page 154),

$$\overline{v}_{nx} + z\omega_y - y\omega_z = 0, \qquad \overline{v}_{ny} + x\omega_z - z\omega_x = 0, \qquad \overline{v}_{nz} + y\omega_x - x\omega_y = 0. \quad . \quad (13)$$

These are the equations of the projections of the instantaneous axis of rotation on the three co-ordinate planes.

Let x_1, y_1, z_1 be the intercepts of these projections on the co-ordinate axes. Then in equations (13), making $y = 0$ and $z = 0$ in the last two, we have

$$x_1 = \frac{\overline{v}_{nz}}{\omega_y} = -\frac{\overline{v}_{ny}}{\omega_z};$$

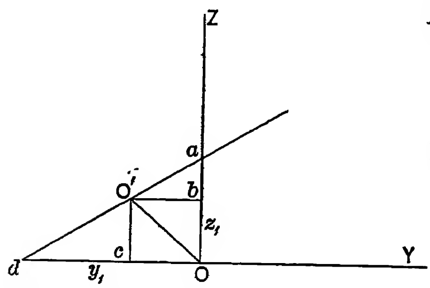
making $x = 0$ and $z = 0$ in the first and last,

$$y_1 = \frac{\overline{v}_{nx}}{\omega_z} = -\frac{\overline{v}_{nz}}{\omega_x}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

making $y = 0$ and $x = 0$ in the first two,

$$z_1 = \frac{\overline{v}_{ny}}{\omega_x} = -\frac{\overline{v}_{nx}}{\omega_y}.$$

Let the perpendicular from the centre of mass O upon the axis of rotation be p, and let p_x, p_y, p_z be its projections on the co-ordinate axes. Let the intersection of p with the axis be O', so that $OO' = p$.



Let us consider the projection of the axis on the plane YZ. In the figure $p_z = Ob$ and $p_y = Oc$. We have, then,

$$p_y = \frac{z_1}{y_1} p_z,$$

where z_1 and y_1 are the intercepts Oa and Od given by equations (14). We have also the distance

$$ab = \frac{z_1}{y_1} p_y.$$

Hence

$$p_z = z_1 - ab = z_1 - \frac{z_1}{y_1} p_y.$$

Substituting the value of p_y, we obtain

$$p_z = \frac{z_1 y_1^2}{z_1^2 + y_1^2}, \qquad p_y = \frac{z_1^2 y_1}{z_1^2 + y_1^2}.$$

Substituting the values of z_1 and y_1 from (14), we have

$$p_z = -\frac{\overline{v}_{nx}\omega_y}{\omega_z^2 + \omega_y^2}, \qquad p_y = \frac{\overline{v}_{nx}\omega_z}{\omega_z^2 + \omega_y^2}.$$

In the same way we can find p_z and p_x, p_x and p_y on the other two co-ordinate planes. We thus have

$$\left.\begin{aligned} p_x &= -\frac{\overline{v}_{ny}\omega_z}{\omega_x^2+\omega_z^2} = \frac{\overline{v}_{nz}\omega_y}{\omega_y^2+\omega_x^2}, \qquad p_y = -\frac{\overline{v}_{nz}\omega_x}{\omega_y^2+\omega_x^2} = \frac{\overline{v}_{nx}\omega_z}{\omega_z^2+\omega_y^2}, \\ p_z &= -\frac{\overline{v}_{nx}\omega_y}{\omega_z^2+\omega_y^2} = \frac{\overline{v}_{ny}\omega_x}{\omega_x^2+\omega_z^2}. \end{aligned}\right\} \quad . \ . \ . \quad (15)$$

Equations (15) give the position of the instantaneous axis of rotation.

We have also, from (13) and (8),

$$\left.\begin{aligned} \overline{v}_x &= \overline{v}_a \cos\alpha - \omega(p_z\cos\beta - p_y\cos\gamma), & \omega_x &= \omega\cos\alpha, \\ \overline{v}_y &= \overline{v}_a \cos\beta - \omega(p_x\cos\gamma - p_z\cos\alpha), & \omega_y &= \omega\cos\beta, \\ \overline{v}_z &= \overline{v}_a \cos\gamma - \omega(p_y\cos\alpha - p_x\cos\beta), & \omega_z &= \omega\cos\gamma. \end{aligned}\right\} \quad . \ . \ . \quad (16)$$

When, therefore, the components of motion, $\overline{v}_x$, $\overline{v}_y$, $\overline{v}_z$, ω_x, ω_y, ω_z, are given for the centre of mass O, we have ω from (7), the direction of the instantaneous axis of rotation from (8), and the position of this axis from (15). We have also the velocity v_a along this axis from (11), and the normal velocity $\overline{v}_n$ from (12).

On the other hand, if the position and direction of the instantaneous axis of rotation are given, together with the velocity $\overline{v}_a$ along it and the angular velocity ω about it, the components of motion are given by equations (16).

THE INVARIANT FOR COMPONENTS OF MOTION.—From (11) we have

$$\overline{v}_a\omega = \overline{v}_x\omega_x + \overline{v}_y\omega_y + \overline{v}_z\omega_z. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

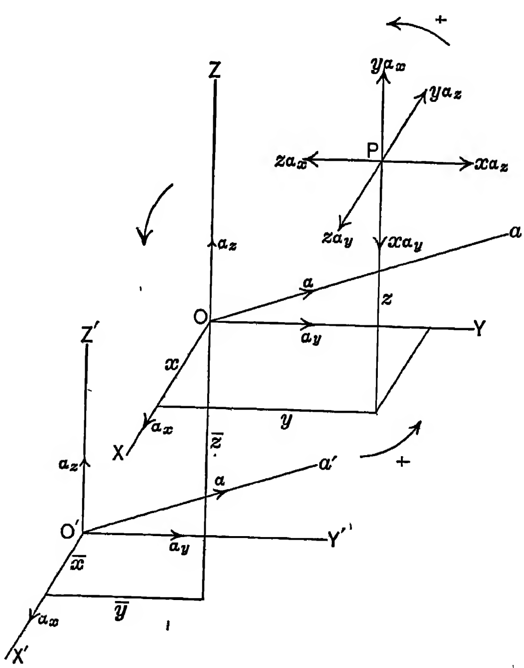
But whatever point we take as origin, ω does not change, and the velocity v_a along the instantaneous axis of rotation does not change. This quantity (17) is therefore called the *invariant of the components of motion.*

If the invariant is zero in any case, we must evidently have either v_a or ω zero. If ω is not zero but the invariant is zero, then the velocity $\overline{v}_a$ along the axis must be zero. In this case the condition

$$\overline{v}_x\omega_x + \overline{v}_y\omega_y + \overline{v}_z\omega_z = 0$$

is the condition for a spin, or angular velocity only, about the instantaneous axis of rotation.

If $\overline{v}_a$ is not zero but the invariant is zero, then ω must be zero, and we have translation only.



Change of Motion of a Point of a Rigid Body—General Analytic Equations.—Let O be the centre of mass of a rigid body. Take any co-ordinate axes X, Y, Z through the centre of mass O, and let the co-ordinates of any point P be x, y, z.

Let the point P have angular acceleration α about any axis Oa, and let the components of α along the co-ordinate axes be α_x, α_y, α_z.

Let the axes X, Y, Z *be fixed in the body and move with it.*

TANGENT ACCELERATION. — Let the axis Oa be fixed. Then $z\alpha_y$ is the tangent acceleration of P parallel to OX in a positive direction due to angular acceleration α_y about OY, and $y\alpha_z$ is the tangent acceleration parallel to OX in a negative direction due to angular acceleration α_z about OZ. In the same way we have, parallel to OY, $x\alpha_z$ positive and $z\alpha_x$ negative, and parallel to OZ, $y\alpha_x$ positive and $x\alpha_y$ negative.

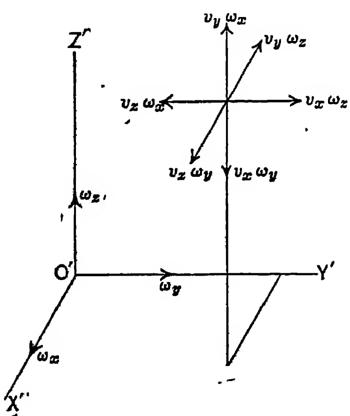
If, then, f_{tx}, f_{ty}, f_{tz} are the component accelerations of the tangent acceleration f_t of the point P due to angular acceleration α about the fixed axis Oa, we have

$$f_{tx} = z\alpha_y - y\alpha_z, \qquad f_{ty} = x\alpha_z - z\alpha_x, \qquad f_{tz} = y\alpha_x - x\alpha_y.$$

For any other axis through any point O' we have $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$ in place of x, y, z, and hence

$$f_{tx} = (\bar{z}+z)\alpha_y - (\bar{y}+y)\alpha_z, \quad f_{ty} = (\bar{x}+x)\alpha_z - (\bar{z}+z)\alpha_x, \quad f_{tz} = (\bar{y}+y)\alpha_x - (\bar{x}+x)\alpha_y. \quad (1)$$

These equations are general. For axis through the centre of mass we have $\bar{x}=0$, $\bar{y}=0$, $\bar{z}=0$.



CENTRAL ACCELERATION.—The central acceleration is due to rotation only, and is not affected by translation. Let, then, v_x, v_y, v_z be the component velocities of the point P *due to rotation only* about an axis $O'a'$ through any point O'.

Then $v_z\omega_y$ is the central acceleration of P parallel to $O'X'$ in a positive direction, and $v_y\omega_z$ is the central acceleration parallel to $O'X'$ in a negative direction. In the same way we have, parallel to $O'Y'$, $v_x\omega_z$ positive and $v_z\omega_x$ negative, and parallel to $O'Z'$, $v_y\omega_x$ positive and $v_x\omega_y$ negative.

If, then, $f_{\rho x}$, $f_{\rho y}$, $f_{\rho z}$ are the component central accelerations of the point P due to rotation only about an axis $O'a'$ through any point O', we have

$$f_{\rho x} = v_z\omega_y - v_y\omega_z, \qquad f_{\rho y} = v_x\omega_z - v_z\omega_x, \qquad f_{\rho z} = v_y\omega_x - v_x\omega_y,$$

where v_x, v_y, v_z are given by equations (1) page 154.

Substituting these values of v_x, v_y, v_z, we have for axis through centre of mass

$$\begin{aligned} f_{\rho x} &= y\omega_x\omega_y + z\omega_x\omega_z - x\omega_y^2 - x\omega_z^2, \\ f_{\rho y} &= z\omega_y\omega_z + x\omega_y\omega_x - y\omega_z^2 - y\omega_x^2, \\ f_{\rho z} &= x\omega_z\omega_x + y\omega_z\omega_y - z\omega_x^2 - z\omega_y^2. \end{aligned}$$

For axis through any point O' we have only to put $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$ in place of x, y, z, and we have then, in general,

$$\left.\begin{aligned} f_{\rho x} &= (\bar{y}+y)\omega_x\omega_y + (\bar{z}+z)\omega_x\omega_z - (\bar{x}+x)\omega_y^2 - (\bar{x}+x)\omega_z^2, \\ f_{\rho y} &= (\bar{z}+z)\omega_y\omega_z + (\bar{x}+x)\omega_y\omega_x - (\bar{y}+y)\omega_z^2 - (\bar{y}+y)\omega_x^2, \\ f_{\rho z} &= (\bar{x}+x)\omega_z\omega_x + (\bar{y}+y)\omega_z\omega_y - (\bar{z}+z)\omega_x^2 - (\bar{z}+z)\omega_y^2. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

These equations are general for axis through any point. For axis through centre of mass we have $\bar{x}=0$, $\bar{y}=0$, $\bar{z}=0$.

DEFLECTING AND DEVIATING ACCELERATIONS.—We see, from page 80, that in equations (2) all terms containing ω_x^2, ω_y^2, ω_z^2 give the component *deflecting* accelerations, the other terms give the component deviating accelerations. If, then, f_r is the deflecting or radial, and f_a the deviating or axial, acceleration, we have for the components of the deflecting acceleration

$$\left.\begin{aligned} f_{rx} &= -(\bar{x}+x)\omega_y^2 - (\bar{x}+x)\omega_z^2, \\ f_{ry} &= -(\bar{y}+y)\omega_z^2 - (\bar{y}+y)\omega_x^2, \\ f_{rz} &= -(\bar{z}+z)\omega_x^2 - (\bar{z}+z)\omega_y^2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

and for the components of the deviating acceleration

$$\left.\begin{aligned} f_{ax} &= (\bar{y}+y)\omega_x\omega_y + (\bar{z}+z)\omega_x\omega_z, \\ f_{ay} &= (\bar{z}+z)\omega_y\omega_z + (\bar{x}+x)\omega_y\omega_x, \\ f_{az} &= (\bar{x}+x)\omega_z\omega_x + (\bar{y}+y)\omega_z\omega_y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

All these equations are general. For axis through centre of mass we have $\bar{x}=0$, $\bar{y}=0$, $\bar{z}=0$.

RESULTANT ACCELERATION.—Let the *centre of mass* O have the acceleration of translation $\bar{f}$ and the components $\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z$. Then the components of the resultant acceleration f are

$$\begin{aligned} f_x &= \bar{f}_x + f_{\rho x} + f_{tx} = \bar{f}_x + f_{ax} + f_{rx} + f_{tx}, \\ f_y &= \bar{f}_y + f_{\rho y} + f_{ty} = \bar{f}_y + f_{ay} + f_{ry} + f_{ty}, \\ f_z &= \bar{f}_z + f_{\rho z} + f_{tz} = \bar{f}_z + f_{az} + f_{rz} + f_{tz}; \end{aligned}$$

or substituting the values of f_{ax}, f_{ay}, f_{az}, f_{rx}, f_{ry}, f_{rz}, f_{tx}, f_{ty}, f_{tz} from equations (4), (3) and (1), we have

$$\left.\begin{aligned} f_x &= \bar{f}_x + (\bar{y}+y)\omega_x\omega_y + (\bar{z}+z)\omega_x\omega_z - (\bar{x}+x)\omega_y^2 - (\bar{x}+x)\omega_z^2 + (\bar{z}+z)\alpha_y - (\bar{y}+y)\alpha_z, \\ f_y &= \bar{f}_y + (\bar{z}+z)\omega_y\omega_z + (\bar{x}+x)\omega_y\omega_x - (\bar{y}+y)\omega_z^2 - (\bar{y}+y)\omega_x^2 + (\bar{x}+x)\alpha_z - (\bar{z}+z)\alpha_x, \\ f_z &= \bar{f}_z + (\bar{x}+x)\omega_z\omega_x + (\bar{y}+y)\omega_z\omega_y - (\bar{z}+z)\omega_x^2 - (\bar{z}+z)\omega_y^2 + (\bar{y}+y)\alpha_x - (\bar{x}+x)\alpha_y. \end{aligned}\right\} (5)$$

These equations are general. For axis through centre of mass we have $\bar{x}=0$, $\bar{y}=0$, $\bar{z}=0$. In any case the resultant acceleration of P is given by

$$f = \sqrt{f_x^2 + f_y^2 + f_z^2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Its line representative passes through P, and its direction cosines are

$$\cos\alpha = \frac{f_x}{f}, \qquad \cos\beta = \frac{f_y}{f}, \qquad \cos\gamma = \frac{f_z}{f}. \quad . \quad . \quad . \quad . \quad (7)$$

RESULTANT ANGULAR ACCELERATION.—The resultant angular acceleration is given by

$$\alpha = \sqrt{\alpha_x^2 + \alpha_y^2 + \alpha_z^2}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Its line representative passes through O, and its direction cosines are

$$\cos\alpha = \frac{\alpha_x}{\alpha}, \qquad \cos\beta = \frac{\alpha_y}{\alpha}, \qquad \cos\gamma = \frac{\alpha_z}{\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

MOMENT OF ACCELERATION.—Let the moment of the acceleration of P about the axis $O'a'$ through any point O' be M'_f. Then for the component moments M'_{fx}, M'_{fy}, M'_{fz} about the axes X', Y', Z' we have

$$\text{about } O'X' \ldots\ldots M'_{fx} = f_z(\bar{y}+y) - f_y(\bar{z}+z),$$

$$\text{“} \quad O'Y' \ldots\ldots M'_{fy} = f_x(\bar{z}+z) - f_z(\bar{x}+x),$$

$$\text{“} \quad O'Z' \ldots\ldots M_{fz} = f_y(\bar{x}+x) - f_x(\bar{y}+y),$$

where f_x, f_y, f_z are given by equations (5). Substituting these values, we have for rotation and translation

$$\left.\begin{aligned}
M'_{fx} = {} & f_z(\bar{y}+y) - \bar{f}_y(\bar{z}+z) + (\bar{y}+y)(\bar{x}+x)(\omega_z\omega_x - \alpha_y) - (\bar{z}+z)(\bar{x}+x)(\omega_y\omega_x + \alpha_z) \\
& + (\bar{z}+z)(\bar{y}+y)(\omega_z^2 - \omega_y^2) + (\bar{y}+y)^2(\omega_z\omega_y + \alpha_x) - (\bar{z}+z)^2(\omega_y\omega_z - \alpha_x), \\
M'_{fy} = {} & f_x(\bar{z}+z) - \bar{f}_z(\bar{x}+x) + (\bar{z}+z)(\bar{y}+y)(\omega\ \omega_y - \alpha_z) - (\bar{x}+x)(\bar{y}+y)(\omega_z\omega_y + \alpha_x) \\
& + (\bar{x}+x)(\bar{z}+z)(\omega_x^2 - \omega_z^2) + (\bar{z}+z)^2(\omega_x\omega_z + \alpha_y) - (\bar{x}+x)^2(\omega_z\omega_x - \alpha_y), \\
M'_{fz} = {} & f_y(\bar{x}+x) - \bar{f}_x(\bar{y}+y) + (\bar{x}+x)(\bar{z}+z)(\omega_y\omega_z - \alpha_x) - (\bar{y}+y)(\bar{z}+z)(\omega_x\omega_z + \alpha_y) \\
& + (\bar{y}+y)(\bar{x}+x)(\omega_y^2 - \omega_x^2) + (\bar{x}+x)^2(\omega_y\omega_x + \alpha_z) - (\bar{y}+y)^2(\omega_x\omega_y - \alpha_z).
\end{aligned}\right\} \quad (10)$$

For rotation only $\bar{f}_x = 0$, $\bar{f}_y = 0$, $\bar{f}_z = 0$. For axis through centre of mass $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$.

ACCELERATION ALONG THE AXIS.—For the acceleration of translation $\bar{f}_a$ along the axis of angular acceleration we have

$$\bar{f}_a = \bar{f}_x \cos\alpha + \bar{f}_y \cos\beta + \bar{f}_z \cos\gamma,$$

or, from (9),

$$\bar{f}_a = \frac{\bar{f}_x\alpha_x + \bar{f}_y\alpha_y + \bar{f}_z\alpha_z}{\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

This is called the *axial acceleration* or the *acceleration of advance*. It is the acceleration with which the body moves along the axis of angular acceleration.

ACCELERATION NORMAL TO THE AXIS OF ANGULAR ACCELERATION.—The components along the co-ordinate axes of the axial acceleration are

$$\bar{f}_a \cos\alpha, \qquad \bar{f}_a \cos\beta, \qquad \bar{f}_a \cos\gamma.$$

If we subtract these from the components $\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z$, we have the components of the acceleration of translation $\bar{f}_n$ normal to the axis of angular acceleration,

$$\left.\begin{aligned}
\bar{f}_{nx} &= \bar{f}_x - \bar{f}_a \cos\alpha = \bar{f}_x - \bar{f}_a \frac{\alpha_x}{\alpha}, \\
\bar{f}_{ny} &= \bar{f}_y - \bar{f}_a \cos\beta = \bar{f}_y - \bar{f}_a \frac{\alpha_y}{\alpha}, \\
\bar{f}_{nz} &= \bar{f}_z - \bar{f}_a \cos\gamma = \bar{f}_z - \bar{f}_a \frac{\alpha_z}{\alpha}.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Equations (8), (9) and (11) give the *screw-twist* about the axis of angular acceleration through the centre of mass. Equations (8) and (9) give the angular acceleration and the direction of the axis of angular acceleration, and equation (11) gives the acceleration $\overline{f}_a$ along this axis. Equations (12) give the components of the acceleration of translation normal to this axis.

INSTANTANEOUS AXIS OF ACCELERATION.—Let x, y, z be the co-ordinates of any point of the instantaneous axis. Since the normal acceleration for any point of this axis is zero, we have

$$\overline{f}_{nx} + z\alpha_y - y\alpha_z = 0, \qquad \overline{f}_{ny} + x\alpha_z - z\alpha_x = 0, \qquad \overline{f}_{nz} + y\alpha_x - x\alpha_y = 0. \qquad (13)$$

These are the equations of the projections of the instantaneous axis of acceleration on the three co-ordinate planes.

Let x_1, y_1, z_1 be the intercepts of these projections on the co-ordinate axes. Then, in equations (13), making $y = 0$ and $z = 0$ in the last two, we have

$$x_1 = \frac{\overline{f}_{nz}}{\alpha_y} = -\frac{\overline{f}_{ny}}{\alpha_z};$$

making $x = 0$ and $z = 0$ in the first and last,

$$y_1 = \frac{\overline{f}_{nx}}{\alpha_z} = -\frac{\overline{f}_{nz}}{\alpha_x};$$

making $y = 0$ and $x = 0$ in the first two,

$$z_1 = \frac{\overline{f}_{ny}}{\alpha_x} = -\frac{\overline{f}_{nx}}{\alpha_y}. \qquad (14)$$

Let the perpendicular from the centre of mass O upon the axis of angular acceleration be p, and let p_x, p_y, p_z be its projections on the co-ordinate axes. Let the intersection of p with the axis be O', so that $OO' = p$. Then, just as on page 157, we have

$$\left.\begin{aligned} p_x &= -\frac{\overline{f}_{ny}\alpha_z}{\alpha_x^2 + \alpha_z^2} = \frac{\overline{f}_{nz}\alpha_y}{\alpha_y^2 + \alpha_x^2}, \qquad p_y = -\frac{\overline{f}_{nz}\alpha_x}{\alpha_y^2 + \alpha_x^2} = \frac{\overline{f}_{nx}\alpha_z}{\alpha_z^2 + \alpha_y^2}, \\ p_z &= -\frac{\overline{f}_{nx}\alpha_y}{\alpha_z^2 + \alpha_y^2} = \frac{\overline{f}_{ny}\alpha_x}{\alpha_x^2 + \alpha_z^2} \end{aligned}\right\} \qquad (15)$$

We have also, from (13) and (9),

$$\left.\begin{aligned} \overline{f}_x &= \overline{f}_a \cos\alpha - \alpha(p_z \cos\beta - p_y \cos\gamma), & \alpha_x &= \alpha\cos\alpha, \\ \overline{f}_y &= \overline{f}_a \cos\beta - \alpha(p_x \cos\gamma - p_z \cos\alpha), & \alpha_y &= \alpha\cos\beta, \\ \overline{f}_z &= \overline{f}_a \cos\gamma - \alpha(p_y \cos\alpha - p_x \cos\beta), & \alpha_z &= \alpha\cos\gamma, \end{aligned}\right\} \qquad (16)$$

When, therefore, the components of change of motion $\overline{f}_x$, $\overline{f}_y$, $\overline{f}_z$, α_x, α_y, α_z are given for the centre of mass O, we have α from (8), the direction of the instantaneous axis of acceleration from (9), and the position of this axis from (15). We have also the acceleration $\overline{f}_a$ along the axis from (11), and the normal acceleration $\overline{f}_n$ from (12).

On the other hand, if the position and direction of the instantaneous axis of acceleration are given, together with the acceleration $\overline{f}_a$ along it and the angular acceleration α about it, the components of change of motion are given by (16).

THE INVARIANT FOR COMPONENTS OF CHANGE OF MOTION.—From (11) we have

$$\overline{f}_a\alpha = \overline{f}_x\alpha_x + \overline{f}_y\alpha_y + \overline{f}_z\alpha_z, \qquad (17)$$

But whatever point we take as origin, α does not change and the acceleration $\overline{f}_a$ along the instantaneous axis of acceleration does not change. This quantity (17) is therefore called the *invariant of the components of change of motion.*

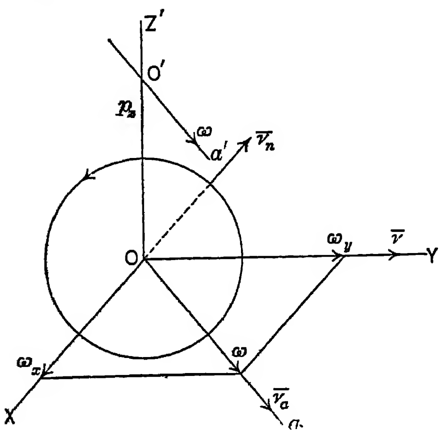
If in any case the invariant is zero, we must evidently have either $\overline{f}_a$ or α zero. If α is not zero but the invariant is zero, then the acceleration $\overline{f}_a$ along the axis must be zero. In this case the condition

$$\overline{f}_x\alpha_x + \overline{f}_y\alpha_y + \overline{f}_z\alpha_z = 0$$

is the condition for a twist, or angular acceleration only about the instantaneous axis of acceleration.

If $\overline{f}_a$ is not zero but the invariant is zero, then α must be zero and we have no angular acceleration.

Examples.—(1) *A baseball rotates about an axis through its centre of mass, with angular velocity ω, and its centre of mass has a velocity $\overline{v}$ making an angle with ω. Find the resultant motion.*



ANS. Take $\overline{v}$ coinciding with the axis of Y, and the plane of $\overline{v}$ and ω as the plane of XY.

The components of motion are

$$\overline{v}_x = 0, \quad \overline{v}_y = \overline{v}, \quad \overline{v}_z = 0, \quad \overline{v}_a = \overline{v}\cos\beta,$$
$$\omega_x = \omega\cos\alpha, \quad \omega_y = \omega\cos\beta, \quad \omega_z = 0,$$

where α and β are the angles of ω with the axes of X and Y.

For the position of the instantaneous axis of rotation we have, from equations (12), page 155, and equations (15), page 157,

$$\overline{v}_{nx} = -\overline{v}\cos\beta, \quad \overline{v}_{ny} = \overline{v} - \overline{v}\cos^2\beta, \quad \overline{v}_{nz} = 0,$$
$$p_x = 0, \quad p_y = 0, \quad p_z = \frac{\overline{v}_{ny}}{\omega_x} = \frac{\overline{v}\cos\alpha}{\omega} = \frac{\overline{v}_a}{\omega},$$

where $\overline{v}_a$ is the component of the velocity along ω, or $\overline{v}_a = \overline{v}\cos\alpha$.

The motion of the ball is then a screw-spin consisting at any instant of rotation ω about an axis $O'a'$ parallel to the axis of rotation at the centre of mass O at the distance p_z from the centre of mass, and a velocity of translation $\overline{v}_a$ along this axis. Or rotation ω about the axis Oa, through the centre of mass, velocity $\overline{v}_a$ along this axis and $\overline{v}_n$ of translation of this axis, where $\overline{v}_n = -\overline{v}\cos\beta$.

Disregarding the action of gravity, the centre of mass O moves in the resultant of $\overline{v}_a$ and $\overline{v}_n$, or along OY with uniform velocity $\overline{v}$.

Since, owing to gravity, the ball falls vertically, the centre of mass O moves in a vertical curve, the plane of which intersects the plane XY in a straight line OY.

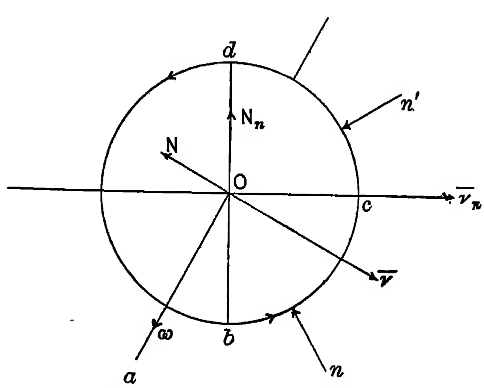
(2) *Ball-players assert that this intersection is not a straight line, but a curve. Show how this can be.*

ANS. No account has been taken of the resistance of the air.

1st. CURVATURE NORMAL TO THE PLANE OF ω AND $\overline{v}$.—Owing to rotation ω about the axis Oa and the velocity $\overline{v}_n$ of translation of this axis, the velocity of all points of the advancing quadrant Obc will be greater than for corresponding points of the receding quadrant Ocd.

The normal pressure n on the advancing quadrant Obc will then be greater than the normal pressure n' on the receding quadrant Ocd.

The resultant pressure N at the centre O makes an angle with $\overline{v}_n$ and its component N_n normal to $\overline{v}_n$ and to the plane of ω, and $\overline{v}$ is always away from the advancing quadrant Obc and causes curvature in this direction.

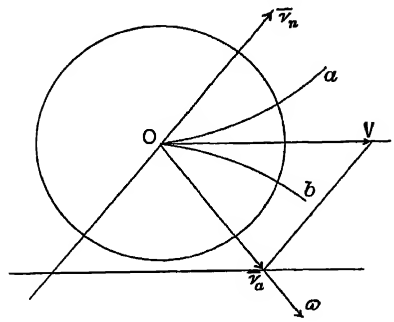


It is generally supposed that the curvature is due to the ball rolling upon a cushion of compressed air

in front of it, but, as we see, the direction of N_n *is always opposite to the direction in which the ball would thus tend to roll.*

2d. CURVATURE IN THE PLANE OF ω AND $\overline{v}$.—The air causes a retardation of $\overline{v}_a$ and $\overline{v}_n$, that is of translation along the axis and translation of the axis.

If these retardations were proportional to the velocities there would be no curvature in the plane of ω and $\overline{v}$.



But these retardations are more nearly proportional to the squares of the velocities. Hence the greater component is retarded proportionally more than the lesser, and we have curvature in the direction of the lesser. If, then, $\overline{v}$ makes an angle with ω less than 45°, we have curvature Oa in the direction of $\overline{v}_n$; if greater than 45°, we have curvature Ob in the direction of $\overline{v}_a$.

Thus by giving the ball a spin the pitcher is able to make it curve right or left *in the plane* of ω and $\overline{v}$, and at the same time a curve at right angles to this plane has also been proved.

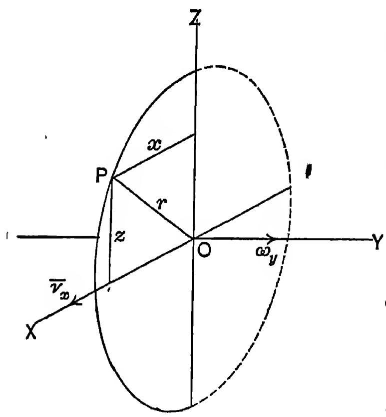
We have, then, an acceleration $\overline{f}$ right or left in the plane, and an acceleration $\overline{f_n}$ parallel to N_n in the first figure at right angles to this plane.

The curvature will always, then, be opposite in direction to the direction in which the ball would tend to roll on the forward cushion of compressed air. So far as any such action exists it decreases the curvature.

If, then, $\overline{v}$ is at right angles to ω, or makes an angles of 45° with it, there is no curvature in the plane of $\overline{v}$ and ω at that instant, but in all cases and at every instant there is curvature at right angles to this plane opposite to the direction in which the ball would tend to roll on the forward cushion of compressed air, and diminished more or less by this tendency.

(3) *A vertical circle of radius $r = 2$ ft. rotates about a fixed horizontal axis through the centre of mass at right angles to the plane of the circle with angular velocity of 3 radians per sec. Find the velocity and central acceleration for any point in general.*

ANS. (Compare with example (1), page 146.) Take co-ordinate axes as shown in the figure. Then $r = 2$ ft., $\omega = \omega_y = +3$ radians per sec., $\omega_x = 0$, $\omega_z = 0$, $\overline{v}_x = 0$, $\overline{v}_y = 0$ $\overline{v}_z = 0$. The components of motion, then, are known. Let x and z be the co-ordinates of any point P. For all points $y = 0$. Then, from equations (1), page 154, we have



$$v_x = z\omega_y, \qquad v_y = 0, \qquad v_z = -x\omega_y. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The resultant velocity is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = r\omega_y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and its direction cosines are

$$\cos\alpha = \frac{v_x}{v} = +\frac{z}{r}, \qquad \cos\beta = \frac{v_y}{v} = 0, \qquad \cos\gamma = \frac{v_z}{v} = -\frac{x}{r}. \quad . \quad . \quad (3)$$

These are the same equations as on page 146.

From equations (15), page 157, we see that the axis OY is the instantaneous axis, or the spontaneous and instantaneous axes of rotation coincide. From equations (17), page 157, we see that the invariant is zero, and we have a spin only about the instantaneous axis

From equations (2), page 158, we have for the component central accelerations, since $\overline{x} = 0$, $\overline{y} = 0$, $\overline{z} = 0$, and $y = 0$,

$$f_{px} = -x\omega_y^2, \qquad f_{py} = 0, \qquad f_{pz} = z\omega_y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The resultant central acceleration is

$$f_p = \sqrt{f_{px}^2 + f_{py}^2 + f_{pz}^2} = r\omega_y^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

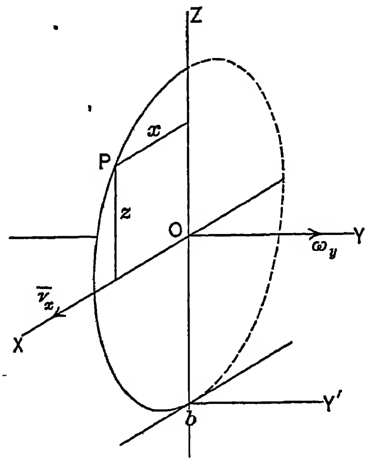
and its direction cosines are

$$\cos\alpha = \frac{f_{px}}{f_p} = -\frac{x}{r}, \qquad \cos\beta = \frac{f_{py}}{f_p} = 0, \qquad \cos\gamma = \frac{f_{pz}}{f_p} = -\frac{z}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These are the same equations as in example (1), page 146 From equations (4), page 159, we see that the component deviating accelerations are $f_{ax} = 0$, $f_{ay} = 0$, $f_{az} = 0$, or the plane of rotation does not change in direction.

(4) *A vertical circle of radius $r = 2$ ft. rolls on a horizontal straight line. The centre moves parallel to that line with a velocity of 6 ft. per sec. Find the angular velocity and the velocity and central acceleration at any point.*

ANS. (Compare example (2), page 146.) Take co-ordinate axes as shown in the figure. Then $r = 2$ ft. and the components of motion are



$$\bar{v}_x = +6 \text{ ft. per sec.,} \qquad \bar{v}_y = 0, \qquad \bar{v}_z = 0.$$
$$\omega_x = 0, \qquad \omega_y = \omega, \qquad \omega_z = 0.$$

Since the circle rolls, $r\omega_y = \bar{v}_x$, and hence

$$\omega_y = \frac{v_x}{r} = +3 \text{ radians per sec.}$$

From equations (4), page 154, the component velocities of any point P whose co-ordinates are x, z, are

$$v_x = \bar{v}_x + z\omega_y = r\omega_y + z\omega_y, \qquad v_y = 0, \qquad v_z = -x\omega_y. \quad . \quad . \quad (1)$$

The resultant velocity is

$$v = \sqrt{v_x^2 + v_y^2} = \omega_y\sqrt{(z+r)^2 + x^2} = r'\omega_y, \quad . \quad . \quad . \quad (2)$$

where r' is the radius vector of the point relative to the bottom point b.

The direction cosines of v are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v} = 0, \qquad \cos\gamma = \frac{v_z}{v}. \quad . \quad . \quad . \quad (3)$$

These are the same equations as in example (2), page 146.

From equations (15), page 151, we have for the co-ordinates of the instantaneous axis of rotation

$$p_x = 0, \qquad p_y = 0, \qquad p_z = -\frac{v_x}{\omega_y} = -r.$$

The axis bY' through the bottom point b parallel to OY is then the instantaneous axis of rotation.

From equation (17), page 157, we see that the invariant is zero and we have a spin only about the instantaneous axis.

From equations (2), page 158, we have for the component central accelerations, since $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$ and $y = 0$,

$$f_{px} = -x\omega_y^2, \qquad f_{py} = 0, \qquad f_{pz} = -z\omega_y^2. \quad . \quad . \quad . \quad (4)$$

The resultant is

$$f_p = \sqrt{f_{px}^2 + f_{pz}^2} = r\omega_y^2, \quad . \quad . \quad . \quad (5)$$

and its direction cosines are

$$\cos\alpha = \frac{f_{px}}{f_p} = -\frac{x}{r}, \qquad \cos\beta = \frac{f_{py}}{f_p} = 0, \qquad \cos\gamma = \frac{f_{pz}}{f_p} = -\frac{z}{r}. \quad . \quad . \quad . \quad (6)$$

These are the same equations as in example (2), page 146.

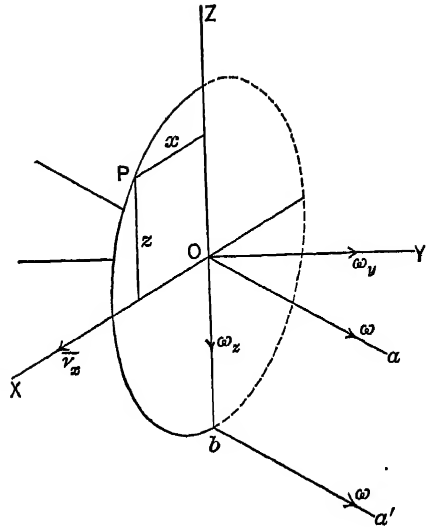
From equations (4), page 159, we see that the component deviating deflections are $f_{ax} = 0$, $f_{ay} = 0$, $f_{az} = 0$, or the plane of rotation does not change in direction.

(5) *Let a vertical circle of radius $r = 2$ ft roll on a horizontal plane. The centre moves with a velocity of 6 ft. per sec. At the same time let the plane of the circle rotate about a vertical diameter with an angular velocity of 2 radians per sec. downwards. Find the angular velocity about the horizontal axis, and the velocity and central acceleration of any point.*

ANS. (Compare example (3), page 147.) Take co-ordinate axes as shown in the figure. Then $r = 2$ ft. and the components of motion are $\overline{v}_x = +6$ ft. per sec., $\overline{v}_y = 0$, $\overline{v}_z = 0$, $\omega_x = 0$, ω_y; $\omega_z = -2$ radians per sec.

Since the circle rolls,

$$r\omega_y = \overline{v}_x, \quad \text{or} \quad \omega_y = \frac{v_x}{r} = +3 \text{ radians per sec.}$$



From equations (4), page 154, the component velocities of any point P whose co-ordinates are x, z, are

$$v_x = \overline{v}_x + z\omega_y = r\omega_y + z\omega_y, \quad v_y = x\omega_z, \quad v_z = -x\omega_y. \quad (1)$$

The resultant velocity is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{[(z+r)^2 + x^2]\omega_y^2 + x^2\omega_z^2}$$
$$= \sqrt{r'^2\omega_y^2 + x^2\omega_z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where r' is the radius vector of the point relative to the bottom point b.

The direction cosines of v are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v}, \qquad \cos\gamma = \frac{v_z}{v}. \quad . \quad . \quad (3)$$

These are the same equations as in example (3), page 147.

From equations (15), page 157, we have for the co-ordinates of the instantaneous axis of rotation

$$p_x = 0, \quad p_y = \frac{r\omega_y\omega_z}{\omega_y^2 + \omega_z^2} = -\frac{12}{13} \text{ ft.}, \quad p_z = -\frac{r\omega_y^2}{\omega_y^2 + \omega_z^2} = -\frac{18}{13} \text{ ft.}$$

The instantaneous axis passes through this point and is parallel to the spontaneous axis Oa. It therefore passes through the bottom point b.

From equations (17), page 157, we see that the invariant is zero and we have a spin only about the instantaneous axis.

From equations (2), page 158, we have for the component central accelerations, since $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$ and $y = 0$,

$$f_{px} = -x\omega_y^2 - x\omega_z^2, \qquad f_{py} = z\omega_y\omega_z, \qquad f_{pz} = -z\omega_y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The resultant is

$$f_p = \sqrt{f_{px}^2 + f_{py}^2 + f_{pz}^2} = \sqrt{r^2\omega_y^4 + z^2\omega_y^2\omega_z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and its direction cosines are

$$\cos\alpha = \frac{f_{px}}{f_p}, \qquad \cos\beta = \frac{f_{py}}{f_p}, \qquad \cos\gamma = \frac{f_{pz}}{f_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These are the same equations as in example (3), page 147.

From equations (3), page 159, we have the component deflecting accelerations

$$f_{rx} = -x\omega_y^2 - x\omega_z^2, \qquad f_{ry} = 0, \qquad f_{rz} = -z\omega_y^2,$$

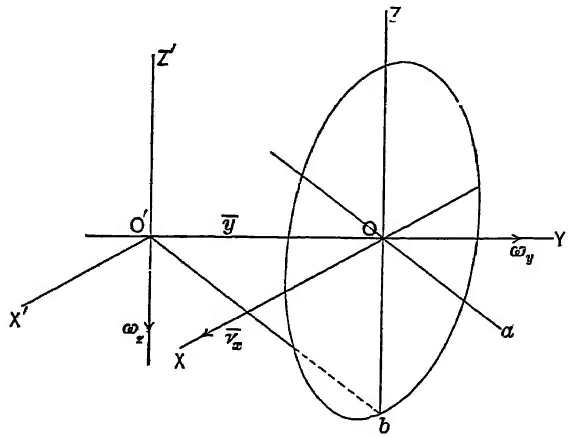
and from equations (4), page 159, we have the component deviating accelerations

$$f_{ax} = 0, \qquad f_{ay} = z\omega_y\omega_z, \qquad f_{az} = 0.$$

The plane of rotation therefore changes in direction.

(6) *Let a vertical circle of radius $r = 2$ ft. roll on a horizontal plane with an angular velocity of 3 radians per sec., while its centre describes a horizontal circle of radius 3 ft. with angular velocity about a fixed axis. Find the angular velocity about the fixed axis; the velocity and central acceleration of any point.*

ANS. (Compare example (4), page 149.) Take co-ordinate axes as shown in the figure. Let $O'Z'$ be the fixed axis Then $r = 2$ ft., $\bar{y} = 3$ ft., and the components of motion are $\bar{v}_x = +6$ ft. per sec., $\bar{v}_y = 0$, $\bar{v}_z = 0$, $\omega_x = 0$, $\omega_y = +3$ radians per sec., ω_z.



Since the circle rolls and the centre rotates about $O'Z'$, we have

$$r\omega_y = \bar{v}_x = -\bar{y}\omega_z \text{ or } \omega_z = -\frac{r\omega_y}{\bar{x}} = -2 \text{ radians.}$$

ω_z is therefore negative as shown in the figure.

From equations (2), page 154, the component velocities of any point P whose co-ordinates are x, z, are, since $\bar{x} = 0$, $\bar{z} = 0$ and $y = 0$,

$$\left.\begin{array}{ll} v_x = z\omega_y - \bar{y}\omega_z = r\omega_y + z\omega_y, & v_y = x\omega_z, \\ v_z = -x\omega_y. & \end{array}\right\} \quad (1)$$

The resultant velocity is

$$v = \sqrt{v_x + v_y^2 + v_z^2} = \sqrt{r'^2\omega_y^2 + x^2\omega_z^2}, \quad \ldots \quad (2)$$

where r' is the radius vector of the point from the bottom point b. The direction cosines of v are

$$\cos\alpha = \frac{v_x}{v}, \qquad \cos\beta = \frac{v_y}{v}, \qquad \cos\gamma = \frac{v_z}{v}. \quad \ldots \quad (3)$$

These equations are the same as in example (4), page 149.

From equations (15), page 157, we have for the co-ordinates of the instantaneous axis of rotation

$$p_x = 0, \qquad p_y = \frac{r\omega_y\omega_z}{\omega_y^2 + \omega_z^2} = -\frac{12}{13} \text{ ft.,} \qquad p_z = -\frac{r\omega_y^2}{\omega_y^2 + \omega_z^2} = -\frac{18}{13} \text{ ft.}$$

The instantaneous axis passes through this point and is parallel to the spontaneous axis Oa. It therefore passes through O' and the bottom point b.

From equations (17), page 157, we see that the invariant is zero and we have a spin only about the instantaneous axis.

From equations (2), page 158, we have for the component central accelerations, since $\bar{x} = 0$, $\bar{z} = 0$ and $y = 0$,

$$f_{\rho x} = -x\omega_y^2 - x\omega_z^2, \qquad f_{\rho y} = -\bar{y}\omega_z^2 + z\omega_y\omega_z, \qquad f_{\rho z} = -z\omega_y^2. \quad \ldots \quad (4)$$

The resultant is

$$f = \sqrt{f_{\rho x}^2 + f_{\rho y}^2 + f_{\rho z}^2}, \quad \ldots \quad (5)$$

and its direction cosines are

$$\cos\alpha = \frac{f_{\rho x}}{f_\rho}, \qquad \cos\beta = \frac{f_{\rho y}}{f_\rho}, \qquad \cos\gamma = \frac{f_{\rho z}}{f_\rho}. \quad \ldots \quad (6)$$

These are the same equations as in example (4), page 149

From equations (3), page 159, we have the component deflecting accelerations

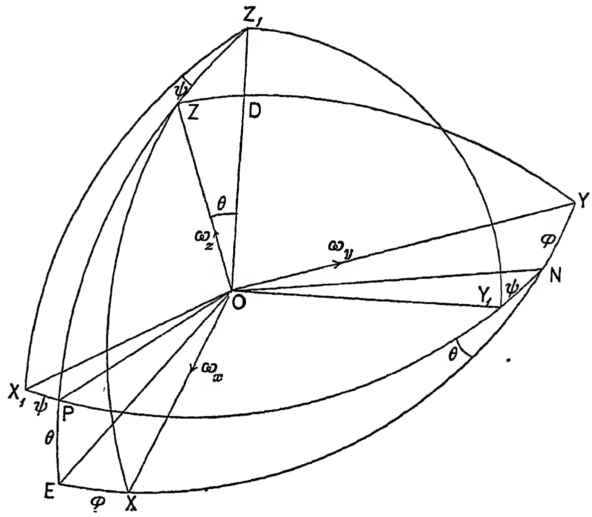
$$f_{rx} = -x\omega_y^2 - x\omega_z^2, \qquad f_{ry} = -\bar{y}\omega_z^2, \qquad f_{rz} = -z\omega_y^2,$$

and from equations (4), page 159, we have the component deviating accelerations

$$f_{ax} = 0, \qquad f_{ay} = z\omega_y\omega_z, \qquad f_{az} = 0.$$

The plane of rotation therefore changes in direction.

Euler's Geometric Equations.—Let OX, OY, OZ be rectangular co-ordinate axes *fixed in the body* and therefore rotating with it, and let the body rotate about some axis *fixed in the body* and therefore making invariable angles with these axes, so that the component angular velocities are ω_x, ω_y, ω_z.



Let OX_1, OY_1, OZ_1 be rectangular co-ordinate axes whose *directions in space are invariable*. For instance, the axis OZ_1 may be always directed towards the north pole or parallel to the earth's axis, then X_1Y_1 is the plane of the celestial equator.

Let the point O be taken as the centre of a sphere of radius r. Let X_1, Y_1, Z_1, X, Y, Z be the points in which this sphere is pierced by the fixed and moving axes.

Let the axes OX, OY, OZ have the initial positions OX_1, OY_1, OZ_1. Turn the body,
1st, about OZ_1 through the angle $X_1Z_1P = \psi$, so that OX_1 moves to OP, and OY_1 to ON.
2d, about ON through the angle $PNE = \theta$, so that OP moves to OE, and OZ_1 to OZ.
3d, about OZ through the angle $EOX = \phi$, so that OE moves to OX, and ON to OY.

It is required to find the geometric relations between ϕ, θ, ψ and ω_x, ω_y, ω_z. These geometric relations are called EULER'S GEOMETRIC EQUATIONS.

The line ON is called the *line of nodes*, θ is the *obliquity* and ψ the *precession*.

The angular velocity of Z perpendicular to the plane of ZOZ_1 or about OZ_1 at any instant is $\frac{d\psi}{dt}$. This is called the *angular velocity of precession*. The angular velocity of Z along ZZ_1 or about ON is at the same instant $\frac{d\theta}{dt}$. This is called the *angular velocity of nutation*. The angular velocity of X relative to E, or Y relative to N, at the same instant is $\frac{d\phi}{dt}$.

Draw ZD perpendicular to OZ_1. Then $ZD = r \sin \theta$, and the linear velocity at any instant of Z perpendicular to the plane of ZOZ_1 is $r \sin \theta \cdot \frac{d\psi}{dt}$, and along ZZ_1 at the same instant it is $r\frac{d\theta}{dt}$. The linear velocity at the same instant of Z along YZ is $r\omega_x$, and along ZX it is $r\omega_y$.

We have, then, directly from the figure,

$$r\frac{d\theta}{dt} = r\omega_y \cos \phi + r\omega_x \sin \phi,$$

$$r \sin \theta \cdot \frac{d\psi}{dt} = r\omega_y \sin \phi - r\omega_x \cos \phi,$$

or, since r cancels out,

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_y \cos \phi + \omega_x \sin \phi, \\ \sin \theta \cdot \frac{d\psi}{dt} &= \omega_y \sin \phi - \omega_x \cos \phi. \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

Combining these two equations, we have

$$\left.\begin{aligned} \omega_x &= \frac{d\theta}{dt}\cdot\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi, \\ \omega_y &= \frac{d\theta}{dt}\cdot\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have also the linear velocity of E perpendicular to the plane of Z_1OE, equal to $r\cos\theta\cdot\frac{d\psi}{dt}$, and of X relative to E along EX, $r\frac{d\phi}{dt}$. We have, then, for the velocity of X along XY

$$r\omega_z = r\frac{d\psi}{dt}\cos\theta + r\frac{d\phi}{dt},$$

or

$$\omega_z = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (2) and (3) are Euler's Geometric Equations. They give the relations between ω_x, ω_y, ω_z and the angular velocity $\frac{d\psi}{dt}$ of OZ about OZ', $\frac{d\theta}{dt}$ of OZ about ON, the line of nodes, $\frac{d\phi}{dt}$ of OY relative to ON.

Auxiliary Angles.—From the spherical angles of the figure, page 167, considering N as a vertex in each, we have for the direction cosines of the moving axes, with reference to the fixed,

$$\left.\begin{aligned} \cos XOX_1 &= -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, \\ \cos YOX_1 &= -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta, \\ \cos ZOX_1 &= \sin\theta\cos\psi, \\ \cos XOY_1 &= \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta, \\ \cos YOY_1 &= \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, \\ \cos ZOY_1 &= \sin\theta\sin\psi, \\ \cos XOZ_1 &= -\sin\theta\cos\phi, \\ \cos YOZ_1 &= \sin\theta\sin\phi, \\ \cos ZOZ_1 &= \cos\theta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the angles which the axes Z_1, Z and ON make with the axes X, Y and Z we have

$$\left.\begin{aligned} \cos Z_1OX &= -\cos\phi\sin\theta, \\ \cos Z_1OY &= \sin\phi\sin\theta, \\ \cos Z_1OZ &= \cos\theta, \\ \cos ZOX &= 0, \\ \cos ZOY &= 0, \\ \cos ZOZ &= 1, \\ \cos NOX &= \sin\phi, \\ \cos NOY &= \cos\phi, \\ \cos NOZ &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

DYNAMICS. GENERAL PRINCIPLES.

CHAPTER I.

FORCE. NEWTON'S LAWS OF MOTION.

Dynamics.—We have given in the preceding pages the principles of KINEMATICS, or the measurable relations of space and time only, that is of pure motion. But we have to deal in nature with *material bodies* and *force*.

That science which treats of the measurable relations of force and of those measurable relations of matter, space and time involved in the study of the change of motion of bodies due to force, is called DYNAMICS (*δύναμις*, *force*).

Material Particle.—We have already seen (page 20) that, whatever the constitution of matter may be, we can consider a body as composed of an indefinitely large number of indefinitely small PARTICLES, so small that each may be treated as a point.

We denote the mass of such a particle by m, and the sum of the masses of all the particles of a body or the entire mass of a body by $\bar{\mathrm{m}}$, so that

$$\bar{\mathrm{m}} = \Sigma m.$$

Impressed Force.—It is a fact of universal experience that no particle of matter is able of itself to change its own motion. If, then, a particle is at rest it must remain at rest unless acted upon from without. If it is moving at any instant in a given direction with a given speed, it cannot change either its speed or direction, that is, *its velocity is uniform*, unless acted upon from without. This action from without to which the change of velocity in any case can always be attributed we call IMPRESSED FORCE.

Newton's First Law of Motion.—A body is a collection of particles. If there are no impressed forces, each particle must then be at rest or move with uniform speed in a straight line, and hence the body itself has *motion of translation in a straight line.*

This fact was expressed by Newton as follows:

Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled to change that state by impressed forces.

This is known as "Newton's first law of motion." It implicitly defines force as that which causes change of motion of matter.

Inertia.—We may also express this law by saying that all matter is *inert*, that is, has no power of itself to change its state of rest or motion. This property of matter we call INERTIA, and Newton's first law we may call the "law of inertia." We recognize, then, not only extension and impenetrability, but also inertia as essential properties of matter. That is, matter occupies space, two bodies cannot occupy the same space at the same time, and all matter is inert.

Force proportional to Acceleration.—Acceleration f of a point we have already illustrated and defined (page 76) as time-rate of change of velocity when the interval of time is indefinitely small. We can only measure force by its effects. These effects are apparently different in different cases, but in all cases when analyzed they are found to consist either in change of velocity of particles or changes of form or volume of a body. As change of form or volume of a body is due to change of relative position and therefore change of velocity of the particles, we see that in all cases the effect of force is to cause acceleration. The impressed force on a particle must then be proportional to the acceleration f of the particle, *and have the same direction.*

Force proportional to Mass.—Consider a body composed of a number N of particles, and let the acceleration of each particle be f and in the same direction, so that the body has motion of translation. Then the force on each particle is proportional to f and the entire force on the body is proportional to Nf *and in the direction of* f. But the number N of particles is proportional to the entire mass $\bar{\mathbf{m}}$ of the body (page 20). The force on the body is then proportional to the mass as well as the acceleration.

We have, then, for a particle of mass m or a translating body of mass $\bar{\mathbf{m}}$ the impressed force F in the direction of f and given by

$$F = c\bar{\mathbf{m}}f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c is a constant.

Unit of Force.—Equation (1) expresses the fact that force is proportional both to mass and to acceleration.

We see from (1) that we shall always have

$$F = \bar{\mathbf{m}}f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if we take $c =$ equal to unity and

$$[F] = [m] \times [f].$$

That is, equation (2) holds, provided we take as our unit of force *that uniform force which will give one unit of mass one unit of acceleration in the direction of the force.*

This is called "*Gauss's absolute unit*" or the "absolute unit of force," because it furnishes a standard force in any system, independent of the force of gravity at different localities.

In the foot-pound-second or "F. P. S." system, then, the absolute unit of force is that uniform force which will give a mass of one lb. a change of velocity in the direction of the force of one ft.-per-sec. in a second. This has been called by Prof. James Thompson the POUNDAL. It is, then, the English absolute unit of force.

The French absolute unit of force is that uniform force which will give one kilogram a change of velocity in the direction of the force of one metre-per-second per second.

In the centimeter-gram-second or "C. G. S." system the absolute unit of force is the uniform force which will give one gram a change of velocity in the direction of the force of one centimeter-per-sec. per sec. This is called the DYNE.

Weight of a Body.—The weight of any body is the force with which the earth attracts it. This force must vary, then, with the acceleration g due to gravity, and this, as we have seen (page 100), varies with the locality. The weight of a body, then, varies with the locality, while its mass of course remains invariable.

If, then, $\bar{\mathbf{m}}$ is the mass of a body and W its weight, we have from (2), for the weight in absolute units,

$$W = \mathbf{m}g \text{ absolute units.}$$

But if $\overline{\text{m}}$ is taken as one unit of mass, then W is numerically equal to g, or

ONE LB. WEIGHS g POUNDALS,

ONE GRAM WEIGHS g DYNES,

according to the system we use.

Since g (page 100) is about 32 ft.-per-sec. per sec., the average weight of one lb. is about 32 poundals, or

ONE POUNDAL IS THE WEIGHT OF ABOUT HALF AN OUNCE.

Strictly speaking, it is the weight of $\frac{1}{g}$th part of a lb., where g must be taken for the locality.

Example.—An athlete throwing a hammer of 100 lbs. at New Haven and at Edinburgh throws a heavier weight at the latter place, by about 4 poundals, or the weight of 2 ounces more. The mass is the same and would "weigh" the same on an equal armed balance in both places. But a spring-balance would show a greater strtech at Edinburgh.

Gravitation Unit of Force.—We have seen (page 170) that the absolute unit of force is that force which will give one unit of mass one unit of acceleration. In the F. P. S. system this is the poundal; in the C. G. S. system the dyne. It is the unit used in physical measurements and generally when small quantities are of importance and great accuracy is desired.

In ordinary mechanical problems this unit is inconveniently small. Also very great accuracy is not a requisite. It is therefore usual in mechanics to express a force at any locality by comparing it with the weight of the unit of mass at that locality. The weight of the unit of mass *at the locality* is then taken as the unit of force. Such a unit is evidently not constant, but varies with the locality. The variation is so small, however, that it can be disregarded in ordinary mechanical problems.

Thus if the mass of a translating body is $\overline{\text{m}}$ lbs. and the acceleration of the centre of mass is $\bar{f}$, we have for the force, if $\bar{f}$ is taken in ft.-per-sec. per sec.,

$$F = \overline{\text{m}}\bar{f} \textit{ poundals.}$$

But since at any locality where the acceleration of gravity is g the weight of one lb. is g poundals, if we take this as the unit of force we have

$$F = \frac{\overline{\text{m}}\bar{f}}{g} \textit{ pounds,}$$

where g has a variable value according to the locality. The result is that for the same mass $\overline{\text{m}}$ and acceleration $\bar{f}$ we have a slightly varying force according to locality, which is of course absurd, strictly speaking. The variation, however, being small, is disregarded.

Some writers avoid this objection by taking a constant value for g, say $32\frac{1}{6}$. That is, the weight of the unit of mass *at some prescribed locality* is taken as the unit of force. This is indeed a constant force, but still a force *which depends upon locality*, whereas the absolute unit is independent of locality, as it should be.

Whichever method we adopt, either the weight of the unit of mass at any locality or at some prescribed locality, this weight is called the GRAVITATION UNIT OF FORCE and is expressed in "*pounds*."

When, then, we speak of a "force of ten pounds" or a force of ten kilograms," we mean the force of gravity at a given place upon a mass of ten lbs. or ten kilograms. The expression is strictly incorrect, because "pound" and "kilogram" denote mass and not force.

The expression is thus a brief and allowable locution for the phrase "force of attraction of the earth for a mass of ten lbs." at a specified locality.

A "force of ten pounds" means, then, a force of $10g$ poundals, where g is the acceleration of gravity in feet-per-sec. per sec. at the place considered. A "force of ten grams" means a force of $10g$ dynes, where g is the acceleration of gravity in centimeters-per-sec. per sec. at the place considered.

In all cases, then,

$$F = \overline{\mathrm{m}}\bar{f}$$

gives force in poundals or dynes according to the system used, and

$$F = \frac{\overline{\mathrm{m}}\bar{f}}{g}$$

gives force in gravitation measure, that is in "pounds" or "grams." Or, expressed in words instead of symbols,

MASS (*in lbs.*) $\times$ ACCELERATION (*in ft.-per-sec. per sec.*) = FORCE IN DIRECTION OF ACCELERATION (*in poundals*).

If we divide the force thus found by g in ft.-per-sec. per sec., we obtain the force in gravitation units (*pounds*) for the locality for which g is taken.

MASS (*in grams*) $\times$ ACCELERATION (*in centimeters-per-sec. per sec.*) = FORCE IN DIRECTION OF ACCELERATION (*in dynes*).

If we divide the force thus found by g in centimeters-per-sec. per sec., we obtain the force in gravitation units (*grams*) for the locality for which g is taken.

Thus if a mass of 25 lbs. has an acceleration in any direction of 6.4 ft.-per-sec per sec., the force in that direction is $25 \times 6.4 = 160$ *poundals*, or 160 times the force necessary to give a mass of one lb. an acceleration of one foot per sec. in a second. This is a definite and constant force no matter what the locality. If g for any given locality is 32 ft.-per-sec. per sec., we can speak of this as a force of $\frac{25 \times 6.4}{32}$ pounds or a "*force of 5 pounds*," meaning the force of gravity upon a mass of 5 lbs. at the locality for which g is 32.

Momentum and Impulse.—We have seen (page 76) that the acceleration f in any direction is equal to the limiting time-rate of change of velocity in that direction. Thus if v_1 is the initial and v the final velocity in any given direction in the indefinitely small time dt, we have for the acceleration *in that direction*

$$f = \frac{v - v_1}{dt}.$$

If, then, $\overline{\mathrm{m}}$ is the mass of the translating body, the force is

$$F = \frac{\overline{\mathrm{m}}(v - v_1)}{dt}, \quad \text{or} \quad Fdt = \overline{\mathrm{m}}(v - v_1). \qquad (3)$$

The product of the mass m of a particle or the mass $\overline{\mathrm{m}}$ of a translating body by its velocity v in any direction is called its MOMENTUM *in that direction.*

The product of a force F in any direction by the indefinitely small time dt is called the IMPULSE in that direction.

Newton's Second Law of Motion.—Newton expressed the relation of equation (3) as follows:

Change of motion is proportional to the motive force, and takes place in the direction of the straight line in which the force acts.

By "motion" Newton here refers, not to velocity but to mass-velocity, or what we have just designated as "momentum," and his "change of motion" is change of momentum. This law, then, is the statement of equation (3), and, using modern terms, we can restate it as follows:

Change of momentum in any direction is proportional to the impressed force in that direction, and equal to the impulse of the impressed force in that direction.

The first law defines force. The second law tells us how to measure force.

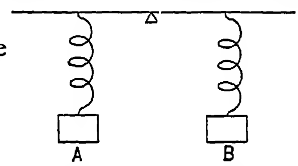
Measurement of Mass.—Newton's second law also tells us how to measure mass. Thus if f is the acceleration in any direction of a translating body of mass $\bar{\mathrm{m}}$, the impressed force F in the direction of the acceleration is

$$F = \bar{\mathrm{m}}f.$$

If, then, any two bodies are known to be acted upon by the same force and have the same acceleration, their masses must be equal.

Equal masses are those to which the same force gives the same acceleration.

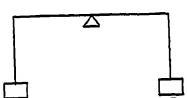
Now we know by experiment that the attraction of the earth gives to all bodies falling in vacuum at any given locality the same acceleration.



Let, then, a mass A be suspended by a spring at any given place and cause an elongation λ of that spring.

At the same place let another mass B be suspended from the same spring, and suppose the observed elongation λ is the same as before. Then the force of gravity on each body is the same, for this is proportional to the elongation of the spring. Also, this force acting upon the bodies produces, as we know by experiment, the same acceleration. Hence in both cases the same force gives the same acceleration and the masses A and B are equal.

Also, when two bodies A and B exactly balance on an equal-armed balance, we also know that the force of gravity on each is the same (page 134).



We have, then, two bodies to which the same force gives the same acceleration, and hence the masses are equal.

By means of the balance, then, we have a convenient means, universally adopted, by which we can readily duplicate the standard mass and fractions of it. Then, by finding how many standard masses balance any given body, we can determine its mass relatively to the standard.

Mass Independent of Gravity.—The intensity of the force of gravity, or the attraction of the earth for a body, varies with the locality and the height above sea level. But evidently the mass of a body or the quantity of matter it contains remains unchanged by locality, and would be the same even if beyond the attraction of the earth.

The equal-armed balance, however, will correctly determine the mass in all localities, because if two bodies exactly balance in one locality they will balance in any other, since the force of gravity, whatever it may be, is always the *same for each* wherever they balance.

In the case of the spring, however, in the first experiment of the preceding article, the graduation is only correct for a given locality. A certain elongation which corresponds to the weight of a pound in one locality will not hold for another locality. The spring measures the *weight* or attraction of the earth for a body in the given locality, the balance measures the mass correctly whatever the locality.

When we speak of the *mass* of one pound, we refer, then, to a definite quantity of matter.

When we speak of the *weight* of a pound, we refer to a variable *force*, viz., the attraction of the earth at a given locality for a mass of one pound.

Notation for Mass.—It is customary when the mass of a body is 2, 3 or 4 times that of the standard, to write it 2 lbs., 3 lbs., 4 lbs. Here the abbreviation "lbs." stands for the latin word LIBRA (*balance*) and thus indicates that the determination has been made *by the balance*.

To avoid confusion, it will be as well to adhere to this notation. Thus "4 lbs." means a *mass*, while "4 pounds" means the *weight* of 4 lbs. at some specified locality. The expression "4 lbs." should really read "4 libras," but as the word "libra" has never come into use, no one would understand, and "4 lbs." must be read, therefore, "four pounds." But as the abbreviation "lbs." is in common use, we can at least make the distinction to the eye if not to the ear.

In the C. G. S. system the distinction is complete, for we speak of "4 grams" when we mean mass, and "4 dynes" when we mean force, grams being measured by the balance, and dynes by the spring or "dynamometer."

The term "weighing" which is applied in common language to the operation of balancing should not be allowed to mislead. "Weighing" a body by a balance always determines its mass and not its weight. This latter is the force of gravity upon it.

Newton's Third Law of Motion.—When one body presses against or pulls another, it is itself pressed or pulled by this other with an equal and opposite force.

When one body has its momentum changed by the action of another, the other body has the same change of momentum in the opposite direction.

When one body attracts another, this other attracts it with equal and opposite force.

Considering the entire phenomenon of this mutual action and reaction between two bodies, we have Newton's third law:

To every action there is always an equal and opposite reaction; or the mutual actions of any two bodies are always equal and opposite.

Remarks on Newton's Laws.—These three laws of motion were enunciated by Newton in 1687. Simple as they appear, the science of Dynamics made no essential progress until they were recognized.

These laws are statements of facts of nature, not *a priori* deductions, and they are verified by the accord of the results deduced from them with observed phenomena. The proof thus furnished in Astronomy, Dynamics and Applied Mechanics is of such a nature that these laws are regarded as rigorously true, and deductions made from them are accepted even when such deductions cannot be directly verified by experiment.

Stress.—From Newton's third law we see that the exertion of force upon a body is only one side of the entire phenomenon, which includes the simultaneous exertion of equal and opposite forces between two bodies.

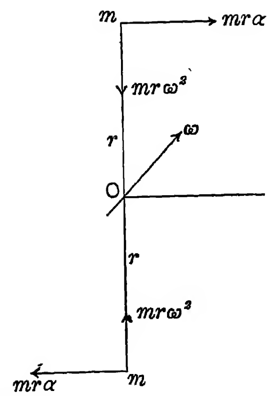
When we fix our attention upon one only of these bodies and, disregarding the other, consider only its action upon the first, we call this action the *impressed force* upon the first. But when we have both bodies in mind and wish to be understood as viewing this force as one of the two mutual, equal and opposite actions between the bodies, or between two particles of a body, we call it a STRESS.

Stress, then, is always an internal force, while impressed force or force in general is always external to the body or system under consideration.

Motion of Centre of Mass.—Suppose a body to rotate about an axis through the centre of mass O with angular velocity ω and angular acceleration α at any instant.

For any particle distant r from O we have the central acceleration (page 78)

$$f_r = r\omega^2$$



and the tangential acceleration (page 85)

$$f_t = r\alpha.$$

If the mass of the particle is m, we have then, from equation (2), page 170, the central force

$$F_r = mf_r = mr\omega^2$$

and the tangential force

$$F_t = mf_t = mr\alpha.$$

But if O is the centre of mass, for every particle of mass m at a distance $+r$ on one side of O there is a particle of equal mass at the same distance $-r$ on the other side. The tangential forces are then parallel, equal and opposite in pairs, and the central forces are equal and opposite in pairs.

Hence *for a rotating body the algebraic sum at any instant of all the particle forces in any direction is zero if the axis of rotation passes through the centre of mass.*

If, then, a body has rotation only about an axis through the centre of mass, we have the force F in any direction at any instant

$$F = \Sigma mf = 0.$$

Suppose, now, a body to have motion of translation only. Then every particle of mass m has the same acceleration in the same direction at the same instant, and if $\bar{f}$ is the acceleration of the centre of mass, we have

$$F = \Sigma mf = \overline{\mathrm{m}}\bar{f}.$$

Now we have seen, page 145, that the motion of a body in general consists at any instant of angular velocity ω about an axis through the centre of mass, and velocity of translation $\bar{v}$ of the centre of mass. Also the change of motion at any instant consists of angular acceleration α about an axis through the centre of mass, and acceleration of translation $\bar{f}$ of the centre of mass. We have also just proved that for the rotation alone $F = 0$, and for the translation alone $F = \overline{\mathrm{m}}\bar{f}$.

In all cases, then, *the motion of the centre of mass of a body is the same as if it were a particle of mass equal to that of the body, all the forces acting upon the body being transferred without change in direction or intensity to this particle.*

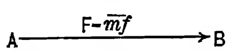
It is this property which makes the centre of mass of such importance in mechanics. So far as the motion of the centre of mass of a body is concerned, we can consider it as a particle of mass equal to the mass of the body and acted upon by all the forces which act upon the body, each force unchanged in magnitude and direction.

CHAPTER II.

RESOLUTION AND COMPOSITION OF FORCES.

Line Representative of Force.—We see, then, that the force on a particle or on a body acts in the direction of the acceleration and is proportional to the acceleration of the particle or of the centre of mass of the body.

Force, then, has magnitude and direction and is therefore, like acceleration itself, a vector quantity and can be represented, like acceleration (page 76), by a straight line.

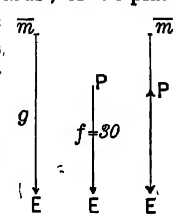


Thus the length of the line AB represents the magnitude of the force $F = \overline{m}f$. Its point of application is A, and its direction of action is indicated by the arrow and is always the same as that of the acceleration $\bar{f}$.

Resolution and Composition of Forces.—We have, then, the triangle and polygon of forces just the same as for accelerations (page 76), and we can resolve a force into two components in any two directions, or we can find the resultant of any number of forces, just as for accelerations.

We have also relative force, just as we have relative acceleration (page 76), with similar notation.

Examples.—(1) *A mass of 20 lbs. rests upon a horizontal platform which has an acceleration $f = 30$ ft.-per-sec. per sec. Find its pressure on the platform when the acceleration is downwards and upwards.* (*Take $g = 32$.*)



ANS. Acceleration of the mass relative to the earth is denoted by $\overline{m}E = g$) downwards; of the platform relative to the earth by $PE = 30$ down. Acceleration of the earth relative to the platform is then $EP = 30$ up. Hence acceleration of mass relative to platform is $\overline{m}P = g - f = 2$ ft.-per-sec. per sec. The pressure is then $\overline{m}(g - f) = 40$ poundals, or $\frac{40}{g} = 1.25$ pounds gravitation measure.

If the acceleration f is upwards, we have in the same way the pressure $\overline{m}\,(g + f) = 1240$ poundals, or $\frac{1240}{g} = 38.75$ pounds gravitation measure.

(2) *A mass hung from a spring-balance in an elevator at rest registers exactly 10 lbs. When the elevator starts, it is observed to register for a moment 10.25 lbs. Find the acceleration of the elevator at that instant.* (*$g = 32$.*)

ANS. Acceleration $f = \frac{g}{40} = 0.8$ ft.-per-sec. per sec. upwards.

(3) *A mass of 20 lbs. rests on a horizontal plane which is made to ascend, first, with a constant velocity of 1 ft. per sec.; second, with a velocity increasing at the rate of 1 ft.-per-sec. per sec. Find in each case the pressure on the plane.* (*$g = 32$.*)

ANS. In the first case the pressure is the weight of 20 lbs. or 640 poundals. In the second case the acceleration relative to the plane is $g + 1 = 33$ ft.-per-sec. per sec. Hence pressure is $20 \times 33 = 660$ poundals, or $\frac{660}{32} = 20\frac{5}{8}$ pounds.

(4) *A spring-balance is graduated correctly for a place where $g = 32.2$. It is transported to a place where $g = 32$, and when a mass is hung on it there it registers 1.6 lbs. Find the correct value of the mass.*

ANS. If $\overline{m}$ is the actual mass, $\overline{m} \times 32$ is the actual weight in poundals of that mass at the place where it is weighed.

If the balance is graduated correctly for a place where $g = 32.2$, the mass it indicates $\times$ 32.2 ought to equal the actual weight. Hence

$$\overline{m} \times 32 = 1.6 \times 32.2, \quad \text{or} \quad m = 1.61 \text{ lbs.}$$

(5) *Let a uniform force of 2 pounds act on a body of 40 lbs. mass for half a minute. Find the velocity acquired and the space passed through.* $(g = 32.)$

ANS. Since the force is uniform, f is constant in direction and magnitude. The force is the weight of 2 lbs. or $2g$ poundals. By the equation of force, $40f = 2g$. Hence $f = \frac{g}{20} = 1.6$ ft.-per-sec. per sec. Since f is uniform, the equations (page 92) apply and we have

$$v = ft = 48\,ft. \text{ per sec.}, \qquad s = \tfrac{1}{2}ft^2 = 720\,ft.$$

(6) *A body acted upon by a uniform force describes in ten seconds, starting from rest, a distance of 25 ft. Compare the force with the weight of the body, and find the velocity acquired.* $(g = 32.)$

ANS. $s = \frac{1}{2}ft^2$ (page 92). Hence $f = \frac{2s}{t^2} = \frac{50}{100} = 0.5$ ft.-per-sec. per sec.; $v = ft = 5\,ft.$ per sec.; $F = \overline{m}f$, or $\frac{F}{\overline{m}g} = \frac{f}{g} = \frac{0.5}{32} = \frac{1}{64}$.

(7) *A force equal to the weight of one lb. acts upon a mass of 18 lbs. free to slide on a smooth horizontal plane. The force is parallel to the plane. When the distance described is 50 ft. find the time and the velocity acquired.* $(g = 32.)$

ANS. The force $F = 1 \times g$ poundals. Since $F = \overline{m}$, we have $1 \times g = 18 \times f$, or $f = \frac{g}{18} = \frac{16}{9}$ ft.-per-sec. per sec. From equations page 92, $v = ft$, $s = \frac{1}{2}ft^2$, hence $t = 7\frac{1}{2}$ sec., $v = 13\frac{1}{3}\,ft.$ per sec.

(8) *Forces of 20 and 30 units acting on two bodies produce accelerations of 40 and 50 units respectively. Show that the masses are as 10 to 12.*

(9) *Two forces produce in two bodies accelerations of 25 and 30 units respectively. Show that if the masses are equal, the forces are as 5 to 6; and if the forces are equal, the masses are as 6 to 5.*

(10) *A balloon is ascending vertically with a velocity which is increasing at the rate of 3 ft.-per-sec. per sec. Find the apparent weight of 1 lb. weighed in the balloon by means of a spring-balance.* $(g = 32.2.)$

ANS. 1.093 pounds.

(11) *A mass $\overline{m}$ lies on a smooth horizontal plane. A uniform horizontal force F is continuously applied. How long will it take to move the mass s ft. from rest? Take $\overline{m} = 2240$ lbs., $F = 28$ pounds, $s = 5$ ft.* $(g = 32.)$

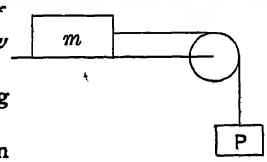
ANS. $F = \overline{m}f$ poundals; hence $f = \frac{F}{\overline{m}}$, or $f = \frac{28g}{2240} = \frac{1}{80}g$ ft.-per-sec. per sec.

If $g = 32$ ft.-per-sec. per sec., we have $f = \frac{2}{5}$ ft.-per-sec. per sec.

Since f is uniform, we have (page 92)

$$s = \frac{1}{2}ft^2, \quad \text{or} \quad 5 = \frac{1}{2} \times \frac{2}{5}t^2, \quad \text{or} \quad t = 5 \text{ sec.}$$

(12) *Let the mass $\overline{m} = 2240$ lbs. be moved by a rope which passes over the edge of the plane on a pulley and sustains a mass $P = 28$ lbs. at its other end. Disregarding all friction and mass of pulley and rope and supposing the rope perfectly flexible and inextensible, find how long it will take to move the mass $\overline{m}$ a distance $s = 5$ ft. from rest.* $(g = 32.)$



ANS. The student should carefully compare this example with the preceding and following.

Here the tension on the rope is $F = \overline{m}f$ poundals, where f is the acceleration of $\overline{m}$. Since P has the same acceleration downward, the resultant acceleration of P is $g - f$. Hence the tension on the rope is also $P(g - f)$ poundals. Therefore

$$\overline{m}f = P(g - f), \quad \text{or} \quad f = \frac{Pg}{P + \overline{m}} = \frac{28g}{2268} = \frac{32}{81} \text{ ft.-per-sec. per sec.}$$

Or we may obtain the same result as follows: The *moving force* is the weight of P or the attraction of gravity for P, or Pg poundals, or the weight of 28 lbs., as in Ex. 11. The mass moved is $P + \bar{m}$. Hence

$$(P + \bar{m})f = Pg, \quad \text{or} \quad f = \frac{Pg}{P + \bar{m}}.$$

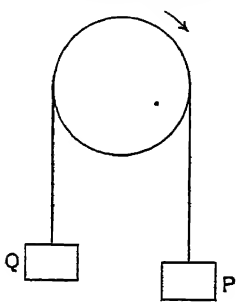
We have for uniform acceleration (page 192)

$$s = \frac{1}{2}ft^2, \quad \text{or} \quad 5 = \frac{1}{2} \times \frac{32}{81}t^2, \quad \text{or} \quad t = 5.051 \text{ sec.}$$

The tension on the rope is $\bar{m}f$ or $P(g - f)$ or

$$\frac{P\bar{m}g}{P + \bar{m}} = \frac{2240 \times 32}{81} \text{ poundals, or the weight of } \frac{P\bar{m}}{P + \bar{m}} = \frac{2240}{81} = 27\tfrac{53}{81} \text{ lbs.}$$

(13) *Two masses $P = 2240$ lbs. and $Q = 2212$ lbs. are hung by means of a perfectly flexible inextensible rope over a pulley. Disregarding all friction and the mass of pulley and rope, how long will it take for each mass to move through $s = 5$ ft. from rest? ($g = 32$.)*



ANS. The student should carefully compare this with the two preceding examples. The tension on the descending side is $P(g-f)$, on the ascending side $Q(g+f)$, where f is the acceleration. Hence

$$P(g - f) = Q(g + f), \quad \text{or} \quad f = \frac{(P - Q)g}{P + Q} = \frac{224}{1113} \text{ ft.-per-sec. per sec.}$$

Or we may obtain the same result as follows: The weight of P is Pg poundals. The weight of Q is Qg poundals. The *moving force* is $Pg - Qg$ or $(P - Q)g$ poundals, or the weight of 28 lbs., as in Ex. 11 and 12. The mass moved is $P + Q$. Hence

$$(P + Q)f = (P - Q)g, \quad \text{or} \quad f = \frac{(P - Q)g}{P + Q}.$$

Since $s = \frac{1}{2}ft^2$, we have $\quad 5 = \frac{1}{2} \times \frac{224}{1113}t^2$, or $t = 7.04$ sec.

The tension on the rope is $Q(g + f)$ or $P(g - f)$ or $\frac{2QPg}{P + Q}$ poundals, or the weight of $\frac{2QP}{P + Q} = 2225.9$ lbs.

NOTE.—The moving force in Ex. 11, 12, 13 is the weight of 28 lbs. In Ex. 11 the mass moved is $\bar{m} = 2240$ lbs., hence $28g = \bar{m}f$. In Ex. 12 the mass moved is $P + \bar{m} = 2268$ lbs., hence $28g = (P + \bar{m})f$. In Ex. 13 the mass moved is $(P + Q) = 4452$ lbs., hence $28g = (P + Q)f$. In all cases, *moving force = mass moved × acceleration.*

The pressure on the axle is the sum of the two tensions, or $(P + Q)g - (P - Q)f$. If the pulley is not allowed to rotate, the pressure upon the axle would be the weight of P and Q, or $(P + Q)g$. The pressure on the axle during motion is therefore less than when at rest.

CHAPTER III.

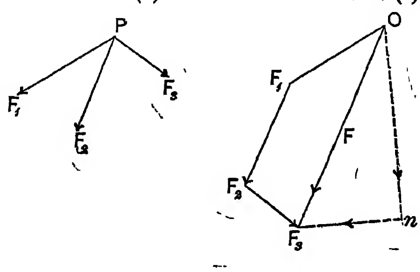
CONCURRING FORCES. TWO NON-CONCURRING FORCES. MOMENT OF A FORCE. RESOLUTION AND COMPOSITION OF MOMENTS. RESULTANT OF TWO PARALLEL FORCES.

Concurring and Non-Concurring Forces.—Forces which act at the same point are called CONCURRING forces. If in the same plane, they are COPLANAR concurring forces. Forces which act at different points are called NON-CONCURRING forces. If in the same plane, they are coplanar.

Resultant for any number of Concurring Forces.—Let any number of forces F_1, F_2, F_3, etc., act at a common point P, Fig. (a). Lay off these forces in order so as to obtain the force polygon $OF_1F_2F_3$, Fig. (b). Then the line OF_3 necessary to close the polygon, taken as acting the opposite way round, or from O to F_3, gives the direction and magnitude of the resultant F.

FIG. (a).

FIG. (b).



This resultant must of course act at the same point P as the forces themselves.

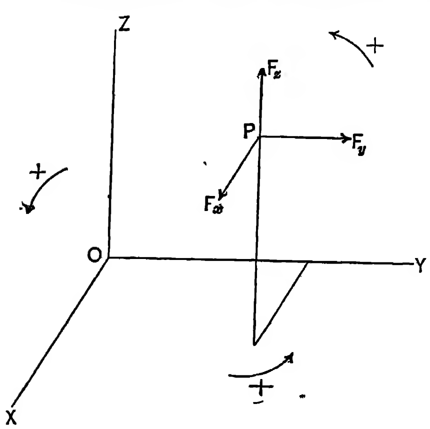
We see, then, that any number of forces acting upon the same point, whether in the same plane or not, can be reduced to a single resultant force acting at this point.

We see also from Fig. (b) that the component On or nF_3 of the resultant F in any direction is equal to the algebraic sum of the components of the forces in that direction.

We see also that if the forces are all parallel the force polygon, Fig. (b), becomes a straight line, and the resultant F is parallel to the components and equal in magnitude to their algebraic sum, or

$$F = \Sigma F.$$

Analytical Determination of Resultant for Concurring Forces.—The same equations and conventions hold for finding the resultant force as for acceleration (page 77) or velocity (page 67) We have therefore only to replace v in equations (1), (2), (3), (4), page 67, by F with the proper subscripts.



We have, then, for the components of the resultant

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + \ldots = \Sigma F \cos \alpha, \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + F_3 \cos \beta_3 + \ldots = \Sigma F \cos \beta, \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + F_3 \cos \gamma_3 + \ldots = \Sigma F \cos \gamma. \end{aligned}\right\} \quad (1)$$

For the magnitude of the resultant

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the direction cosines of the resultant

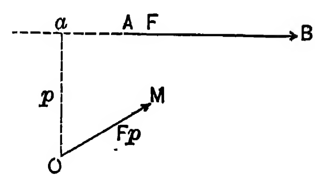
$$\cos\alpha = \frac{F_x}{F}, \quad \cos\beta = \frac{F_y}{F}, \quad \cos\gamma = \frac{F_z}{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have also

$$F_r = F_x \cos\alpha + F_y \cos\beta + F_z \cos\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Examples.—*Students should solve examples (1) and (2), page 68, for forces instead of velocities.*

Moment of a Force.—Let AB be the line representative of a force F acting at the point A.



Take any point O and draw Oa perpendicular to AB, intersecting AB produced if necessary, at a. Let the length of this perpendicular OA be p. Then the product Fp is called the MOMENT of the force F relative to O. The point O is called the CENTRE OF MOMENTS, and the perpendicular p is called the LEVER-ARM for F relative to O.

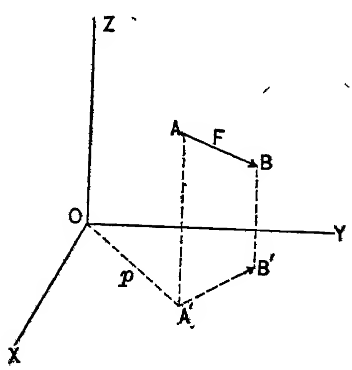
Line Representative of Moment.—This moment has both magnitude and direction and can therefore be represented by a straight line and an arrow, just like moment of acceleration, page 89.

Thus the line representative of the moment in the preceding figure is a straight line OM, passing through the centre of moments O, at right angles to the plane of BAO, whose magnitude is Fp, with an arrow so directed that when we look in the direction of the arrow the rotation indicated by the direction of F relative to O is seen clockwise.

When we speak of the "direction" of a moment we mean the direction of its line representative.

Thus if the force F lies in a vertical east and west plane and points east, then if O is below F, the moment is Fp and its direction north. If O is above F, the moment is Fp and its direction south.

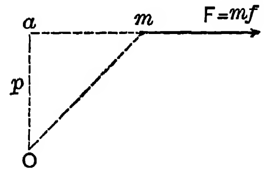
Resolution and Composition of Moments.—We see, then, that a force moment is a vector quantity and all the principles of pages 88, 89 hold.



We have, then, component and resultant moments and the triangle and polygon of moments, just as for displacement (page 54), velocity (page 66), acceleration (page 76) and force (page 176). Also, the moment of a resultant force is equal to the algebraic sum of the moments of its components.

Moment about an Axis.—Hence, just as on page 88, if OZ is a given axis and AB a force F acting at the point A, the moment of F relative to the axis OZ is the same as the moment $A'B' \times p$ of the component $A'B$ in a plane perpendicular to the axis OZ relative to the point of intersection O, of the axis and plane.

Significance of Force Moment.—Force is proportional to acceleration and acts in the same direction (page 176). We have seen, page 89, that the moment of an acceleration is twice the areal acceleration of the radius vector. Hence the moment of a force is *proportional* to the areal acceleration of the radius vector.



Thus if m is the mass of a particle, f its acceleration, and F the force, we have

$$F = mf.$$

The moment fp of the acceleration is twice the areal acceleration of the radius vector Om.

The moment $Fp = mfp$ of the force is therefore proportional to the areal acceleration of the radius vector.

Unit of Force Moment.—We must evidently take for the unit of force moment one unit of force with a lever-arm of one unit of length. If we take feet and poundals, our unit is then one "poundal-foot," or one poundal with a lever-arm of one foot. If we take feet and pounds, our unit is one "*pound-foot.*" So, also, we may have the "*dyne-foot*" or the "*kilogram-foot.*"

Thus a force of 10 pounds with a lever-arm of 3 feet gives a moment of 30 *pound-feet* or $30g$ *poundal-feet.*

Resultant of two Non-Concurring Forces.—Let two forces F_1, F_2 act at the points A_1, A_2, Fig. (a). Then the magnitude and direction of the resultant F are found by the triangle of forces, Fig. (b), just as for concurring forces, page 179. But the position of the resultant in the plane of F_1 and F_2 has still to be determined.

FIG. (a).

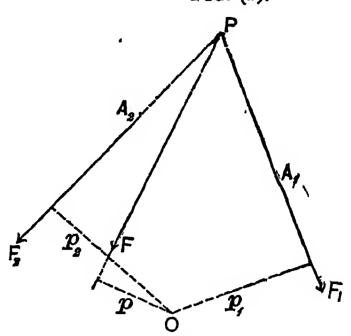
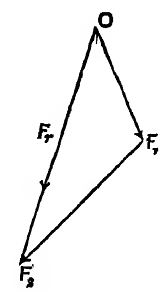


FIG. (b).



Take a point O anywhere in this plane as a centre of moments, and draw the lever-arms p_1, p_2 and p

Then, since the moment of the resultant relative to any point is equal to the algebraic sum of the moments of the components, we have in general

$$Fp = F_1p_1 + F_2p_2. \qquad (1)$$

[Regard must be paid to sign. Thus if we take counter-clockwise rotation as positive, we have for the case of the figure

$$Fp = -F_1p_1 + F_2p_2.]$$

Equation (1) holds good no matter where the centre of moments O is taken in the plane of the forces. Let us then take it at P, the point of intersection of F_1 and F_2 prolonged. For this point the lever-arms p_1, p_2 are zero. Since we must always have the moment of the resultant equal to the algebraic sum of the moments of the components, $Fp = 0$ and the lever-arm p must be zero. The resultant F must therefore pass through the point P, and the system reduces to two concurring forces acting at P. Hence

(1) *A force acting at any point can be considered as acting at any point in its line of direction.*

(2) *The resultant of two non-concurring forces lies in the plane of the forces, passes through the point of intersection of the forces produced, and can also be considered as acting at any point in its line of direction.*

Resultant of Two Parallel Forces.—Let us take next two parallel forces, F_1, F_2, acting at the points A_1 and A_2, either in the same direction, Fig. (a), or in opposite directions, Fig. (b).

FIG. (a).

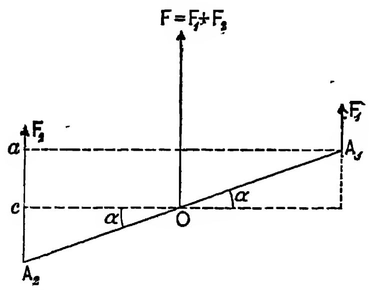
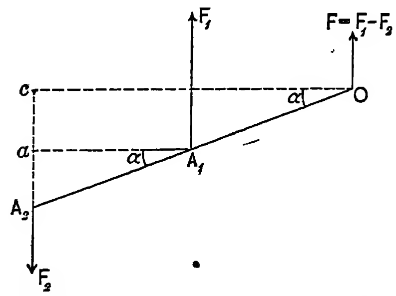


FIG. (b).



The resultant F is parallel to the forces and given in magnitude by

$$F = F_1 + F_2, \quad . \; . \; . \quad (1)$$

where F_1 and F_2 are to be taken with their proper signs, (+) if acting up, (−) if acting down. Thus, in Fig. (a), $F = F_1 + F_2$, and in Fig. (b) $F = F_1 - F_2$.

In order to find the position of the resultant F, join the points of application A_1 and A_2 and let O be the point where the resultant F_r intersects the line A_1A_2. Through O and A_1 draw Oc and A_1a perpendicular to F_2, and let the angle cOA_2 or aA_1A_2 be denoted by α.

If now we take A_2 or any point on the line representative of F_2 as a point of moments, we have the moment of F_2 for this point zero. The moment of F_1 is $F_1 \times A_1a = F_1 \times A_1A_2 \cos \alpha$. The moment of F is $F \times Oc = F \times OA_2 \cos \alpha$. Since the moment of the resultant is equal to the algebraic sum of the moments of the components, we have

$$F \times OA_3 \cos \alpha = F_1 \times A_1A_2 \cos \alpha.$$

Hence

$$F \times OA_2 = F_1 \times A_1A_2, \quad \text{or} \quad OA_2 = \frac{F_1}{F} \cdot A_1A_2 \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

If we take A_1 or any point on the line representative of F_1 as a point of moments, we have the moment of F_1 for this point zero. The moment of F is $F \times OA_1 \cos \alpha$. The moment of F_2 is $F_2 \times A_1A_2 \cos \alpha$. We have then

$$F \times OA_1 \cos \alpha = F_2 \times A_1A_2 \cos \alpha.$$

Hence

$$F \times OA_1 = F_2 \times A_1A_2, \quad \text{or} \quad OA_1 = \frac{F_2}{F} \cdot A_1A_2. \; . \; . \; . \; . \; . \; . \quad (3)$$

If we take any point on the line representative of F as a point of moments, the moment of F for this point is zero. Hence the algebraic sum of the moments of the components for this point must also be zero. We have then

$$F_1 \times OA_1 = F_2 \times OA_2, \quad \text{or} \quad \frac{F_1}{F_2} = \frac{OA_2}{OA_1}. \; . \; . \; . \; . \; . \; . \quad (4)$$

We see from (2) and (3) that the distances OA_1 and OA_2 depend only upon the magnitudes of F_1 and F_2 and the distance A_1A_2 between their points of application, and not at all upon the angle α or upon the common direction of F_1 and F_2. For the same forces F_1 and F_2 and the same points of application A_1 and A_2 the resultant F will then always pass through the same point O, *no matter what the direction of the parallel forces may be.* This point O is therefore called the CENTRE of the two parallel forces.

We have then the following principle:

The resultant of two parallel forces F_1, F_2, acting at the extremities of a straight line A_1A_2, is in their plane and equal in magnitude to their algebraic sum. It acts parallel to the forces in the direction of the larger force, and always acts at a point O on the straight line A_1A_2 or on this line produced, which divides this line into segments inversely as the forces. Or the products of the forces into the adjacent segments are equal.

This principle is known as the "*law of the lever.*" When the forces act in the same direction, as in Fig. (*a*), the resultant lies *within the components.* When the forces act in opposite directions, as in Fig. (*b*), the resultant lies *outside the components and on the side of the larger.*

Examples.—(1) *Two parallel forces F_1, F_2, of 17 and 33 pounds respectively, act in the same direction, and their points of application A_1, A_2 are 8 ft. apart. Find the resultant and the distances OA_1, OA_2 of its point of application.*

ANS. $F = 50$ pounds parallel to the forces, and acting in the same direction. $OA_1 = 5.28$ ft., $OA_2 = 2.72$ ft.

(2) *Find the resultant and the point O when the forces in the preceding example act in opposite directions.*

ANS. $F = 16$ pounds in the direction of the larger force. $OA_1 = 16.5$ ft., $OA_2 = 8.5$ ft.

(3) *Two parallel forces F_1, F_2 of 12.5 and 25 pounds act in the same direction upon two points. The resultant acts at a distance of 4 ft. from F_1. Find the distance between the forces.*

ANS. 6 ft.

(4) *Resolve a force $F = 52$ pounds into two parallel forces acting in the same direction, F_1 and F_2,* (*a*) *when the distances from F are 2 and 3 ft.;* (*b*) *when $F_1 = 20$ pounds at a distance of 2 ft.*

ANS. (*a*) $F_1 = 31.2$ pounds, $F_2 = 20.8$ pounds. (*b*) $F_2 = 32$ pounds at a distance from F of 1.25 ft.

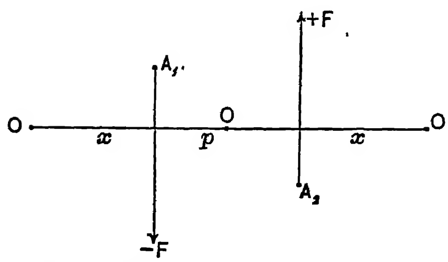
(5) *Resolve a force $F = 20$ pounds into two parallel forces F_1, F_2, one of which, F_1, acts opposite to F_2:* (*a*) *when the forces are distant from F 8 and 3 ft.,* (*b*) *when F_1 is 30 pounds and distant from F 6 ft.*

ANS. (*a*) $F_1 = 12$ pounds, $F_2 = 32$ pounds. (*b*) $F_2 = 50$ pounds at a distance of 3.6 ft.

CHAPTER IV.

FORCE-COUPLES. EFFECT OF FORCE-COUPLE ON A RIGID BODY.

Force-Couple.—Two equal and parallel forces acting in opposite directions and *not in the same line* constitute a FORCE-COUPLE.



Thus the two equal parallel forces $+F$, $-F$, acting at A_1 and A_2, constitute a force-couple.

The perpendicular distance p between the forces is called the ARM of the couple.

The plane of the two forces is the plane of the couple.

Moment of a Force-Couple.—Take any point O in the plane of the couple on the left of $-F$, distant x from $-F$. For any and every such point we have the moment (taking counter-clockwise rotation positive)

$$F(x+p) - Fx = Fp.$$

Take any point O in the plane of the couple on the right of $+F$, distant x from $+F$. For any and every such point we have the moment

$$F(x+p) - Fx = Fp.$$

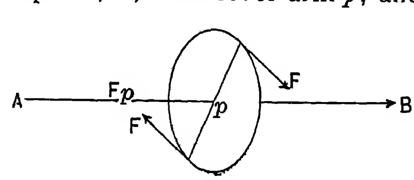
Take any point O in the plane of the couple between the forces distant x from $+F$ and $p-x$ from $-F$. For any and every such point we have the moment

$$Fx + F(p-x) = Fp.$$

In general, then, *the moment of a force-couple relative to any point in its plane is constant and equal to the product Fp of the arm p by one of the forces.*

Line Representative of a Force-Couple.—The moment of a force-couple is of course represented by a straight line, just like the moment of a force in general (page 181), and when we speak of the "direction" of a couple we mean the direction of its line representative.

Thus the line AB represents the magnitude Fp of a couple F, F, with lever-arm p, and the direction of the couple is that of the arrow at B, so that looking from A to B in the direction of the arrow, the rotation is seen clockwise. The plane of the couple is at right angles to AB. The line representative may be drawn at right angles to this plane *through any point of the plane we please*, since the moment is the same at all points.



Resolution and Composition of Couples.—We have then for couples the same principles as for moments in general (page 88). Thus we have component and resultant couples, and the triangle and polygon for couples, just as for displacement (page 54), velocity (page 66), acceleration (page 76) and force (page 176).

The following corollaries are at once evident:

COR. 1.—A couple can be turned in its own plane so that the forces have any desired direction, or it can be shifted to any position in its own plane. For so long as the plane and the arm and forces are unchanged the moment is unchanged, and the line representative has the same magnitude and direction and can pass through any point of the plane.

COR. 2.—All couples whose planes are parallel and moments equal are equivalent. For the line representative is the same for all.

COR. 3.—Any couple can be replaced by another in the same plane of the same direction and moment and having any desired arm or any desired force. For if the plane and moment are unchanged, the line representative is unchanged.

COR. 4.—Any number of couples in the same plane or in parallel planes can be reduced to a single resultant couple in that plane or a parallel plane whose moment is the algebraic sum of the moments of the couples. For the line representatives of all are parallel and can all be taken as acting at the same point of the plane. The resultant is therefore the algebraic sum.

Resultant of a Force-Couple.—A force-couple is only a special case of two parallel forces which we have discussed already on page 183.

Thus for two parallel forces F_1 and F_2 acting at points A_1, A_2 we have found for the resultant in general

$$F = F_1 + F_2,$$

and for the distance of the point of application C of the resultant from A_1 along the line A_1A_2

$$A_1C = \frac{F_2}{F} A_1A_2.$$

Hence for the perpendicular distance from F_1 to the resultant we have

$$\text{distance} = \frac{F_2}{F} \cdot p,$$

where p is the lever-arm or perpendicular distance between the forces.

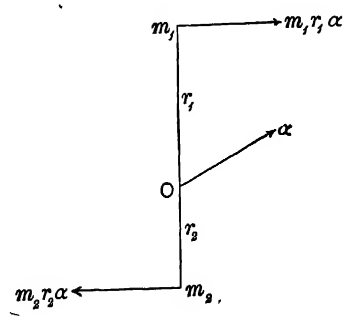
Now, for a couple, F_1 and F_2 are both equal to F and opposite in direction. Hence $F = 0$ and

$$\text{distance} = \frac{Fp}{0} = \frac{M}{0} = \pm \infty,$$

where M is the moment of the couple. That is, *the resultant of a force-couple is a force of zero at an infinite distance*, plus or minus according to the sign of the moment.

A couple *cannot, therefore, be replaced, nor can it be held in equilibrium by a single force, but only by another couple.*

This is also evident from the fact that a single force has a different moment at different points, the moment varying as the lever arm, while a couple has the same moment for all points in its plane. It cannot, then, be replaced by a single force.



Effect of a Force-Couple on a Rigid Body.—Let a rigid body have angular acceleration α about an axis through its centre of mass O. Let m_1 and m_2 be the masses of two particles of the body situated in a line passing through the axis of acceleration and at right angles to it. Let the distance of m_1 be r_1, and of m_2 be r_2, and let the centre of mass of the two particles be on the axis through O, so that

$$m_1 r_1 = m_2 r_2.$$

Since all particles of the body have the angular acceleration α about the axis, the acceleration of m_1 is $f_1 = r_1\alpha$, and of m_2 is $f_2 = r_2\alpha$. The force upon m_1 is then

$$m_1 f_1 = m_1 r_1 \alpha,$$

and the force upon m_2 is

$$m_2 f_2 = m_2 r_2 \alpha,$$

and these two forces are parallel and opposite. But, since $m_1 r_1 = m_2 r_2$, these two forces are not only parallel and opposite, but equal. They therefore constitute a couple.

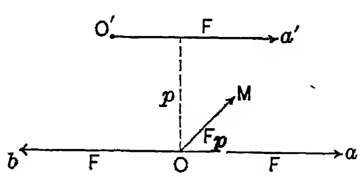
Now the entire body is composed of such pairs of particles, and each pair therefore constitutes a couple. All these couples are in parallel planes, and all have the same direction. They can, then, be reduced to a single resultant couple having the same direction, whose moment is the sum of the moments of all the components (COR. 4, page 186). Hence

If a rigid body has angular acceleration about an axis through its centre of mass, the resultant is a force couple in a plane at right angles to this axis.

Conversely,

If a rigid body is acted upon by a force couple, the effect is to cause angular acceleration about an axis through the centre of mass at right angles to the plane of the couple.

Resultant for Non-Concurring Forces acting on a Rigid Body.—Let a force F, whose line representative is $O'a'$, act at any point O' of a rigid body.



Let O be any point of the body. If at the point O we apply two equal and opposite forces $Oa = F$ and $Ob = F$, both parallel and equal to $O'a' = F$, the previous motion of the body is evidently not affected.

Hence the single force $O'a' = F$ can be replaced by the same force, $Oa = F$, acting at any given point O, and a force couple $O'a'$ and Ob. Let p be the arm of the couple. Then the moment of the couple is $M = Fp$, and its line representative is OM at right angles to the plane of the couple.

When we speak of the direction of a moment or force couple we always mean the direction of its line representative.

Hence (compare page 151) we have the following principles:

(*a*) A single force F acting at any point of a rigid body can be replaced by an equal and parallel force F acting at any given point O and a couple whose moment is $M = Fp$ and whose line representative through O is at right angles to the plane of the couple.

(*b*) Conversely, the resultant of a couple M and a force F in the plane of the couple is a single equal and parallel force in that plane at a distance $p = \frac{M}{F}$.

(*c*) If any number of forces act upon a rigid body, each acting at a different point and in a different direction, then by (*a*) we can replace each one by an equal and parallel force acting at any point O, and a couple whose line representative through O is at right angles to the force.

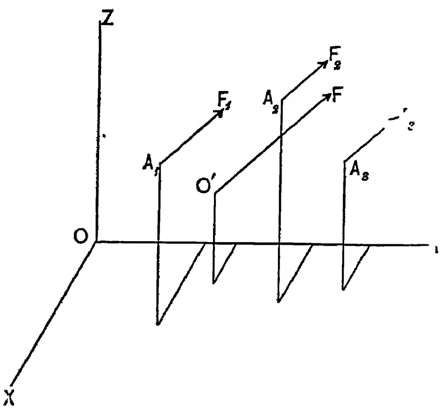
We can reduce all the forces at O by the polygon of forces (page 176) to a single resultant force F at O, and all the couples at O by the polygon of moments (page 88) to a single resultant couple M whose line representative *is not necessarily at right angles to F.* Hence, generally,

Any system of forces acting on a rigid body can be reduced at any point O to a single resultant force F and a single resultant couple M whose line representative is not necessarily at right angles to F.

CHAPTER V.

CENTRE OF PARALLEL FORCES. CENTRE OF MASS.

Centre of Parallel Forces.—Let F_1, F_2, F_3, etc., be any number of parallel forces acting at the points A_1, A_2, A_3, etc., of a rigid body and given by the co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2), etc., and making the angles α, β, γ with the co-ordinate axes.



Then the resultant F is parallel to the forces and equal to their algebraic sum, or

$$F = F_1 + F_2 + F_3 + \ldots = \Sigma F. \quad . \quad . \quad (1)$$

In taking the algebraic sum in (1), forces acting in one direction are taken as positive, in the other direction as negative.

If ΣF is not zero, we have a single resultant. It remains to find its position.

For the moments M_x, M_y, M_z of the forces about the axes of X, Y, Z we have

$$\left.\begin{aligned} M_x &= \Sigma F \cos\gamma \,.\, y - \Sigma F \cos\beta \,.\, z = \cos\gamma \Sigma Fy - \cos\beta \Sigma Fz, \\ M_y &= \Sigma F \cos\alpha \,.\, z - \Sigma F \cos\gamma \,.\, x = \cos\alpha \Sigma Fz - \cos\gamma \Sigma Fx, \\ M_z &= \Sigma F \cos\beta \,.\, x - \Sigma F \cos\alpha \,.\, y = \cos\beta \Sigma Fx - \cos\alpha \Sigma Fy. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

Let the point of application O' of the resultant be given by the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$. If we transfer the origin O to this point, and take co-ordinate axes X', Y', Z' at this point parallel to X, Y, Z, we can replace x, y, z in equations (2) by $x - \bar{x}$, $y - \bar{y}$, $z - \bar{z}$, and we have for the moments of the forces about the axes X', Y', Z' through the point of application O' of the resultant

$$\left.\begin{aligned} M_x' &= \cos\gamma \Sigma F(y - \bar{y}) - \cos\beta \Sigma F(z - \bar{z}), \\ M_y' &= \cos\alpha \Sigma F(z - \bar{z}) - \cos\gamma \Sigma F(x - \bar{x}), \\ M_z' &= \cos\beta \Sigma F(x - \bar{x}) - \cos\alpha \Sigma F(y - \bar{y}). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But the moment of the resultant is equal to the algebraic sum of the moments of the components, and for the point of application of the resultant its moment is zero. Hence $\bar{x}$, $\bar{y}$, $\bar{z}$ must have such values as to make equations (3) zero.

We have then

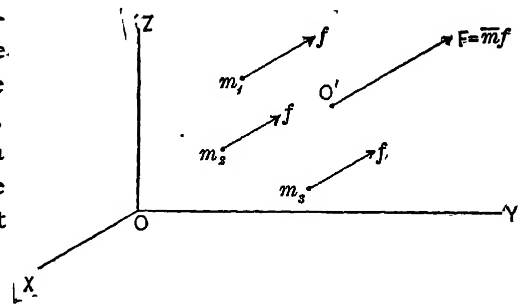
$$\left.\begin{aligned} \Sigma F(x - \bar{x}) &= \Sigma Fx - \bar{x}\Sigma F = 0, \quad \text{or} \quad \bar{x} = \frac{\Sigma Fx}{\Sigma F}, \\ \Sigma F(y - \bar{y}) &= \Sigma Fy - \bar{y}\Sigma F = 0, \quad \text{or} \quad \bar{y} = \frac{\Sigma Fy}{\Sigma F}, \\ \Sigma F(z - \bar{z}) &= \Sigma Fz - \bar{z}\Sigma F = 0, \quad \text{or} \quad \bar{z} = \frac{\Sigma Fz}{\Sigma F}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equations (4), then, give the position of the point of application for the resultant of a system of parallel forces for any assumed origin and co-ordinate axes.

This point is called the CENTRE OF PARALLEL FORCES. We see that its position is independent of the direction of the forces and depends only upon the forces and their points of application.

Properties of Centre of Mass.—When a body has motion of translation all particles of the body move in parallel paths with the same velocity, in the same direction, at the same instant. The acceleration of every particle is therefore the same and in the same direction at any instant.

Let m_1, m_2, m_3, etc., be the masses of particles of a translating body, each one therefore having the same acceleration f in the same direction. The forces on the particles are $m_1 f$, $m_2 f$, $m_3 f$, etc., and these forces constitute a system of parallel forces. Let $\text{m} = \Sigma m$ be the mass of the entire body, and F the resultant force.



Then, by the preceding article, we have

$$F = m_1 f + m_2 f + m_3 f + \ldots = f\Sigma m = \text{m} f, \quad \ldots \ldots \quad (1)$$

and for the point of application O' of this resultant we have the co-ordinates

$$\bar{x} = \frac{f\Sigma mx}{\text{m}f}, \quad \bar{y} = \frac{f\Sigma my}{\text{m}f}, \quad \bar{z} = \frac{f\Sigma mz}{\text{m}f},$$

or, since f is constant,

$$\bar{x} = \frac{\Sigma mx}{\text{m}}, \quad \bar{y} = \frac{\Sigma my}{\text{m}}, \quad \bar{z} = \frac{\Sigma mz}{\text{m}}. \quad \ldots \ldots \ldots \quad (2)$$

But we have seen, page 21, that these values of $\bar{x}$, $\bar{y}$, $\bar{z}$ give the position of the centre of mass of the body.

Hence *the center of mass of a body coincides with the point of application of the resultant of that system of parallel forces which acts upon all the partieles of the translating body. If* m *is the mass of the body and* $\bar{f}$ *the common acceleration, the resultant* F *is given by* $F = \text{m}\bar{f}$.

Conversely, *if a force F act at the centre of mass of a body of mass* m, *it will cause in every particle the same acceleration f, in the same direction,* given by $\bar{f} = \frac{F}{\text{m}}$. *That is, it will cause acceleration of translation only.*

We have seen, page 188, that any number of forces acting in any directions on a body can be reduced to a force F acting at *any* point and a couple. Take this point as the centre of mass. Then we have a force F acting at the centre of mass and a couple. But we have seen, page 187, that the effect of a couple is to cause angular acceleration about an axis through the centre of mass. It does not, then, affect the motion of the centre of mass itself. We have also just seen that the effect of the force F acting at the centre of mass is to cause in every particle an acceleration $\bar{f}$ in the same direction given by $\bar{f} = \frac{F}{M}$. Since, then, the acceleration of the centre of mass is not affected by the couple, we have for the acceleration

$\bar{f}$ of the centre of mass for any number of forces in any directions, $= \bar{f}\, \frac{F}{\overline{\mathfrak{m}}}$. Hence $F = \overline{\mathfrak{m}}\bar{f}$, where (page 188) F is the resultant of all the forces acting upon the body, each one considered as transferred to the centre of mass, without change in magnitude or direction.

Hence *the acceleration $\bar{f}$ of the centre of mass of a body is the same as if all the forces acting on the body were applied to the entire mass $\overline{\mathfrak{m}}$ concentrated there.*

It is this property which makes the centre of mass of such importance in Mechanics. So far as the motion *of the centre of mass* of a body is concerned, we can always consider it as a particle of mass equal to the mass of the body and acted upon by all the forces which act upon the body, unchanged in magnitude and direction.

We have also the following properties of the centre of mass:

1. The attraction of the earth for a body whose longest dimension is insignificant compared to the earth's radius is practically a parallel force mg on every particle of the body of mass m. The entire weight of the body, $\overline{\mathfrak{m}}g$. acts, in such case, practically at the centre of mass for all positions of the body, and a body acted upon by its weight only has motion of translation.

Hence the centre of mass is often called erroneously the "centre of gravity" (page 207).

2. If a rigid body at rest is supported at its centre of mass and is acted upon by gravity only, it will remain at rest in all positions.

CHAPTER VI.

NON-CONCURRING FORCES IN GENERAL. ANALYTIC EQUATIONS.

Resultant for Non-Concurring Forces.—We have seen (page 188) that, generally, any system of forces acting on a rigid body can be reduced at any point O to a single resultant force F and a single resultant couple M, whose line representative is not necessarily at right angles to F.

The force F does not change in magnitude or direction no matter where the point O is taken. The couple M, however, changes in magnitude and direction with the position of the point O. The line representative of the couple M *at the centre of mass* is the spontaneous axis of angular acceleration (page 152).

Since M is not necessarily at right angles to F, we can resolve it at any point O into a component M_a along F and a component M_n normal to F. But by (*b*), page 188, we can reduce F and M_n to an equal and parallel force at a distance $p = \frac{M_n}{F}$. Since the couple M_a at O can be replaced by the same couple along F (page 186), we have a force F and a couple M_a whose line representative coincides with F. Hence, generally,

Any system of forces acting on a rigid body can be reduced at any point O to a resultant force F, a couple M_a along F and a couple M_n normal to F. Or to a force F at a distance $p = \frac{M_n}{F}$ and a couple M_a along F at this distance.

Again, since M is not necessarily at right angles to F, we can resolve F at any point O into a component F_a along M at that point and a component F_n normal to M at that point. But by (*b*), page 188, we can reduce F_n and M to an equal and parallel force F_n at a distance $p = \frac{M}{F_n}$. Hence, generally,

Any system of forces acting on a rigid body can be reduced at any point O to a couple M, a force F_a along M and a force F_n normal to M. Or to a force F_a at O and a normal force F_n at a distance $p = \frac{M}{F_n}$.

As M changes in magnitude and direction with the point O, F_a and F_n will change also. But the resultant F is unchanged in magnitude and direction no matter where O is taken.

The line representative of M at the centre of mass is the spontaneous axis of angular acceleration, and the instantaneous axis is parallel to it (page 150).

Effect of any System of Forces acting on a Rigid Body.—Since we can take the point O where we please, let us always take it *at the centre of mass.*

We have seen (page 190) that a force acting at the centre of mass of a rigid body causes acceleration of translation $\bar{f}$ of the body in the direction of the force. Also (page 187) that

a force-couple acting on a rigid body causes angular acceleration α of the body about the line representative through the centre of mass as axis.

Therefore the force F_a along M at the centre of mass causes acceleration of translation $\bar{f}_a$ of the body along the spontaneous axis of angular acceleration. The normal force F_n at the centre of mass causes acceleration of translation $\bar{f}_n$ of this axis. The couple M at the centre of mass causes angular acceleration α about the spontaneous axis of acceleration.

But we have seen (page 151) that $\bar{f}_a$, $\bar{f}_n$ and α reduce to $\bar{f}_a$ and α about the instantaneous axis of acceleration, or to a screw-twist. Hence, generally,

The effect of any system of forces acting on a rigid body at any instant is to cause a screw-twist, or angular acceleration about the instantaneous axis and acceleration of translation along this axis.

Wrench. Screw-Wrench.—A force-couple acting on a rigid body we call a *wrench*. It causes angular acceleration about its line representative through the centre of mass (page 187). The line representative is the *axis* of the wrench and may be taken anywhere parallel to itself (page 186).

A force-couple or wrench, and a force parallel to the axis of the wrench, we call a *screw-wrench.*

The preceding principles may then be expressed as follows:

Any system of forces acting on a rigid body reduces in general to a resultant force F at the centre of mass, and a resultant couple or wrench M with axis at the centre of mass.

Any system of forces acting on a rigid body reduces in general to a screw-wrench M_a about an axis coinciding with F at a distance from the centre of mass given by $p = \frac{M_n}{F}$, where M_n is the component of M normal to F.

Any system of forces acting on a rigid body reduces in general to a screw-wrench M with axis at the centre of mass, a force F_a coinciding with this axis and a force F_n normal to it. Or to a force F_a at the centre of mass, and a normal force F_n at a distance from the centre of mass given by $p = \frac{M}{F_n}$.

The effect of any system of forces acting on a rigid body is, generally, to cause a screw-twist about the instantaneous axis of acceleration.

This axis is parallel to the spontaneous axis of acceleration, that is to M at the centre of mass.

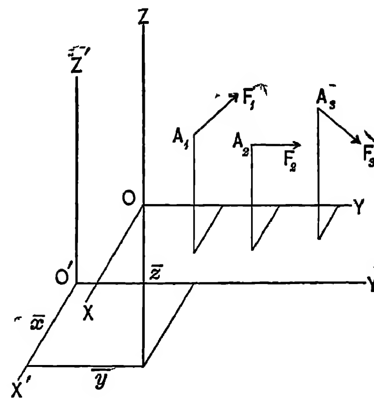
Dynamic Components of Motion.—We have just seen (page 191) that any number of forces acting upon a rigid body reduces to a single resultant force F at the centre of mass and a resultant couple M at the centre of mass. Let us take co-ordinate axes through the centre of mass O, and let the components of F be F_x, F_y, F_z, and the components of M be M_x, M_y, M_z.

The motion of the body under the action of forces is then known if these six quantities are known. These six quantities

$$F_x,\ F_y,\ F_z,\ M_x,\ M_y,\ M_z,$$

with reference to co-ordinate axes through the centre of mass, are therefore called the dynamic COMPONENTS OF MOTION of the body.

General Analytic Equations.—Let any number of forces F_1, F_2, F_3 act at points A_1, A_2, A_3 of a rigid body given by the co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2), etc. Let F_1 make with the co-ordinate axes the angles $(\alpha_1, \beta_1, \gamma_1)$, F_2 the angles $(\alpha_2, \beta_2, \gamma_2)$, etc. Take the origin O at the centre of mass.



We can replace each force by an equal parallel force in the same direction at O and a couple with axis through O (page 187). All the forces can then be reduced to a single resultant F, and all the couples to a single couple whose moment is M.

RESULTANT FORCE AT CENTRE OF MASS.—We have for the algebraic sum of all the components along the co-ordinate axes through the centre of mass O

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \Sigma F \cos \alpha, \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \Sigma F \cos \beta, \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \ldots = \Sigma F \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

The resultant force is then

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The line representative passes through the centre of mass O, and its direction cosines are

$$\cos \alpha = \frac{F_x}{F}, \qquad \cos \beta = \frac{F_y}{F}, \qquad \cos \gamma = \frac{F_z}{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations give in any case the components of motion F_x, F_y, F_z, and the magnitude and direction of the resultant force F of the screw-wrench. They hold good, evidently, no matter where the origin is taken, whether at the centre of mass or not.

RESULTANT COUPLE OR WRENCH AT CENTRE OF MASS.—For the moments about the co-ordinate axes through the centre of mass we have

$$\left.\begin{aligned} M_x &= \Sigma F \cos \gamma \,.\, y - \Sigma F \cos \beta \,.\, z, \\ M_y &= \Sigma F \cos \alpha \,.\, z - \Sigma F \cos \gamma \,.\, x, \\ M_z &= \Sigma F \cos \beta \,.\, x - \Sigma F \cos \alpha \,.\, y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The moment of the resultant couple M at the centre of mass is then

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Its line representative passes through the centre of mass O, and its direction cosines are

$$\cos \alpha = \frac{M_x}{M}, \qquad \cos \beta = \frac{M_y}{M}, \qquad \cos \gamma = \frac{M_z}{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These equations give in any case the components of motion M_x, M_y, M_z, and the magnitude and direction of the resultant moment M *at the centre of mass* O. If we take the

origin at any other point, the moment M changes in direction and magnitude. These equations hold good, therefore, only for origin at the centre of mass.

RESULTANT COUPLE OR WRENCH AT ANY POINT.—Instead of taking the origin at the centre of mass O, let us take it at any point O' not at the centre of mass. Take co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$ at the origin O' parallel to OX, OY, OZ at the centre of mass O (see figure, page 193). Let the co-ordinates of the centre of mass O be $\bar{x}$, $\bar{y}$, $\bar{z}$. Then equations (4) still hold if we put $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$ in place of x, y, z.

We have, then, for the moments about the co-ordinate axes at any point O'

$$\left.\begin{aligned} M_x' &= M_x + F_z\bar{y} - F_y\bar{z}, \\ M_y' &= M_y + F_x\bar{z} - F_z\bar{x}, \\ M_z' &= M_z + F_y\bar{x} - F_x\bar{y}. \end{aligned}\right\} \qquad (7)$$

The moment of the resultant couple M' is then

$$M' = \sqrt{M_x'^2 + M_y'^2 + M_z'^2}.$$

Its line representative passes through the origin O', and its direction cosines are

$$\cos\alpha = \frac{M_x'}{M'}, \qquad \cos\beta = \frac{M_y'}{M'}, \qquad \cos\gamma = \frac{M_z'}{M'}.$$

These equations give the components of motion M_x', M_y', M_z' and the magnitude and direction of the resultant moment M' for any point O'. If we take the origin at the centre of mass, they reduce to equations (4), (5) and (6).

MOMENT M_a AT CENTRE OF MASS ALONG THE RESULTANT FORCE.—Let the component of the moment M at the centre of mass along the line representative of the resultant force F be M_a. Then we have

$$M_a = M_x \cos\alpha + M_y \cos\beta + M_z \cos\gamma,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are given by equations (3). We have then

$$M_a = \frac{F_xM_x + F_yM_y + F_zM_z}{F}. \qquad (8)$$

The resultant force F is independent of where we take the origin, and the moment M_a *at the centre of mass* remains the same no matter where we take the origin. Equation (8) therefore holds good no matter where we take the origin, whether at the centre of mass or not.

MOMENT M_n AT CENTRE OF MASS NORMAL TO THE RESULTANT FORCE.—Let the component of the moment M at the centre of mass normal to the line representative of the resultant force F be M_n. The components of M_a along the co-ordinate axes through the centre of mass O are

$$M_a \cos\alpha, \qquad M_a \cos\beta, \qquad M_a \cos\gamma,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are given by equations (3). If we subtract these components from the components M_x, M_y, M_z of the resultant moment M at the centre of mass, we have the components of the normal moment M_n at the centre of mass O

$$\left.\begin{aligned} M_{nx} &= M_x - M_a \cos\alpha = M_x - \frac{M_aF_x}{F}, \\ M_{ny} &= M_y - M_a \cos\beta = M_y - \frac{M_aF_y}{F}, \\ M_{nz} &= M_z - M_a \cos\gamma = M_z - \frac{M_aF_z}{F}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

FORCE F_a AT CENTRE OF MASS ALONG M.—Let the component of the force F along M at the centre of mass be F_a. Then we have

$$F_a = F_x \cos\alpha + F_y \cos\beta + F_z \cos\gamma,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are given by equations (6). We have then

$$F_a = \frac{F_xM_x + F_yM_y + F_zM_z}{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The resultant force F is independent of where we take the origin, and the moment M *at the centre of mass* remains the same no matter where we take the origin. Equation (10) therefore holds good no matter where we take the origin, whether at the centre of mass or not.

FORCE F_n AT CENTRE OF MASS NORMAL TO M.—Let the component of the force F normal to the axis of M at the centre of mass be F_n. The components of F_a along the co-ordinate axes through the centre of mass O are

$$F_a \cos\alpha, \qquad F_a \cos\beta, \qquad F_a \cos\gamma,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are given by equations (6). If we subtract these components from the components F_x, F_y, F_z of the force F, we have the components of F_n:

$$\left.\begin{aligned} F_{nx} &= F_x - F_a \cos\alpha = F_x - \frac{F_aM_x}{M}, \\ F_{ny} &= F_y - F_a \cos\beta = F_y - \frac{F_aM_y}{M}, \\ F_{nz} &= F_z - F_a \cos\gamma = F_z - \frac{F_aM_z}{M}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

POSITION OF RESULTANT FORCE F OF THE SCREW-WRENCH M_a.—We have from (8) the moment of the screw-wrench M_a, and from (2) the force F along its axis, and from (3) the direction of the axis. It remains only to find the position of F relative to the centre of mass.

The moment M_n at the centre of mass is equal to Fp, where p is the distance of F from the centre of mass O. Let x, y, z be the co-ordinates of any point on the line representative of F. Then we have

$$F_zy - F_yz = M_{nx}, \qquad F_xz - F_zx = M_{ny}, \qquad F_yx - F_xy = M_{nz}. \quad . \quad . \quad . \quad (12)$$

These are the equations of the projection of F on the three co-ordinate planes.

Let x_1, y_1, z_1 be the intercepts on the co-ordinate axes of these projections. Then we have from equations (12), making $y = 0$ and $z = 0$ in the last two,

$$x_1 = -\frac{M_{ny}}{F_z} = \frac{M_{nz}}{F_y};$$

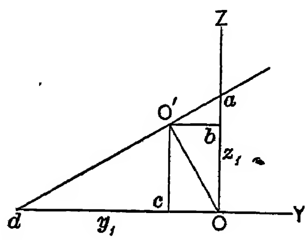
making $x = 0$ and $z = 0$ in the first and last,

$$y_1 = -\frac{M_{nz}}{F_x} = \frac{M_{nx}}{F_y};\qquad\qquad (13)$$

making $y = 0$ and $x = 0$ in the first two,

$$z_1 = -\frac{M_{nx}}{F_y} = \frac{M_{ny}}{F_x}.$$

Let the distance of the force F from the centre of mass O be p, and let p_x, p_y, p_z be its projections on the co-ordinate axes. Let the intersection of p with F be O', so that $OO' = p$.



Let us consider the projection of F on the plane YZ. In the figure $p_z = Ob$ and $p_y = Oc$. We have then

$$p_y = \frac{z_1}{y_1} p_z,$$

where z_1 and y_1 are the intercepts Oa and Od given by equations (13).

We have also the distance

$$ab = \frac{z_1}{y_1} p_y.$$

Hence

$$p_z = z_1 - ab = z_1 - \frac{z_1}{y_1} p_y.$$

Substituting the value of p_y, we obtain

$$p_z = \frac{z_1 y_1^2}{z_1^2 + y_1^2}, \quad p_y = \frac{z_1^2 y_1}{z_1^2 + y_1^2}.$$

Substituting the values of z_1 and y_1 from equations (13), we have

$$p_z = -\frac{F_y M_{nx}}{F_z^2 + F_y^2}, \quad p_y = \frac{F_z M_{nx}}{F_z^2 + F_y^2}.$$

In the same way we can find p_z and p_x, p_x and p_y on the other two co-ordinate planes. We thus have

$$\left.\begin{aligned} p_x &= -\frac{F_z M_{ny}}{F_x^2 + F_z^2} = \frac{F_y M_{nz}}{F_y^2 + F_x^2}, \quad p_y = -\frac{F_x M_{n\cdot}}{F_y^2 + F_x^2} = \frac{F_z M_{nx}}{F_z^2 + F_y^2}, \\ p_z &= -\frac{F_y M_{nx}}{F_z^2 + F_y^2} = \frac{F_x M_{ny}}{F_x^2 + F_z^2}. \end{aligned}\right\} \qquad (14)$$

The screw-wrench M_a is thus completely determined if the components of motion F_x, F_y, F_z, M_x, M_y, M_z are given. From (2) and (3) we have the magnitude and direction of the resultant force F, and from (14) its position. From (8) we have the moment M_a.

COMPONENTS OF MOTION.—Suppose, on the other hand, the screw-wrench M_a is given, that is, we have M_a and F and the position and direction of F given.

From equations (12) and (3) we have

$$\left.\begin{aligned} M_x &= M_a \cos \alpha - F(p_z \cos \beta - p_y \cos \gamma), & F_x &= F \cos \alpha; \\ M_y &= M_a \cos \beta - F(p_x \cos \gamma - p_z \cos \alpha), & F_y &= F \cos \beta; \\ M_z &= M_a \cos \gamma - F(p_y \cos \alpha - y_x \cos \beta), & F_z &= F \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

If, then, the screw-wrench M_a is given, we have from (15) the components of motion.

SCREW-WRENCH M.—The screw-wrench M is already completely determined. Its axis passes through the centre of mass, and its direction is given by equations (6), its moment by (5), and the force F_a along its axis by (10).

POSITION OF THE FORCE F_n.—We have seen (page 193) that the force system reduces to a force F_a along M at the centre of mass, or the spontaneous axis of angular acceleration, and a force F_n normal to this axis, at a distance $p = \frac{M}{F_n}$.

For any point on the line representative of F_n given by the co-ordinates x, y, z we have then

$$F_{nz}y - F_{ny}z = M_x, \qquad F_{nx}z - F_{nz}x = M_y, \qquad F_{ny}x - F_{nx}y = M_z.$$

These are the equations of the projections of F_n on the these co-ordinate planes.

Let x_1, y_1, z_1 be the intercepts on the co-ordinate axes of these projections. Then we have, just as on page 196,

$$x_1 = \frac{M_z}{F_{ny}} = -\frac{M_y}{F_{nz}}, \qquad y_1 = \frac{M_x}{F_{nz}} = -\frac{M_z}{F_{nx}}, \qquad z_1 = \frac{M_y}{F_{nx}} = -\frac{M_x}{F_{ny}}.$$

Proceeding, then, just as on page 196, we have for the co-ordinates p_x, p_y, p_z giving the position of F_n

$$\left.\begin{aligned} p_x &= -\frac{F_{nz}M_y}{F_{nx}^2 + F_{nz}^2} = \frac{F_{ny}M_z}{F_{ny}^2 + F_{nx}^2}, \qquad p_y = -\frac{F_{nx}M_z}{F_{ny}^2 + F_{nx}^2} = \frac{F_{nz}M_x}{F_{nz}^2 + F_{ny}^2}, \\ p_z &= -\frac{F_{ny}M_x}{F_{nz}^2 + F_{ny}^2} = \frac{F_{nx}M_y}{F_{nx}^2 + F_{nz}^2}. \end{aligned}\right\} \quad . \quad (16)$$

Equations (16) give the position of F_n.

The values of M_x, M_y, M_z are given by equations (4); the values of M and F_a by equations (5) and (10).

THE INVARIANT.—From (8) and (10) we have

$$FM_a = F_aM = F_xM_x + F_yM_y + F_zM_z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Whatever point we take for origin the moments M_a and M *at the centre of mass* remain unchanged, and F does not change for any origin.

The quantity

$$F_xM_x + F_yM_y + F_zM_z$$

remains the same, therefore, no matter where we take the origin, and we can write in general

$$FM_a = F_aM = F_xM'_x + F_yM'_y + F_zM'_z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where F_x, F_y, F_z, and F are given by equations (1) and (2); M'_x, M'_y, M'_z by equations (7); M and M_a by equations (5) and (8); and F_a by equation (10).

Since the quantity

$$F_xM'_x + F_yM'_y + F_zM'_z$$

is constant no matter where the origin is taken, it is called the *invariant* for non-concurring forces.

If the invariant is not zero, the force system reduces to the screw-wrench M_a and F given by (8), (2), (3) and (14). Or to two forces F_a and F_n given by (10), (6), (11) and (16).

If the invariant is zero and F and M are not, we have $M_a = 0$ and $F_a = 0$, and F is at right angles to M at the centre of mass. In this case the force system reduces to a single force $F = F_n$ at a point given by equations (16).

If the invariant is zero but M_a is not, then $F = 0$. In this case the force system reduces to a resultant couple or wrench M at the centre of mass.

If the invariant is zero but F_a is not, then $M = 0$ and the force system reduces to a resultant force at the centre of mass.

Thus, for example, let all the forces be coplanar. Take the plane as that of $X'Y'$. Then $F_z = 0$, $M'_x = 0$, $M'_y = 0$ and the invariant is zero. Since F and M are not zero, we have F at right angles to M at the centre of mass. If F is zero, we have then a resultant moment only. If M is zero, we have a resultant force only If neither F nor M are zero, we have a single force F only at a point given by equations (16).

Hence a system of coplanar forces must reduce to *either* a single force *or* a single couple. We cannot have both.

Let all the forces be parallel. Take them all parallel to the axis of Z'. Then $F_x = 0$, $F_y = 0$, $M'_z = 0$ and the invariant is zero. Since F and M are not zero, we have F at right angles to M at the centre of mass, and the force system reduces to a single force F only at a point given by equations (16). If F is zero, we have a resultant moment only. If M is zero, we have a resultant force only.

Hence a system of parallel forces must reduce to either a single force or a single couple. We cannot have both. If the parallel forces are all in the same direction, F cannot be zero and we must have a resultant force only.

Examples.—(1) *Find the resultant for a system of parallel coplanar forces given by*

$$\begin{array}{lll}
F_1 = +\ 33 \text{ lbs.,} & x_1 = +25 \text{ ft.,} & y_1 = +13 \text{ ft.;} \\
F_2 = +\ 20 \text{ ''} & x_2 = -10 \text{ ''} & y_2 = -15 \text{ ''} \\
F_3 = -\ 35 \text{ ''} & x_3 = +15 \text{ ''} & y_3 = -27 \text{ ''} \\
F_4 = -\ 72 \text{ ''} & x_4 = -31 \text{ ''} & y_4 = +17 \text{ ''} \\
F_5 = +120 \text{ ''} & x_5 = +23 \text{ ''} & y_5 = -19 \text{ ''}
\end{array}$$

ANS. $F = +66$ lbs., $p_x = +77.15$ ft., $p = -36.82$ ft.

If the forces are parallel to the axis of Y, $M_z = +5091.9$ lb.-ft.

If the forces are parallel to the axis of X, $M_z = +2430.12$ lb.-ft.

If we look along the line representative of the moment towards the origin, the rotation is seen counter-clockwise.

(2) *Find the resultant for the parallel-force system given by*

$$\begin{array}{llll} F_1 = +\ 60 \text{ lbs.}, & x_1 = 0, & y_1 = 0, & z_1 = 0; \\ F_2 = +\ 70 \text{ “} & x^2 = +1 \text{ ft.}, & y_2 = +2 \text{ ft.}, & z_2 = +3 \text{ ft.}; \\ F_3 = -\ 90 \text{ “} & x_3 = +2 \text{ “} & y_3 = +3 \text{ “} & z_3 = +4 \text{ “} \\ F_4 = -150 \text{ “} & x_4 = +3 \text{ “} & y_4 = +4 \text{ “} & z_4 = +5 \text{ “} \\ F_5 = +200 \text{ “} & x_5 = +4 \text{ “} & y_5 = +5 \text{ “} & z_5 = +6 \text{ “} \end{array}$$

ANS. $F = +90$ lbs., $p_x = +2\frac{8}{9}$ ft., $p_y = +3$ ft., $p_z = +3\frac{1}{9}$ ft.

If the forces are parallel to the axis of Y, we have

$$M_x = +315 \text{ lb.-ft.}, \qquad M_z = +240 \text{ lb.-ft.}, \qquad M_r = 396 \text{ lb.-ft.}$$

The line representative making the angles with the axes of X, Y, Z given by

$$\cos\alpha = +\frac{315}{396}, \qquad \cos\beta = 0, \qquad \cos\gamma = +\frac{240}{396},$$

or

$$\alpha = 322^\circ\ 41'\ 41'', \qquad \beta = 90^\circ, \qquad \gamma = 52^\circ\ 41'\ 41'.$$

If we look along the line representative towards the origin, the rotation is seen counter-clockwise.

(3) *Let a rigid body be acted upon by the coplanar forces*

$$F_1 = 50 \text{ lbs}, \quad F_2 = 30 \text{ lbs.}, \quad F_3 = 70 \text{ lbs.}, \quad F_4 = 90 \text{ lbs.}, \quad F_5 = 120 \text{ lbs.}$$

acting at the points given by

$$\begin{array}{l} x_1 = +\ 5 \text{ ft.}, y_1 = +10 \text{ ft.};\ x_2 = +\ 9 \text{ ft.}, y_2 = +12 \text{ ft.}; \\ x_3 = +17 \text{ ft.}, y_3 = +14 \text{ ft};\ x_4 = +20 \text{ ft.}, y_4 = +13 \text{ ft.}; \\ x_5 = +15 \text{ ft.}, y_5 = +\ 8 \text{ ft.} \end{array}$$

Let the forces make angles with the axes of X and Y given by

$$\begin{array}{l} \alpha_1 = 70^\circ, \beta_1 = 20^\circ;\ \alpha_2 = 60^\circ, \beta_2 = 150^\circ;\ \alpha_3 = 120^\circ, \beta_3 = 30^\circ; \\ \alpha_4 = 150^\circ, \beta_4 = 120^\circ;\ \alpha_5 = 90^\circ, \beta_5 = 0^\circ. \end{array}$$

Find the resultant, etc.

ANS. We have for the components parallel to the axes X and Y:

$$\begin{array}{l} F_x = 50 \cos 70^\circ + 30 \cos 60^\circ - 70 \cos 60^\circ - 90 \cos 30^\circ = -80.842 \text{ lbs.}; \\ F_y = 50 \cos 20^\circ - 30 \cos 30^\circ + 120 + 70 \cos 30^\circ - 90 \cos 60^\circ = +156.626 \text{ lbs.}; \\ F_z = 0. \end{array}$$

The resultant is given in magnitude by

$$F = \sqrt{F_x^2 + F_y^2} = 176.259 \text{ lbs.},$$

and its direction cosines by

$$\cos\alpha = \frac{F_x}{F} = \frac{-80.842}{176.259}, \quad \text{or} \quad \alpha = 117^\circ\ 18'\ 1'';$$

$$\cos\beta = \frac{F_y}{F} = \frac{+156.626}{176.259}, \quad \text{or} \quad \beta = 27^\circ\ 18'\ 1''.$$

We have from equation (4), page 193,

$$\Sigma Fx \cos\beta = +50 \cos 20^\circ \times 5 - 30 \cos 30^\circ \times 9 + 70 \cos 30^\circ \times 17 - 90 \cos 60^\circ \times 20 + 120 \times 15$$
$$= +1931.670 \text{ lb.-ft.};$$

$$\Sigma Fy \cos\alpha = +50 \cos 70^\circ \times 10 + 30 \cos 60^\circ \times 12 - 70 \cos 60^\circ \times 14 - 90 \cos 30^\circ \times 13 = -1152.245 \text{ lb.-ft.}$$

$$M_x = 0, \qquad M_y = 0, \qquad M_z = \Sigma Fx \cos\beta - \Sigma Fy \cos\alpha = +3083.915 \text{ lb.-ft.}$$

Since, then, equation (18), page 198,

$$F_x M_x + F_y M_y + F_z M_z = 0,$$

is satisfied, the forces reduce to a single resultant force.

The moment of this resultant force relative to the origin is

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2} = M_z = +3083.915 \text{ lb.-ft.}$$

Its lever-arm is

$$p = \frac{M}{F} = \frac{3083.915}{176.259} = 17.5 \text{ ft.}$$

The equation of the line of direction of the resultant is

$$y = \frac{F_y}{F}x - \frac{M_z}{F} = -1.95x + 38.14.$$

The co-ordinates of the point of application of the resultant are given from equation (10), page 195:

$$p_x = \frac{\Sigma Fx \cos\beta}{F_y} = \frac{+1931.67}{+156.626} = +12\tfrac{1}{3} \text{ ft.};$$

$$p_y = \frac{\Sigma Fy \cos\alpha}{F_x} = \frac{-1152.245}{-80.842} = +14.25 \text{ ft.}$$

(4) *Find the resultant, etc., for the force system acting on a rigid body given by*

$F_1 = 50$ lbs.,	$\alpha_1 = 60°$,	$\beta_1 = 40°$,	γ_1 acute;
$F_2 = 70$ "	$\alpha_2 = 65°$,	$\beta_2 = 45°$,	γ_2 obtuse;
$F_3 = 90$ "	$\alpha_3 = 70°$,	$\beta_3 = 50°$,	γ_3 acute;
$F_4 = 120$ "	$\alpha_4 = 75°$,	$\beta_4 = 55°$,	γ_4 obtuse.

$x_1 = 0$,	$y_1 = 0$,	$z_1 = 0$;
$x_2 = +1$ ft.,	$y_2 = +4$ ft.,	$z_2 = +7$ ft.;
$x_3 = +2$ "	$y_3 = +5$ "	$z_3 = +8$ "
$x_4 = +3$ "	$y_4 = +6$ "	$z_4 = +9$ "

ANS. We find the angles γ by the formula, page 13,

$$\cos^2\gamma = -\cos(\alpha+\beta)\cos(\alpha-\beta).$$

Then, from page 193, we have

$$F_x = +116.423 \text{ lbs.}, \quad F_y = +214.480 \text{ lbs.}, \quad F_z = -51.057 \text{ lbs.}$$

Therefore the resultant is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = +249.325 \text{ lbs.},$$

and its direction cosines are given by

$$\cos\alpha = \frac{F_x}{F_r}, \quad \cos\beta = \frac{F_y}{F_r}, \quad \cos\gamma = \frac{F_z}{F_r},$$

or $\alpha = 62°\ 9'\ 48''$, $\beta = 30°\ 39'\ 20''$, $\gamma = 101°\ 49'$.

We also have for the moments from equation (4), page 193,

$$M_x = -1838.604, \quad M_y = +928.947, \quad M_z = -86\ 903 \text{ lb.-ft.}$$

The resultant moment about the origin is

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2} = +2061.789 \text{ lb.-ft.},$$

and the direction cosines of its line representative are given by

$$\cos\alpha = \frac{M_x}{M_r}, \quad \cos\beta = \frac{M_y}{M_r}, \quad \cos\gamma = \frac{M_z}{M_r},$$

or $\alpha = 153°\ 5'\ 40''$, $\beta = 63°\ 14'\ 15''$, $\gamma = 92°\ 24'\ 56''$.

Looking along this line representative towards the origin, the direction of rotation is seen counter-clockwise.

The equations of the projection of the resultant on the co-ordinate planes are

$$y = 1.885x + 0.746, \quad x = -2.28z + 18.19, \quad z = -0.238y - 8.57.$$

We see that

$$F_xM_x + F_yM_y + F_zM_z$$

does not in this case equal zero. Hence, page 198, the forces do not reduce to a single resultant force, but to a resultant force and a couple.

The resultant force is, as already found, $F = 249.325$ lbs., and its angles with the co-ordinate axes are as already found.

The co-ordinates of the axis of the couple are given by equation (14), page 196:

$$p_x = \frac{F_yM_z - F_zM_y}{F_r^2} = +0.463 \text{ ft.}, \qquad p_y = \frac{F_zM_x - F_xM_z}{F_r^2} = +1\ 673 \text{ ft.}, \qquad p_z = \frac{F_xM_y - F_yM_x}{F_r^2} = +8.08 \text{ ft.}$$

The resultant couple M_a is given by equation (8), page 194,

$$M_a = \frac{F_xM_x + F_yM_y + F_zM_z}{F} = -\ 41.624 \text{ lb.-ft.}$$

The direction cosines of its line representative are the same as for the resultant F, and looking along this line representative towards the origin the rotation is seen counter-clockwise.

(5) *In the preceding example find what the co-ordinates x_4, y_4, z_4 of the force $F_4 = 120$ lbs. must be in order that all the forces may reduce to a single resultant.*

ANS. We evidently have F_x, F_y, F_z, F_r and the angles α, β, γ unchanged, since changing the point of application of F_4 without changing its direction or magnitude has no effect on the magnitude of the resultant or its direction.

We have then

$$\left.\begin{aligned} M_x &= -\ 659\ 571 - 93.262y_4 - 68.829z_4, \\ M_y &= +\ 369.629 + 31.059z_4 + 93.262x_4, \\ M_z &= -\ 107.036 + 68\ 829x_4 - 31.059y_4. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have as the equation of condition for a single resultant

$$F_xM_x + F_yM_y + F_zM_z = 0,$$

or

$$116.423M_x + 214.48M_y - 51.057M_z = 0,$$

or

$$M_x + 1.842M_y - 0.4386M_z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we obtain

$$(M_x + 659.571)31.059 + (M_y - 369.629)68.829 = (M_z + 107.036)93.262,$$

or

$$M_x + 2.216M_y - 3.003M_z = +\ 481.034, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3) we obtain

$$0.374M_y - 2.564M_z = +\ 481\ 034.$$

If we retain for M_y its value in the preceding example, $+\ 928\ 947$ lb.-ft., we shall have

$$M_z = -\quad 52.108 \text{ lb.-ft.},$$
$$M_x = -\ 1733.95 \quad \text{"}$$

If we substitute these values in (1), we obtain

$$93.262y_4 + 68\ 829z_4 = +\ 1074.4,$$
$$31.059z_4 + 93.262x_4 = +\quad 559.308,$$
$$68.829x_4 - 31.059y_4 = +\quad 54.934.$$

Hence

$$x_4 = -\ 0.333z_4 + \quad 5.997,$$
$$y_4 = -\ 0.738z_4 + 11.520.$$

If then we assume $z_4 = 0$, we have

$$x_4 = +\ 5.997, \qquad y_4 = +\ 11.520.$$

(6) *Using the values of the preceding example, find the point of application of the resultant.*

ANS. We have

$$F_x = +\ 116.423 \text{ lbs.}, \qquad F_y = +\ 214.480 \text{ lbs.}, \qquad F_z = -\ 51\ 057 \text{ lbs}, \qquad F_r = +\ 249.325 \text{ lbs.};$$
$$\alpha = 62°\ 9'\ 48'', \qquad \beta = 30°\ 39'\ 20'', \qquad \gamma = 101°\ 49';$$

$M_x = -1733.975$ lb.-ft., $M_y = +928.947$ lb.-ft., $M_z = -52.108$ lb.-ft., $M_r = +1967.823$ lb.-ft.;

$$\alpha = 151° 47', \quad \beta = 61° 49' 53'', \quad \gamma = 91° 31' 3''.$$

The co-ordinates p_x, p_y, p_z of the point of application of the resultant are given by

$$-1733.975 = F_z p_y - F_y p_z = -51.057\bar{y} - 214.480\bar{z},$$
$$+928.947 = F_x p_z - F_z p_x = +116.423\bar{z} + 51.057\bar{x},$$
$$-52.108 = F_y p_x - F_x p_y = 214.480\bar{x} - 116.423\bar{y}.$$

Hence we obtain

$$p_x = -2.2802_z + 18.194,$$
$$p_y = -4.2008_z + 33.961.$$

If we assume $p_z = 0$, we have then

$$p_x = +18.194 \text{ ft.}, \quad p_y = +33.961 \text{ ft.}$$

If we introduce, then, a fifth force, $F_5 = +249.325$ lbs., whose direction makes with the axes the angles

$$\alpha_5 = 117° 50' 12'', \quad \beta_5 = 149° 20' 40'', \quad \gamma_5 = 78° 11',$$

acting at a point whose co-ordinates are $p_x = +18.194$ ft. and $p_y = 33.961$ ft., $p_z = 0$, we have a system of forces in equilibrium.

(7) *Find the resultant, etc., for the parallel-force system given by*

$$\begin{array}{llll}
F_1 = +\ 60 \text{ lbs.}; & x_1 = 0, & y_1 = 0, & z_1 = 0; \\
F_2 = +\ 70 \text{ “} & x_2 = +1 \text{ ft.}, & y_2 = +2 \text{ ft.}, & z_2 = +3 \text{ ft.}; \\
F_3 = -\ 90 \text{ “} & x_3 = +2 \text{ “} & y_3 = +3 \text{ “} & z_3 = +4 \text{ “} \\
F_4 = -\ 150 \text{ “} & x_4 = +3 \text{ “} & y_4 = +4 \text{ “} & z_4 = +5 \text{ “} \\
F_5 = +\ 200 \text{ “} & x_5 = +4 \text{ “} & y_5 = +5 \text{ “} & z_5 = +6 \text{ “}
\end{array}$$

ANS. $F = \Sigma F = +90$ lbs.; $p_x = \dfrac{\Sigma Fx}{F} = +2\frac{2}{3}$ ft., $p_y = \dfrac{\Sigma Fy}{F} = +3$ ft., $p_z = \dfrac{\Sigma Fz}{F} = +3\frac{1}{3}$ ft.

CHAPTER VII.

FORCE OF GRAVITATION. CENTRE OF GRAVITY.

Force of Gravitation.—The "law of gravitation," as formulated by Newton, asserts that *every particle of matter attracts every other particle with a force which acts in the straight line joining the particles, and whose magnitude is directly proportional to the product of the masses of the particles, and inversely proportional to the square of the distance between them.*

If, then, m_1 and m_2 are the masses of two particles, and r is the distance between them, the mutual force of attraction F is given by

$$F = k\frac{m_1 m_2}{r^2}, \qquad \qquad (1)$$

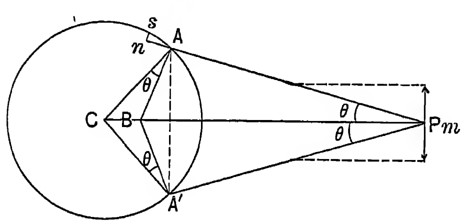
where k is a constant to be determined by experiment, and is called the *constant of gravitation.*

For absolute accuracy and universal generality, as well as for far-reaching consequences, this statement is without parallel in the history of science. The facts that by means of it the motions of all the bodies of the solar system are completely explained, that their past and future positions can be told; that the existence of Neptune was deduced from the assumption that certain disturbances in the motion of Uranus were due to the attraction of an unknown planet according to this law, all go to prove that the law holds with absolute accuracy, so far as the action upon each other of large masses *separated by distances which are great compared with their linear dimensions is concerned.*

The enunciation of the law expressly confines it to such cases, since only when the linear dimensions of the attracting bodies are insignificant compared to the distance between them can we conceive them as "particles."

We shall, however, show in the next article that if bodies are homogeneous and spherical, this limitation may be removed and the "distance between them" is the distance between their centres of mass.

Attraction of a Homogeneous Shell or Sphere.—Let the circle ACA', with centre at C, represent a uniform thin homogeneous spherical shell whose surface density is δ. Suppose a particle at P whose mass is m. Join C and P. Take any point A of the shell and draw CA and AP. Let AP make the angle θ with CP, and draw a line AB through A, making the same angle θ with CA.



Then in the two triangles CAB and CAP we have the side CA and the angle at C common to both, and the angles at A and P equal by construction. These triangles are therefore similar and we have

$$\frac{AB}{AP} = \frac{CA}{CP}.$$

Now let As represent any small elementary area of the spherical surface, and An its projection normal to AB.

Let ω square radians (page 7) denote the conical angle subtended at B by An. Then the area denoted by An is equal to $\overline{AB}^2 . \omega$, and the area denoted by As is equal to $\dfrac{\overline{AB}^2 . \omega}{\cos\theta}$, since the angle $nAs = BAC = \theta$, and the angle snA is a right angle.

The mass of the elementary area denoted by As is then $\dfrac{\delta\overline{AB}^2 . \omega}{\cos\theta}$, and the attraction of this mass for the particle of mass m at P is, by Newton's law,

$$k\frac{m . \delta\overline{AB}^2 . \omega}{\overline{AP}^2 \cos\theta},$$

and acts in the line AP.

If we draw AA' perpendicular to CP, we have evidently the same attraction between the equal elementary mass at A' and the particle of mass m at P acting in the line $A'P$.

We can resolve each of these equal forces into a component along the line CP and at right angles to CP at P. Since the angles APC and $A'PC$ are each equal to θ, the two components at right angles to CP at P are equal and opposite and therefore produce no effect upon P. The resultant attraction of the two elements at A and A' upon the particle of mass m at P acts then in the line CP and is equal to

$$2k\frac{m . \delta\overline{AB}^2 . \omega}{\overline{AP}^2 \cos\theta}\cos\theta = 2k\frac{m . \delta\overline{AB}^2 . \omega}{\overline{AP}^2},$$

or, since $\dfrac{AB}{AP} = \dfrac{CA}{CP}$, the resultant attraction is

$$2k\frac{m . \delta\overline{CA}^2 . \omega}{\overline{CP}^2}.$$

But $\overline{CA}^2 . \omega$ is the area of the elementary area at A or A', and $2k\dfrac{m\delta}{\overline{CP}^2}$ is constant for all pairs of elements A and A'. The total attraction of the shell for the particle of mass m at P acts, then, in the line CP and is equal to

$$2k\frac{m . \delta}{\overline{CP}^2}\Sigma\overline{CA}^2 . \omega,$$

where the summation is to be taken for an entire hemisphere. But $\Sigma\overline{CA}^2 . \omega$ for a hemisphere is $2\pi\overline{CA}^2$, and hence the attraction is equal to

$$F = k\frac{4\pi\delta\overline{CA}^2 . m}{\overline{CP}^2} = k\frac{m\overline{\mathrm{m}}}{\overline{CP}^2},$$

where $\overline{\mathrm{m}} = 4\pi\delta\overline{CA}^2$ is the total mass of the spherical shell.

We see, then, that the spherical shell attracts a particle of mass m at any outside point P, just as if its entire mass were condensed at the centre of the shell.

If instead of a homogeneous spherical shell we have a solid homogeneous sphere, we may consider it as composed of an indefinite number of concentric homogeneous spherical shells, each of which attracts the mass at P as if its entire mass were condensed at its centre.

Hence *the attraction of a homogeneous spherical shell or of a homogeneous sphere upon a particle at any outside point is the same as if the entire mass of the shell or sphere were condensed in a point at the centre.*

We can therefore consider a homogeneous shell or sphere as a particle of equal mass at the centre, so far as its attraction upon an outside particle is concerned.

COR.—If the sphere is not homogeneous, but the density of every point at the same distance from the centre is the same, we may still consider the sphere as composed of homogeneous spherical concentric shells, each one of which attracts an outside mass as if its entire mass were condensed at the centre. Hence the same holds true for the sphere.

Value of Constant of Gravitation.—We have seen that Newton's law is expressed by

$$F = k\frac{m_1 m_2}{r^2}, \quad \ldots \ldots \ldots \quad (1)$$

where the constant k must be determined by experiment. This determination we are now able to make.

Thus for a body of mass m_1 at any point on the earth's surface where the acceleration is g, we know that the force of gravity is $F = m_1 g$ poundals. Let m_0 be the mass of the earth, and let r_0 be the radius of the earth at the place of experiment.

If we consider the earth as a sphere whose density is either constant or the same at all points at equal distances from the centre, then, as we have just seen, we may consider it as a particle of equal mass at the centre of mass so far as its attraction upon any outside particle is concerned, and the centre of mass is the centre of figure.

The earth is not strictly spherical, but its deviation from sphericity is very slight. Also, the density is not constant nor strictly the same at points equidistant from the centre. But the small distance between the centre of mass and that point at which in any case of attraction we may consider its mass condensed is insignificant compared to its radius. So far as its attraction for any outside particle is concerned, we may then still consider it as a particle of equal mass at the centre of mass, and the centre of mass as the centre of figure.

Also, since the dimensions of any body with which we experiment at the earth's surface are insignificant compared to the radius, we may consider any such body as a particle. Hence in equation (1) we can take r as the radius of the earth r_0, and m_2 as the mass m_0 of the earth, and we then have

$$m_1 g = k\frac{m_1 m_0}{r_0^2}.$$

Hence we have

$$k = \frac{g r_0^2}{m_0}. \quad \ldots \ldots \ldots \quad (2)$$

Substituting this value of κ in (1), we have

$$F = \frac{m_1 m_2}{r^2} \cdot \frac{r_0^2}{m_0} g. \quad \ldots \ldots \ldots \quad (3)$$

Equation (3) gives the force of attraction F between two masses m_1 and m_2 at any distance r, in absolute units. Thus if m_0, m_1, m_2 are taken in lbs., r_0 and r in feet, and g in ft.-per-sec. per sec., equation (3) gives F in poundals.

If we take mass in grams and distance in centimeters, we have F in dynes.

If we divide by g, we have in gravitation units

$$F = \frac{m_1 m_2}{r^2} \cdot \frac{r_0^2}{m_0} \quad \ldots \ldots \ldots \quad (4)$$

If M is the mass of the sun and m the mass of a planet, we have, from (3), for the mutual force of attraction

$$F = \frac{Mm}{r^2} \cdot \frac{r_0^2}{m_0} g.$$

Since force equals mass × acceleration, if we divide this by M, we have for the acceleration SO of the sun relative to a fixed point O $\quad S \longrightarrow O$

$$\frac{m}{r^2} \cdot \frac{r_0^2}{m_0} g.$$

If we divide by m, we have for the acceleration PO of the planet relative to a fixed point O $\quad O \longleftarrow P$

$$\frac{M}{r^2} \cdot \frac{r_0^2}{m_0} g$$

opposite in direction to SO. We have then for the acceleration PS of the planet relative to the sun $PO + OS$, or $\quad S \longleftarrow O \longleftarrow P$

$$\frac{M + m}{r^2} \cdot \frac{r_0^2}{m_0} g.$$

At a distance r_0 we have, making $r = r_0$, the acceleration f_0 of the planet relative to the sun

$$f_0 = \frac{M + m}{m_0} \cdot g. \quad \ldots \ldots \ldots \quad (5)$$

This is the value of f_0 to be used in our equations for planetary motion, page 123.

Astronomical Unit of Mass.—That mass which attracts an equal mass at unit distance with unit force is called the ASTRONOMICAL UNIT OF MASS.

Let $[F]$ denote the unit of force, and $[L]$ the unit of length. Then, from equation (3), we have for the astronomical unit of mass $[m]$

$$[F] = \frac{[m]^2}{[L]^2} \cdot \frac{r_0^2}{m_0} g.$$

Hence we obtain for the astronomical unit of mass $[m]$

$$[m] = \sqrt{\frac{m_0 [F][L]^2}{g r_0^2}}. \quad \ldots \ldots \ldots \quad (6)$$

Equation (6) gives by definition the astronomical unit of mass.

The mass m_0 of the earth is about 11920×10^{21} lbs. The mean radius of the earth r_0 is about 21×10^6 ft. Taking these values and $g = 32$ ft.-per-sec. per sec., we have, from (6), for the astronomical unit of mass

$$[m] = 29063 \text{ lbs.}$$

Taking m_0 equal to 6.14×10^{27} grams, r_0 equal to 6.37×10^8 cm. and $g = 981$ cm.-per-sec. per sec., we have

$$[m] = 3928 \text{ grams.}$$

If we adopt the astronomical unit of mass, we can then write simply the numeric equation

$$F = \frac{m_1 m_2}{r^2},$$

where m_1 and m_2 are the number of astronomical units of mass in the two particles, r the number of units of length in the distance between them, and F the number of absolute units of force in the attraction.

Centre of Gravity.—When a body attracts and is attracted by all external bodies, whatever their distance and position, as though its mass were condensed in a single point fixed relatively to the body, that point is properly called the CENTRE OF GRAVITY (page 23).

A body which has a centre of gravity is said to be *centrobaric* or *barycentric*. In general bodies are not centrobaric if the law of attraction follows Newton's law, that is, if the force is inversely as the square of the distance.

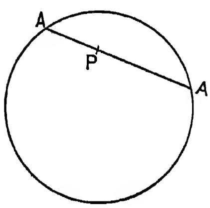
As we have seen, page 205, a homogeneous spherical shell or a homogeneous sphere is centrobaric and the centre of gravity is at the centre of mass. So also for a non-homogeneous sphere whose density is the same at all equidistant points from the centre.

In general if a body has a centre of gravity, it must always coincide with the centre of mass, because the attraction of an infinitely distant body upon it constitutes a system of parallel particle forces, and the centre for such a system is (see page 190) at the centre of mass.

All bodies, then, have a centre of mass, but, as we have seen, only a few bodies have a centre of gravity.

Examples.—(1) *Show that the attraction of a thin spherical shell of uniform thickness and density upon a particle inside is zero.*

ANS. Let P be the particle of mass m_1. Take any point A on the spherical surface Join AP and produce to A'. If from all points of a small element of the surface at A lines be drawn through P, they will mark off a corresponding element at A'. Both these elements subtend the same conical angle (page 7), ω square radians. The area of the element at A is then $\overline{AP}^2 . \omega$ (page 7), and the area of the element at A' is $\overline{A'P}^2 . \omega$. If δ is the uniform surface density, the mass of the element at A is $m_2 = \delta \overline{AP}^2 . \omega$ and the mass of the element at A' is $m_2' = \delta \overline{A'P}^2 . \omega$. The attraction of the element at A for a particle of mass m_1 at P is then (page 203)



$$\frac{km_1 \delta \overline{A'P}^2 . \omega}{\overline{AP}^2} = km_1 \delta \omega$$

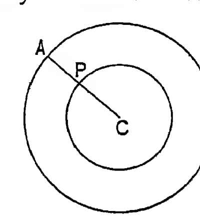
and acts in the line PA. The attraction of the element at A' for the particle of mass m at P is

$$\frac{km_1 \delta \overline{A'P}^2 . \omega}{\overline{A'P}^2} = \kappa m_1 \delta \omega$$

and acts in the line PA'. The resultant attraction upon the particle at P of the pair of elements at A and A' is then zero. The whole shell consists of such pairs of elements. Hence the resultant attraction of the shell on a particle at P is zero.

(2) *Show that the attraction of a homogeneous sphere on a particle within it is directly proportional to its distance from the centre.*

ANS. Let P be a particle of mass m_1 situated within a homogeneous sphere at any distance PC from the centre C. Then, from the preceding example, we know that the attraction upon the particle at P due to the shell outside of the sphere whose radius is PC is zero. The attraction upon the particle of mass m_1 at P is then due to the attraction of the sphere whose radius is PC. The volume of the sphere is $\frac{4}{3}\pi \overline{PC}^3$. If δ is the uniform density, the mass of this sphere is $\frac{4}{3} \delta \pi \overline{PC}^3$. Its attraction for a particle of mass m_1 at P is (page 205) the same as if the entire mass of the sphere were condensed at the centre, or (page 203) $km_1 \frac{\frac{4}{3}\delta\pi \overline{PC}^3}{\overline{PC}^2} = km_1 . \frac{4}{3}\delta\pi . \overline{PC}.$



The attraction is therefore directly proportional to the distance PC of the particle from the centre.

(3) *Assuming the earth to be a homogeneous sphere, compare its attraction on a given mass at a distance from its centre equal to one half its radius, with the attraction when the given mass is at a distance equal to twice the radius.*

ANS. 2 to 1.

(4) *Find in dynes the attraction of two homogeneous spheres, each of 100 kilograms mass, with their centres 1 metre apart.*

ANS. 0.0648 dynes nearly.

(5) *How far would a body fall toward the earth in one second from a point at a distance from the earth's surface equal to the radius of the earth?*

ANS. The acceleration is inversely as the square of the distance. We have then $g' : g :: r^2 : 4r^2$, of $g' = \frac{1}{4}g$. That is, the acceleration is one fourth of the acceleration at the surface.

The distance is then $s = \frac{1}{2}g't^2$, or, taking $g = 32$ ft.-per-sec. per sec. and $t = 1$, $s = 4$ ft.

(6) *The moon's mass is 136×10^{21} lbs.; the moon's radius, 5.70×10^6 ft.; the mass of the earth, 11920×10^{21} lbs.; the radius of the earth, 21×10^6 ft. Find how far a stone at the moon's surface would fall in a second, the attraction of the earth being neglected.*

ANS. If M is the mass of the moon and m that of the stone, the force of attraction, if R is the radius or the moon, is, from equation (3), page 205,

$$F = \frac{mM}{R^2} \cdot \frac{r^2}{E}g.$$

The acceleration of the stone at the moons's surface is then

$$g' = \frac{F}{m} = \frac{gr^2}{E} \cdot \frac{M}{R^2} = \frac{32 \times 21^2 \times 10^{12} \times 136 \times 10^{21}}{11920 \times 10^{21} \times (5.7)^2 \times 10^{12}} = 5 \text{ ft.-per-sec. per sec.}$$

The distance, then, is $\frac{1}{2}g't^2$, or, taking $t = 1$ sec., $s = 2.5$ ft.

(7) *Suppose the earth to contract until its diameter is 6000 miles, what would be the effect on the weight of an inhabitant? The diameter of the earth to be taken at 8000 miles.*

ANS. Increased in the ratio of 16 to 9.

(8) *If the mass of the sun is 300,000 times the mass of the earth, and its radius is 100 times the radius of the earth, find the attraction at the surface of the sun of a mass which at the surface of the earth is attracted by the force of one pound weight.*

ANS. $30g$ poundals, or the attraction of the earth for 30 lbs.

(9) *The diameter of Jupiter is 10 times that of the earth, and its mass 300 times. By how much per cent of his former weight would the weight of a man be increased by being removed to the surface of Jupiter?*

ANS. By 200 per cent. He would weigh by a spring-balance three times as much as before. The same number of standard pounds would, however, balance him in a lever-balance. The standard pound at Jupiter would be attracted by a force three times as great as the earth's attraction here. The lever-balance weight which gives his mass is unchanged.

(10) *Find the acceleration due to the attraction of the earth at the distance of the moon. Assume $g = 32$ at the earth's surface, the diameter of the moon's orbit 480,000 miles, the diameter of the earth 8000 miles.*

ANS. 0.0089 ft.-per-sec. per sec.

STATICS. GENERAL PRINCIPLES.

CHAPTER I.

EQUILIBRIUM OF FORCES. DETERMINATION OF MASS.

Statics.—That portion of Dynamics which treats of balanced forces, and of bodies at rest under the action of balanced forces, is called STATICS. Statics is thus a special case of Dynamics which it is convenient to consider separately.

Equilibrium of Concurring Forces.—Any number of forces acting upon a point or particle are said to be concurring forces. If the resultant of any system of concurring forces is zero, the forces are said to be in equilibrium.

The effect of the resultant force is to cause acceleration of the particle. When the forces are in equilibrium, then, the particle has no acceleration, and if the particle is at rest it is said to be in *static* equilibrium.

But if the resultant force is zero, all the concurring forces must evidently reduce to two equal and opposite forces, or, what is the same thing, any one of the forces must be equal and opposite to the resultant of all the others.

Hence the algebraic sum of the components of all the forces in any three rectangular directions must be zero.

We have then, from equations (1), page 194, for the conditions of equilibrium of concurring forces

$$F_x = \Sigma F \cos \alpha = 0, \qquad F_y = \Sigma F \cos \beta = 0, \qquad F_z = \Sigma F \cos \gamma = 0.$$

We obtain, then, the following obvious results from the condition for equilibrium of concurring forces, which will be found useful in special cases:

(1) If two concurring forces are in equilibrium, they must be equal in magnitude and opposite in direction.

(2) If three concurring forces are in equilibrium, they must all act in the same plane. For the resultant of any two must act in their plane and be equal and opposite to the third.

(3) If three concurring forces are represented in magnitude and direction by the sides of a triangle taken the same way round, the resultant is zero and the forces are in equilibrium.

(4) Hence, if three concurring forces are in equilibrium, each one is proportional to the sine of the angle between the other two.

(5) If three concurring forces are in equilibrium and their directions are represented by the sides of a triangle taken the same way round, their magnitudes will also be represented by the sides of that triangle, and *vice versa*.

(6) If any number of concurring coplanar forces are represented in magnitude and direction by the sides of a plane closed polygon taken the same way round, they are in equilibrium. If their magnitudes are given by the sides of the polygon, their directions are also given by the directions of the sides.

But if the directions only of the forces are given by the sides of the plane polygon, it does not follow that the sides of this polygon represent the magnitudes, because any number of plane polygons with parallel sides may be drawn, the magnitudes of the sides varying.

(7) If three concurring forces in different planes are represented by the three edges of a parallelopipedon, the diagonal taken the opposite way round will represent the resultant in direction and magnitude. This is called the *parallelopipedon of forces.*

Examples.—(1) *Find the resultant of forces of 7, 1, 1, 3 units, represented by lines drawn from one angle of a regular pentagon towards the other angles taken in order.*

ANS $\sqrt{74}$ units.

(2) *P and Q are two component forces at right angles, whose resultant is R. S is the resultant of R and P. If $Q = 2P$, what is S?*

ANS. $S = 2P\sqrt{2}$.

(3) *Component forces P, Q, R are represented in direction by the sides of an equilateral triangle taken the same way round. Find the magnitude of the resultant.*

ANS. $\sqrt{P^2 + Q^2 + R^2 - QR - PR - PQ}$.

(4) *A weight of 10 tons is hanging by a chain 20 feet long. Find how much the tension in the chain is increased by the weight being pulled out by a horizontal force to a distance of 12 feet from the vertical.*

ANS. By 2.5 tons.

(5) *A weight of 4 pounds is suspended by a string, and is acted upon by a horizontal force. If in the position of equilibrium the tension of the string is 5 pounds, what is the horizontal force?*

ANS. 3 lbs.

(6) *A mass of 10 lbs. is supported by strings of lengths 3 and 4 feet attached to two points in the ceiling 5 feet apart. What is the tension of each string?*

ANS. 8 lbs. and 6 lbs.

(7) *A particle is acted on by a force whose magnitude is unknown, but whose direction makes an angle of 60° with the horizon. The horizontal component of the force is 1.35 dynes. Determine the total force and its vertical component.*

ANS. 2.7 dynes and 2.34 dynes.

(8) *Three forces proportional to 1, 2, 3, act on a point. The angle between the first and second is 60°, between the second and third 30°. Find the angle which the resultant makes with the first.*

ANS. About 67°.

(9) *Three cords are tied together at a point. One is pulled in a northerly direction with a force of 6 pounds, and another in an easterly direction with a force of 8 pounds. With what force must the third be pulled in order to keep the whole at rest?*

ANS. 10 pounds, at an angle with the horizon whose tang $= \frac{3}{4}$.

Static — Molar — Dynamic and Molecular Equilibrium.—Forces acting at different points of a body are said to be non-concurring forces.

We have seen (page 188) that any number of non-concurring forces acting upon a rigid body can be reduced to a resultant force and a resultant couple at the centre of mass.

The effect of the resultant force (page 190) is to cause linear acceleration of translation of the body. The effect of the resultant couple (page 187) is to cause angular acceleration of the body about an axis through the centre of mass.

If both resultant force and couple are zero and the body is at rest, it will remain at rest and then it is said to be in *static* equilibrium. If every point has the same velocity of translation, this velocity of translation will be uniform, and the body is then in *molecular* equi-

librium. If the body has angular velocity about an axis through the centre of mass, this angular velocity is unchanged and the body is said to be in *molar* equilibrium.

If only the resultant force is zero, but not the couple, the centre of mass is either at rest or moves with uniform velocity, but the angular velocity changes. The body is then said to be in *dynamic* equilibrium.

Equilibrium of Non-Concurring Forces.—When both the resultant force and couple are zero the forces are in equilibrium. In such case all the forces must evidently reduce to two equal and opposite forces in *the same straight line*, or what is the same thing, any one of the forces must be equal and opposite to the resultant of all the others, and lie in the same straight line with it.

We have, then, for the equilibrium of non-concurring forces two necessary conditions.

1st. *The algebraic sum of the components of all the forces in any three rectangular directions must be zero.*

From equations (1), page 194, we have, then,

$$F_x = \Sigma F \cos \alpha = 0, \qquad F_y = \Sigma F \cos \beta = 0, \qquad F_z = \Sigma F \cos \gamma = 0. \quad . \quad . \quad (1)$$

When these conditions are complied with, there is no resultant force on the centre of mass of the body, and any one force is equal and opposite to the resultant of all the others but does not necessarily lie in the same straight line with it.

2d. *The algebraic sum of the component moments in any three rectangular planes must be zero.*

From equations (4), page 194, we have, then,

$$\left.\begin{aligned} M_x &= \Sigma F \cos \gamma . y - \Sigma F \cos \beta . z = 0, \\ M_y &= \Sigma F \cos \alpha . z - \Sigma F \cos \gamma . x = 0, \\ M_z &= \Sigma F \cos \beta . x - \Sigma F \cos \alpha . y = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When these conditions are complied with there is no resultant moment at the origin. But it does not necessarily follow that there is no moment at any other point unless the resultant force is also zero.

When, then, both equations (1) and (2) are satisfied the forces are in equilibrium, and the body, if at rest, remains at rest and is in *static equilibrium*. If every point has the same velocity of translation, it is in *molecular* equilibrium. If the body has angular velocity about an axis through the centre of mass, it is in *molar* equilibrium.

If only equations (1) are satisfied but not (2), the forces reduce to a couple. The body is then in dynamic equilibrium.

If only equations (2) are satisfied but not (1), we have a single resultant force *passing through the origin.*

COR. 1.—If the forces are all co-planar, let XY be their plane. Then $z = 0$, $\gamma = 90°$, and we have

$$F_x = \Sigma F \cos \alpha = 0, \qquad F_y = \Sigma F \cos \beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M_z = \Sigma F \cos \beta . x - \Sigma F \cos \alpha . y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is

1st. *The algebraic sum of the components of all the forces in any two rectangular directions in the plane of the forces must be zero.*

2d. *The algebraic sum of the moments of the forces about any point in this plane must be zero.*

If the first condition only is satisfied, we have a couple.

If the second only is satisfied, there is a single resultant *passing through the origin.*

COR. 2.—If three forces acting at different points of a rigid body are in equilibrium, *their lines of direction must intersect in a common point and the forces must be coplanar.*

For the resultant of any two must pass through their point of intersection and lie in their plane. The third must be equal and opposite to this resultant and lie in the same straight line.

COR. 3.—If the forces are parallel, take their common direction parallel to the axis of Y. Then $\cos\alpha = 0$, $\cos\gamma = 0$, $\cos\beta = 1$, $F_x = 0$, $F_z = 0$, $F_y = \Sigma F$, and we have

$$\Sigma F = 0, \quad \ldots \quad (1)$$

$$\left.\begin{aligned}\Sigma Fx &= 0,\\ \Sigma Fy &= 0.\end{aligned}\right\} \quad \ldots \quad (2)$$

That is:

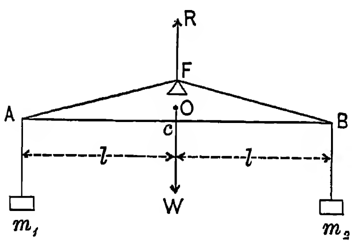
1st. *The algebraic sum of the forces must be zero.*

2d. *The algebraic sum of the component moments of the forces in any two rectangular planes must be zero.*

If the first condition only is satisfied, we have a couple.

If the second condition only is satisfied, the resultant passes through the origin and coincides with the axis of Y.

Determination of Mass by the Balance.—Let two masses $\overline{\mathrm{m}}_1$ and $\overline{\mathrm{m}}_2$ be suspended from the arms of an equal-armed balance, the fulcrum F being in a vertical through the centre of mass O of the balance. Let the balance before the masses $\overline{\mathrm{m}}_1$ and $\overline{\mathrm{m}}_2$ are applied be horizontal and at rest. Then the balance is in static equilibrium, and its weight W acting at the centre of mass O must be equal and opposite to the upward pressure of the fulcrum and in the same line.



Now let the masses $\overline{\mathrm{m}}_1$ and $\overline{\mathrm{m}}_2$ be suspended, the length of arm $AC = BC$ being l, and suppose the balance still remains horizontal and at rest. It is then still in static equilibruim and the acting forces are in equilibrium. These forces are the weight W of the balance, the weights $\overline{\mathrm{m}}_1 g$ and $\overline{\mathrm{m}}_2 g$ of the masses, and the upward reaction R of the fulcrum.

We have, then:

1st. Algebraic sum of the forces equal to zero, or

$$R - W - \mathrm{m}_1 g - \mathrm{m}_2 g = 0, \quad \text{or} \quad R = W + \mathrm{m}_1 g + \mathrm{m}_2 g.$$

2d. Algebraic sum of the moments about any point zero. Take F as the point. Then

$$\mathrm{m}_1 g \cdot l - \mathrm{m}_2 g l = 0, \quad \text{or} \quad \mathrm{m}_1 = \mathrm{m}_2.$$

Hence for equilibrium *the masses must be equal.* (See page 173.)

Examples.—(1) *A circular table weighing m lbs. has three equal legs at equidistant points on its circumference. The table is placed on a level floor. Neglecting the mass of the legs and considering table and floor and legs rigid, find the mass which, placed anywhere on the table, will just bring it to the point of overturning.*

ANS m lbs.

(2) *If the table has four legs at equidistant points on the circumference, find the mass that will just bring it to the point of overturning.*

ANS. 2.4 m.

(3) *The centre of mass of a ladder weighing 50 lbs. is 12 ft. from one end, which is fixed. What force must a man apply at a distance of 6 ft. from this end to just raise the ladder?*

ANS. 100 lbs.

(4) *A horizontal beam of length l is supported at its ends. It is acted upon by vertical downward forces F_1, F_2, F_3 acting at points of application A_1, A_2, A_3, dividing the beam into segments b, c, d, e. Find the resultant pressures R_1, R_2 at the right and left supports, neglecting the weight of the beam.*

ANS. R_1 and R_2 act upwards, and are given by

$$R_1 = \frac{F_1(l-b) + F_2(d+e) + F_3 e}{e}, \qquad R_2 = \frac{F_3(l-e) + F_2(b+c) + F_1 b}{e}.$$

(5) *A mass of 6 lbs. hangs on the arm of a safety-valve at a distance of 18 inches from the fulcrum. The valve-spindle is attached at 1 inch from the fulcrum. Disregarding friction and the weight of the arm, find the steam-pressure for static equilibrium.*

ANS. 108 pounds.

(6) *In a wheel and axle the radius of the axle is r, and of the wheel R. A mass Q hangs by a rope wound round the axle. Find the force P acting tangent to the wheel in order to hold Q suspended, disregarding friction.*

ANS. $P = \frac{Qr}{R}$.

(7) *A shopkeeper has correct weights but an untrue balance, one arm of which is a and the other b. He serves out to each of two customers, according to his balance, m lbs. of a commodity, using first one scale-pan and then the other. Does he gain or lose, and how much?*

ANS. Loses $m\frac{(a-b)^2}{ab}$ lbs.

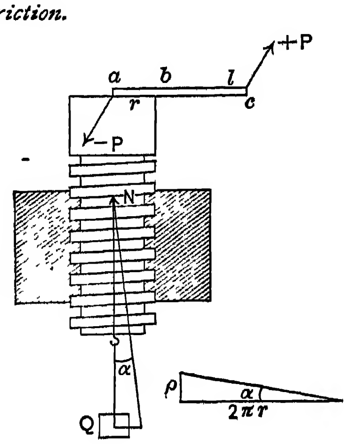
(8) *The arms of a balance are unequal, and one of the scale-pans is loaded. A body the true mass of which is m lbs. appears when placed in the loaded pan to weigh m_1 lbs., and when placed in the other m_2 lbs. Find the ratio of the arms and the mass with which the pan is loaded.*

ANS. Ratio of arms $= \frac{m_1 - m}{m - m_2}$; mass $= \frac{m^2 - m_1 m_2}{m_1 - m}$.

(9) *A balance is in equilibrium and unloaded. A body in one scale-pan is found to balance m_1 lbs. In the other scale-pan it balances m_2 lbs. Find the true mass.*

ANS. $\sqrt{m_1 m_2}$.

(10) *Find the condition for static equilibrium for a screw, neglecting friction.*



ANS. Let $+P$, $-P$ be the couple applied at the top with lever-arm $ac = l$; let $ab = r$ be the radius of the screw, and p the pitch or distance between the threads, so that if the screw be developed we have an inclined plane, which has a rise p for a base $2\pi r$. Let α be the inclination of the screw to the horizontal, so that

$$\tan \alpha = \frac{p}{2\pi r}.$$

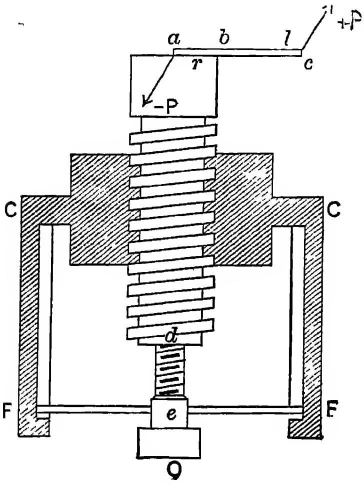
Let Q be the mass supported. We can resolve Q into a normal to the thread $\frac{Q}{\cos \alpha}$ and a horizontal force $Q \tan \alpha$. The normal is balanced by the upward normal pressure N on the threads. The horizontal components at every two diametrically opposite points of the thread are equal and opposite. We have then a couple $Q \tan \alpha \times r$ balanced by Pl, or

$$Q \tan \alpha \times r = Pl; \therefore P = \frac{Qr}{l} \tan \alpha.$$

Inserting the value for $\tan \alpha$,

$$P = \frac{Qp}{2\pi l}$$

(11) *The differential screw consists of a screw $a\,d$ which works in a fixed nut CC. The screw is hollow and has a thread cut inside in which a screw $d\,e$ works. This screw $d\,e$ is prevented from turning by the rod FF, whose ends slide in vertical grooves. Find the condition for static equilibrium, neglecting friction.*



ANS. Let p_1 and p_2 be the pitch of the screws, and α the common inclination of the threads, and r_1 and r_2 the radii of the screws. When the arm $ac = l$ turns through 2π radians screw ad rises a distance p_1 and screw de falls a distance p_2.

We have then just as before, in the preceding example,

$$Q \tan \alpha \times r_1 - Q \tan \alpha \times r_2 = Pl,$$

Hence

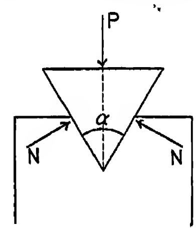
$$P = \frac{Q(r_1 - r_2) \tan \alpha}{l},$$

or, since $\tan \alpha = \dfrac{p_1}{2\pi r_1} = \dfrac{p_2}{2\pi r_2}$,

$$P = \frac{Q(p_1 - p_2)}{2\pi l}.$$

If $p_2 = 0$, we have the simple screw. By making p_1 and p_2 nearly equal, we can have P very small compared to Q.

(12) *Let the force P act normally upon the middle of the back of an isosceles wedge. Find the condition of static equilibrium, neglecting friction.*



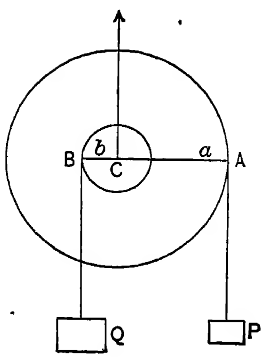
ANS. The pressure N on each side must be normal. Let α be the angle of the wedge. Then

$$P - 2N \sin \frac{\alpha}{2} = 0, \quad \text{or} \quad P = 2N \sin \frac{\alpha}{2}.$$

Hence

$$N = \frac{P}{2 \sin \frac{\alpha}{2}}.$$

If α is very small, N may be made very great.



(13) *In a wheel and axle the radius of the wheel is $CA = a$, and of the axle $CB = b$. Find the conditions for static equilibrium when a mass P hung from the wheel balances a mass Q hung from the axle, neglecting friction and rigidity of ropes.*

ANS. The weight of P is Pg; of Q, Qg poundals. Let R be the upward pressure of the journal bearings. Then

$$R - Pg - Qg = 0, \quad \text{or,} \quad R = (Pg + Qg) \text{ poundals, or, in gravitation units,}$$
$$R = (P + Q).$$

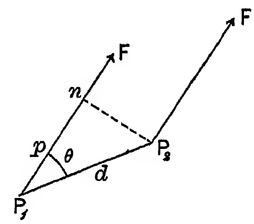
Taking moments about the centre C, we have

$$Qgb - Pga = 0, \quad \text{or} \quad Q = \frac{a}{b} P.$$

CHAPTER II.

WORK. VIRTUAL WORK.

Work.—The product of a uniform force by the component displacement along the line of the force of its point of application, is called the WORK of the force.



Thus let F be a force acting at the point P_1, and let the displacement be $d = P_1P_2$. Let F be uniform during displacement, and let the projection P_1n along the line of the force of the displacement be $p = d \cos \theta$, where θ is the angle of the displacement with F.

Then the work of F is

$$\pm Fp = \pm Fd \cos \theta.$$

If the projection p is in the direction of the force, the work is positive and the force is said to do work. If the projection p is opposite to the direction of the force, the work is negative and work is done against the force.

Now $F \cos \theta$ is the projection of the force along the line of the displacement. We can therefore define work generally as *the product of a uniform force by the component displacement along the line of the force* ($\pm F . d \cos \theta$), *or the product of the displacement by the component force along the line of the displacement* ($\pm d . F \cos \theta$).

Work of Resultant.—Let F_1, F_2, F_3, etc., Fig 1, be any number of forces acting on a point P which has an indefinitely small displacement ds in any given direction. Then during this displacement the forces remain unchanged in magnitude and direction.

FIG. (1).

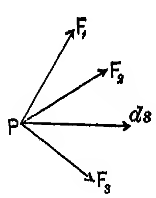
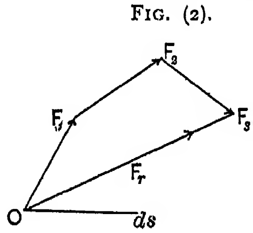


FIG. (2).



Let F_1, F_2, F_3, etc., Fig. 2, be the line representatives of the forces. The resultant is then F_r given in magnitude and direction by the closing line of the force polygon.

The projection of F_r along the line of the displacement ds is then evidently equal to the algebraic sum of the projections of all the components.

Hence *the work of the resultant for any indefinitely small displacement, is equal to the algebraic sum of the works of the components.*

If the forces do not change in magnitude or direction with the displacement, then the same principle will hold for any displacement large or small.

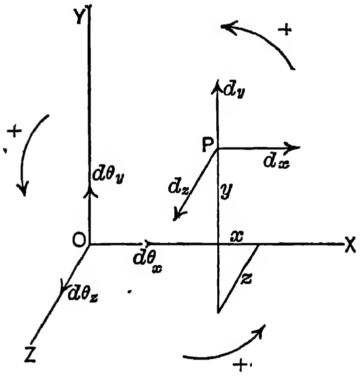
Work—Non-Concurring Forces.—Let a rigid body acted upon by any system of non-concurring forces F_1, F_2, F_3, etc. have an indefinitely small displacement. Let p_1, p_2, p_3, etc., be the displacement of each point of application along the line of the force at that point.

Then the work, no matter whether the displacement is one of translation or of rotation or of both combined, is given by

$$work = F_1p_1 + F_2p_2 + F_3p_3 + \ldots = \Sigma Fp. \quad . \quad . \quad . \quad (1)$$

In equation (1) each term is to be taken positive or negative according as the projections p_1 p_2, p_3, etc., are in the direction of the corresponding forces or in the opposite direction.

Let us suppose, first, that the displacement is one of translation and given by ds.



Take any origin O and co-ordinate axes X, Y, Z. Let F_1, F_2, F_3, etc., act at points P_1, P_2, P_3, etc., given by the co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2), etc. Let the direction cosines of F_1, F_2, F_3, etc., be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc.

Then the components of F_1 are $F_1 \cos \alpha_1$, $F_1 \cos \beta_1$, $F_1 \cos \gamma_1$; of F_2, $F_2 \cos \alpha_2$, $F_2 \cos \beta_2$, $F_2 \cos \gamma_2$, and so on. Also the components in the direction of the axes are given (page 194) by

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \Sigma F \cos \alpha, \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \Sigma F \cos \beta, \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \ldots = \Sigma F \cos \gamma, \end{aligned}\right\} \quad . \quad . \quad (2)$$

and the moments about the axes are given (page 194) by

$$\left.\begin{aligned} M_x &= \Sigma F \cos \gamma \,.\, y - \Sigma F \cos \beta \,.\, z, \\ M_y &= \Sigma F \cos \alpha \,.\, z - \Sigma F \cos \gamma \,.\, x, \\ M_z &= \Sigma F \cos \beta \,.\, x - \Sigma F \cos \alpha \,.\, y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the components of the displacement ds be dx, dy, dz.

Since the work of any force is equal to the algebraic sum of the works of its components, we have for translation

$$\left.\begin{aligned} F_1p_1 &= F_1 \cos \alpha_1 \,.\, dx + F_1 \cos \beta_1 \,.\, dy + F_1 \cos \gamma_1 \,.\, dz, \\ F_2p_2 &= F_2 \cos \alpha_2 \,.\, dx + F_2 \cos \beta_2 \,.\, dy + F_2 \cos \gamma_2 \,.\, dz, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

and so on. By addition, we have, from (1) and (2), for the *work of translation*

$$work = F_x \,.\, dx + F_y \,.\, dy + F_z \,.\, dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now let us suppose the displacement is one of rotation due to an indefinitely small angular displacement $d\theta$ about an axis through O. Let the components of this displacement along the axes be $d\theta_x$, $d\theta_y$, $d\theta_z$.

Then we have

$$\begin{aligned} dx &= z \,.\, d\theta_y - y \,.\, d\theta_z, \\ dy &= x \,.\, d\theta_z - z \,.\, d\theta_x, \\ dz &= y \,.\, d\theta_x - x \,.\, d\theta_y. \end{aligned}$$

Substituting these values in (4) and adding, we have, from (1) and (3), for the *work of rotation*

$$work = M_x \,.\, d\theta_x + M_y \,.\, d\theta_y + M_z \,.\, d\theta_z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For translation and rotation combined we have then, from (5) and (6),

$$work = F_x \,.\, dx + F_y \,.\, dy + F_z \,.\, dz + M_x \,.\, d\theta_x + M_y \,.\, d\theta_y + M_z \,.\, d\theta_z \quad . \quad . \quad . \quad (7)$$

Equations (1) and (7) are then equivalent. Equation (1) is general and includes all cases.

Principle of Virtual Work.—If we put equation (7) equal to zero, since dx, dy, dz, $d\theta_x$, $d\theta_y$, $d\theta_z$ are not zero for any displacement, we must have

$$F_x = 0, \qquad F_y = 0, \qquad F_z = 0,$$
$$M_x = 0, \qquad M_y = 0, \qquad M_z = 0.$$

But we have seen, page 211, *that these are the conditions for equilibrium.*

Hence, *when a rigid body is acted on by any system of non-concurring forces in equilibrium. the algebraic sum of the works of all the forces for any indefinitely small displacement either of translation or rotation or both combined is equal to zero.*

The same holds for a system of rigid bodies connected by inextensible strings.

We have then, from (1), for static equilibrium the condition

$$F_1p_1 + F_2p_2 + F_3p_3 + \ldots = \Sigma Fp = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where p_1, p_2, p_3, etc., are the displacements in the direction of each force, of its point of application, and each term is to be taken positive or negative according as the forces are in the direction of their displacements or in the opposite direction.

But in static equilibrium every point is at rest and there is no real displacement. The principle nevertheless holds good for any supposed indefinitely small displacement. Such a supposed indefinitely small displacement which does not really take place we call a *virtual* displacement. The work of any force for such a displacement is then *virtual* work.

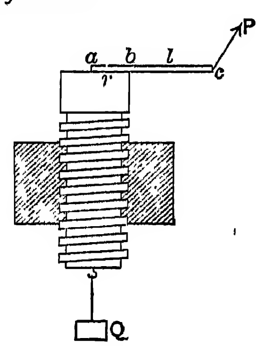
The principle expressed by (8) is called, therefore, the *principle of virtual work*, and may be expressed as follows:

If a rigid body is in static equilibrium, the algebraic sum of the virtual works for any virtual displacement is zero.

If the forces do not change in magnitude and direction with the supposed displacement, the virtual displacement need not be taken indefinitely small.

Examples.—(1) *Find the condition for static equilibrium for a screw, neglecting friction.*

ANS. This is example (10), page 213. We have solved it there by resolution of forces. By virtual work the solution is as follows:



If the arm l turn through a small angle θ, the work of P is $P\,.\,l\theta$. The mass Q is raised a distance $\frac{p\theta}{2\pi}$. Hence by virtual work

$$Pl\theta - Q\frac{p\theta}{2\pi} = 0, \quad \text{or} \quad P = \frac{Qp}{2\pi l}.$$

(2) *Solve the differential screw given in example (11), page 214, by virtual work.*

ANS. We have at once

$$Pl\theta - Q\frac{p_1\theta}{2\pi} + Q\frac{p_2\theta}{2\pi} = 0, \quad \text{or} \quad P = \frac{Q(p_1 - p_2)}{2\pi l}.$$

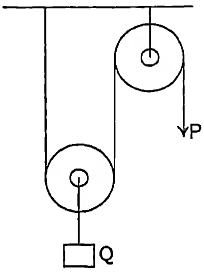
(3) *Solve the wedge given in example (12), page 214, by virtual work.*

ANS. For a small vertical displacement d we have work of P given by Pd. The work of N is $-Nd\sin\frac{\alpha}{2}$. Hence by virtual work

$$Pd - 2Nd\sin\frac{\alpha}{2} = 0, \quad \text{or} \quad P = 2N\sin\frac{\alpha}{2}.$$

(4) *Solve the wheel and axle given in example (13), page 214, by virtual work.*

ANS. For a small angle θ we have



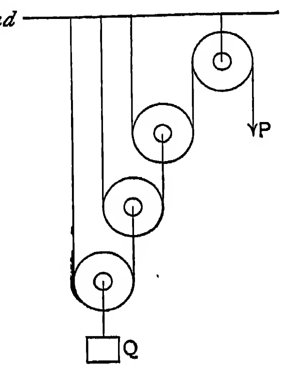
$$Pa\theta - Qb\theta = 0, \quad \text{or} \quad Q = \frac{a}{b}P.$$

(5) *In the single movable pulley shown in the figure find the relation between P and Q for static equilibrium, disregarding friction and rigidity of rope.*

ANS. For a displacement s of P downwards, each rope passing around the movable pulley is diminished $\frac{s}{2}$ and Q is raised $\frac{s}{2}$. We have then

$$Ps - Q\frac{s}{2} = 0, \quad \text{or} \quad P = \frac{Q}{2}.$$

(6) *In the system of pulleys shown in the figure find the relation between P and Q for static equilibrium, disregarding friction and rigidity of ropes.*



ANS. Let m be the mass of each pulley.

For a displacement s of P downwards,

the first movable pulley is raised					$\frac{s}{2}$.
" second "	"	"	"		$\frac{s}{4}$,
" third "	"	"	"		$\frac{s}{8}$,
" nth "	"	"	"		$\frac{s}{2^n}$.

We have then

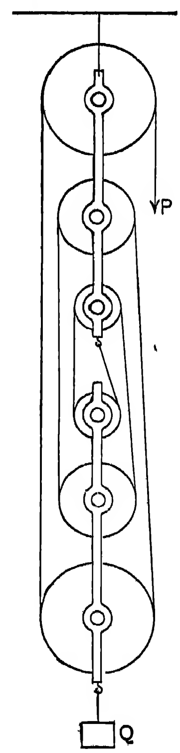
$$Ps - \frac{ms}{2} - \frac{ms}{4} - \ldots \frac{ms}{2n} - \frac{Qs}{2n} = 0,$$

or

$$P - \frac{Q}{2^n} + m\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^n}\right) = \frac{Q + (2^n - 1)m}{2^n},$$

where n is the number of movable pulleys.

(7) *In the system of pulleys shown in the figure find the relation between P and Q for static equilibrium, disregarding friction and rigidity of ropes.*

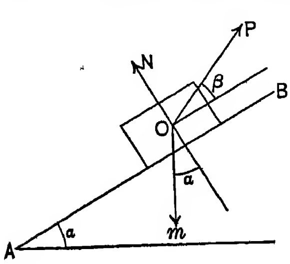


ANS. For a vertical displacement s of P each rope from the lower block will be shortened by s, and the displacement of Q upwards will be $\frac{s}{n}$, where n is the number of ropes from the lower block.

If m is the mass of the lower block, we have

$$Ps - (Q + m)\frac{s}{n} = 0, \quad \text{or} \quad P = \frac{Q + m}{n}.$$

(8) *A body of mass m rests upon a smooth inclined plane AB which makes an angle α with the horizontal and is acted upon by a force P which makes the angle β with the plane. Find the conditions of static equilibrium.*



ANS. Consider the body as a particle at any point O on the plane. We have acting upon the particle the weight of m, the force P and the nominal reaction N of the plane, and these three forces must be in equilibrium. Let the angle $POB = \beta$ be positive when above the plane, and negative when below.

1ST SOLUTION. BY RESOLUTION OF FORCES.—We have, putting the algebraic sum of components at right angles to P equal to zero, for gravitation measures

$$N\cos\beta - m\cos(\alpha+\beta) = 0, \quad \text{or} \quad N = \frac{m\cos(\alpha+\beta)}{\cos\beta}. \quad \ldots\ldots\ldots (1)$$

Putting the algebraic sum of components parallel to the plane equal to zero, we have for gravitation measures

$$P \cos \beta - m \sin \alpha = 0, \quad \text{or} \quad P = \frac{m \sin \alpha}{\cos \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When $\beta = 90^\circ - \alpha$, P is vertical and, from (1), $N = 0$. For any greater value of positive β we have, from (1), N negative and no equilibrium is possible. For negative β we must have β less than 90°. Equations (1) and (2) hold good, then, for all values of β between $+ (90^\circ - \alpha)$ and $- 90^\circ$. Outside of these limits there is no equilibrium.

The minimum value of P is for $\beta = 0$ and equal to $P = m \sin \alpha$.

Again, we can put the algebraic sum of components along the plane and perpendicular to the plane equal to zero. We have then

$$P \cos \beta - m \sin \alpha = 0,$$

$$N + P \sin \beta - m \cos \alpha = 0.$$

From these two equations we can obtain (1) and (2).

Again, we can put the algebraic sum of horizontal and vertical components equal to zero. We have then

$$P \sin (\alpha + \beta) + N \cos \alpha - m = 0,$$

$$P \cos (\alpha + \beta) - N \sin \alpha = 0,$$

and from these equations we can obtain (1) and (2).

2D SOLUTION. BY VIRTUAL WORK.—In order to find P, suppose a virtual displacement d along the plane from O towards B. This displacement is at right angles to N, and hence the virtual work of N is zero. We have then

$$Pd \cos \beta - md \sin \alpha = 0, \quad \text{or} \quad P = \frac{m \sin \alpha}{\cos \beta}.$$

In order to find N, suppose a horizontal virtual displacement d. Then the virtual work of W is zero, and we have

$$Pd \cos (\alpha + \beta) - Nd \sin \alpha = 0, \quad \text{or} \quad N = \frac{P \cos (\alpha + \beta)}{\sin \alpha} = \frac{m \cos (\alpha + \beta)}{\cos \beta}.$$

CHAPTER III.

STATIC FRICTION.

Friction.—Every natural surface offers a resistance to the motion of a body upon it. Part of this resistance is due to ADHESION between the body and surface and part is due to FRICTION.

Friction, then, is always a retarding force or resistance, and acts always in a direction opposite to that in which the body moves or would move if there were no resistance.

When one surface moves upon another, the surfaces in contact are compressed and projecting points and irregularities are bent over, broken off, rubbed down, etc.

The resistance due to friction, therefore, evidently depends upon the materials of which the surfaces are composed, and also upon the roughness or smoothness of the surfaces in contact.

It may also evidently vary for the same surfaces, according to their condition or state or material constitution.

Thus it may not be the same for surfaces of dry wood or iron as for the same surfaces under the same conditions when wet. It may not be the same for two surfaces of wood with their fibres parallel as for the same surfaces under the same conditions when their fibres are not parallel.

Unguents also have a great influence. Such fluid or semi-fluid unguents as oil, tallow, etc., fill up interstices and diminish the effect of irregularities of surfaces, or a film of unguent may be interposed between the surfaces and thus the resistance of friction greatly diminished.

Adhesion.—We must not confound the resistance due to friction with that due to adhesion. Adhesion is that resistance to motion which takes place when two different surfaces come in contact at many points without pressure. Adhesion increases with the area of surface of contact and is independent of the pressure, while, as we shall see (page 221), friction increases with the pressure and is in general independent of the area of surface of contact. When the pressure, then, is very small, adhesion may be great compared with friction.

If, however, the pressure is great, adhesion may be neglected compared to the friction, and the resistance to motion is practically that due to the friction only.

Kinds of Friction.—Surfaces may slide or roll on one another. We distinguish accordingly SLIDING FRICTION and ROLLING FRICTION.

It is also found by experiment that the friction which just prevents motion is greater than that which exists after actual motion takes place. The friction which just prevents motion is called friction of repose or quiescence, or STATIC FRICTION. The friction which exists after actual motion takes place is called friction of motion, or KINETIC FRICTION.

We have, then, two kinds of static friction, viz., STATIC SLIDING FRICTION and STATIC ROLLING FRICTION.

We have also two kinds of kinetic friction, viz., KINETIC SLIDING FRICTION and KINETIC ROLLING FRICTION.

In any case, whether of sliding or rolling, the kinetic friction is always less than the static friction.

We have to do in this portion of our work with static friction only.

Coefficient of Friction.—When two surfaces are in contact and there is friction and normal pressure at every point of contact, the sum of the frictions at every point of contact is the total friction, and the sum of the normal pressures at every point of contact is the total normal pressure.

The ratio of the total friction to the total normal pressure when motion, either sliding or rolling, is *just about to begin*, is called the COEFFICIENT OF STATIC FRICTION, either of sliding or rolling.

The same ratio *after motion has taken place* is called the COEFFICIENT OF KINETIC FRICTION, either of sliding or rolling.

We denote the coefficient of friction in general by μ. We have then, in general, for all cases

$$\mu = \frac{F}{N}, \quad \text{or} \quad F = \mu N,$$

where F is the total friction and N the total normal pressure when motion either sliding or rolling is *just about to begin*, or else when motion either sliding or rolling *has taken place*. In the first case μ is the coefficient of static friction of sliding or rolling. In the second case μ is the coefficient of kinetic friction of sliding or rolling. We have to do in this portion of the work with static friction only.

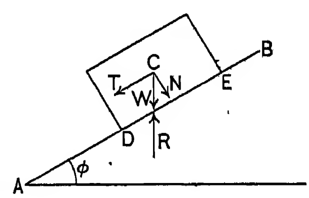
Limiting Equilibrium.—The student should carefully note that

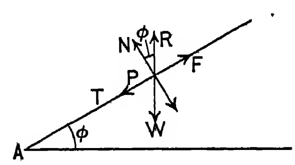
$$F = \mu N$$

does not give the actual resistance of friction in all cases of equilibrium, but only the resistance which exists when the surfaces *are on the point of motion.*

Friction acts always in a direction opposite to the force which tends to cause motion, and so long as there is equilibrium it is always equal in magnitude to this force. But when this force has the magnitude μN motion is *just about to begin*, and the body is said to be in LIMITING EQUILIBRIUM. If this force is less than μN, there will still be equilibrium, whatever its magnitude, and the body is in NON-LIMITING EQUILIBRIUM.

Coefficient of Static Sliding Friction—Experimental Determination.—Let a body of weight W, acting at the centre of mass C, rest in equilibrium upon a rough plane AB, the surfaces of contact being plane.





Then for equilibrium the reaction R of the plane is equal and opposite to W and in the same vertical line, and the sum N of all the normal pressures acting at every point of contact must be equal and opposite to the normal component of W, and the sum F of all the frictions at every point of contact must be equal and opposite to the component T of W parallel to the plane.

We have, then, when sliding is about to begin, for the coefficient of sliding friction

$$\mu = \frac{F}{N},$$

and we see from the figure that $\frac{F}{N}$ is the tangent of the angle which the reaction R makes with the normal when sliding is about to begin. Now the reaction at every point of contact is paral-

lel to R or W, and sliding begins at all points of contact simultaneously. Hence the angle which R makes with the normal when sliding is about to begin is evidently the same as the angle which the plane makes with the horizontal. Therefore

$$\mu = \frac{F}{N} = \tan \phi.$$

We call the angle ϕ which the plane makes with the horizontal when motion is about to begin the ANGLE OF REPOSE.

That is, *the coefficient of static sliding friction is equal to the tangent of the angle of repose.*

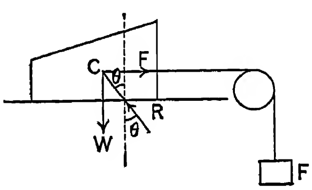
If, then, we place a body upon a rough plane and then gradually incline the plane until sliding just begins, the inclination of the plane at this instant gives the angle of repose ϕ. The tangent of this angle gives the coefficient μ of static sliding friction for plane surfaces.

We obtain the same result by resolution of forces. Thus let ϕ be the inclination of the plane when sliding begins.

Then for equilibrium $W \cos \phi = N$, and $W \sin \phi = F$. Hence

$$\mu = \frac{F}{N} = \tan \phi.$$

We can thus make use of the inclined plane as an apparatus for determining μ by experiment.



Again, if we place a body of weight W on a horizontal plane and measure the horizontal force F just necessary to cause it to begin to slide, we have

$$\mu = \frac{F}{W} = \tan \phi,$$

where ϕ is the angle of the reaction R with the normal *when sliding begins*, or the angle of repose.

Such an apparatus should be so constructed that the friction of the pulley and other resistances due to the string, etc., can be disregarded or else allowed for.

Cone of Friction.—If we revolve the line representative of R in the figure page 221 about the vertical, it will describe a cone. This cone is called the CONE OF FRICTION. The reaction of R is equal and opposite to the resultant of F and W acting at C.

We see, then, that *no force acting at C, however great in magnitude, can cause sliding to begin if its line representative lies within the cone of friction.*

Laws of Static Sliding Friction.—The following laws of static sliding friction have been established by experiment as holding true within the limits indicated:

1. *Other things being the same, within certain limits of the normal pressure, static sliding friction is proportional to the total normal pressure and independent of the area of the surfaces in contact.*

In other words, within the limits of normal pressure referred to, the coefficient of static sliding friction μ *is constant for the same two surfaces in the same condition*, whatever the area of the surfaces of contact and whatever the total normal pressure.

Thus, if the normal pressure N over a given area is increased or decreased, the friction F increases or decreases in the same proportion and $\mu = \frac{F}{N}$ is unchanged.

It follows directly that if the area increases or decreases, N remaining the same, the number of points of contact is correspondingly increased or decreased, but the normal pressure at each point, and therefore the friction at each point, is correspondingly decreased or increased. The sum of all the frictions F remains, then, the same and $\mu = \frac{F}{N}$ is unchanged.

Limitations of the Law.—The limitations of normal pressure referred to are as follows:

If the normal pressure per unit of area approaches the crushing strength or becomes so great as to break up the film of interposing unguent, the friction F increases more rapidly than the normal pressure, and the law fails.

In properly designed structures the normal pressure per unit of area is much less than this limit, and the law applies.

Again, if the normal pressure per unit of area is very small, adhesion may constitute the larger portion of the resistance. This adhesion increases with the area of contact (page 220).

In all practical cases, however, the influence of adhesion may be neglected.

Hence in practical appplications the friction is the only resistance which is considered, and it is assumed that

$$F = \mu N$$

gives the resistance, where μ is in practice a constant for the same two surfaces in the same condition, whatever the area of the surfaces in contact and whatever the total normal pressure N.

2. *Other things being the same, within certain limits of the normal pressure, the static sliding friction of greased surfaces is less than that of ungreased and depends less upon the surfaces than upon the unguent.*

Here, again, if the normal pressure per unit of area becomes so great as to break up the film of interposing unguent, surface comes in contact with surface and the friction may depend more on the surfaces than upon the unguent.

In properly designed structures the normal pressure per unit of area is much less than this, and the law applies.

Again, if the normal pressure per unit of area is very small, adhesion may constitute the larger portion of the resistance, and this adhesion is increased by the unguent.

In all practical cases, however, the influence of adhesion may be neglected.

Hence in practical applications the friction is the only resistance which is considered, and it is assumed that

$$F = \mu N$$

gives the resistance, where μ is in practice a constant for the same two surfaces in the same condition, whatever the area of the surfaces in contact and whatever the total normal pressure N.

Upon these two laws depend the value and use of the values for the coefficient of static sliding friction given in the next article.

Values of Coefficient of Static Sliding Friction.—The following table gives a few values of μ as determined by experiment for static sliding friction.

COEFFICIENTS OF STATIC SLIDING FRICTION $\mu = \tan \phi$.

Substances in Contact.		Condition of Surfaces and Kind of Unguent. Dry.	Wet.	Olive Oil.	Lard.	Tallow.	Dry Soap.	Polished and Greasy.
Wood on wood	minimum	0.30	0 65	....	....	0.14	0.22	0.30
	mean	0.50	0.68	....	0.21	0.19	0.36	0.35
	maximum	0.70	0.71	....	....	0.25	0.44	0.40
Metal on metal	minimum	0.15	....	0.11				
	mean	0.18	....	0.12	0.10	0.11	....	0.15
	maximum	0.24	....	0.16				
Wood on metal		0.60	0.65	0.10	0.12	0.12	....	0.10
Hemp ropes or plaits on wood	minimum	0.50						
	mean	0.63	0.87					
	maximum	0.80						
Leather belts over drums made of	wood	0.47						
	metal	0.54	....	....	....	....	....	0.28
Stone or brick on stone or brick, polished	minimum	0.67						
	maximum	0.75						
Dry masonry and brickwork		0.65						
Masonry and brickwork, damp mortar		0 74						
Timber on stone		0.40						
Iron on stone		0.7 to 0.3						
Masonry on dry clay		0 51						
" " moist clay		0 33						
Earth on earth		0.25 to 1						
Damp clay on damp clay		1 0						

More extensive tables will be found in treatises on Friction. It will be noted that the coefficient of static sliding friction is practically always less than unity. In only one case given in the table, viz., for damp clay on damp clay, is $\mu = 1$, corresponding to an angle of repose of $\phi = 45°$. Rankine gives for "shingle on gravel" a maximum $\mu = 1.11$, corresponding to an angle of repose $\phi = 48°$.

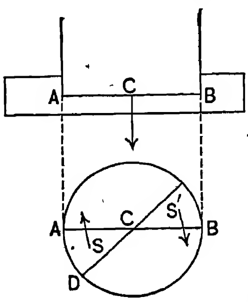
Static Friction for Pivots.—In all cases of the sliding of two surfaces we denote the coefficient of static sliding friction by μ and take the value of μ as given by the table. We have then, *in all cases* of sliding friction, for the friction *when sliding is about to begin*

$$F = \mu N = N \tan \phi,$$

where N is the *total normal pressure* and ϕ is the angle of repose, and μ is given by the table. The direction of the friction is always opposite to the direction of motion if motion were to take place.

The application to pivots is then simple.

1. SOLID FLAT PIVOT.—Let ACB be the base of a solid flat pivot, and N the total normal pressure upon the base.



We have then for the static friction

$$F = \mu N, \quad \ldots \ldots \quad (1)$$

where μ is given by the table.

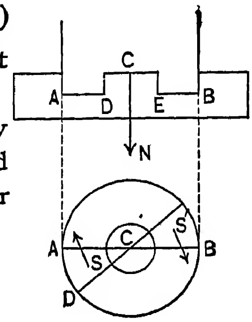
If we divide the base into a very large number of very small equal triangles, such as ACD, the friction on each can be considered as the resultant of equal parallel forces distributed over the surface. The point of application for each triangle is then at the centre of mass for that triangle. The point of application of the entire friction is then at a distance $Cs = \frac{2}{3} r$ from the centre. The *moment* of the entire friction with reference to the axis is then

$$M = \frac{2}{3} \mu N r. \quad \ldots \ldots \quad (2)$$

Since for any point s of the base there is a corresponding point s' for which the friction is equal and opposite, the moment of the friction is the moment of a *couple* and is therefore the same for every point in the plane of the base (page 185).

2. HOLLOW FLAT PIVOT.—If the rubbing surface is a flat ring $ADEB$, we have as before

$$F = \mu N, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$



where N is the total normal pressure on the base, and μ is the coefficient of static sliding friction as given by the table page 224.

Let the outer radius be r_1 and the inner radius r_2. Then any small portion of the base is a circular ring for which the length of chord and arc AD may be taken equal. The centre of mass (page 29) for each small portion is then at a distance Cs from the axis given by

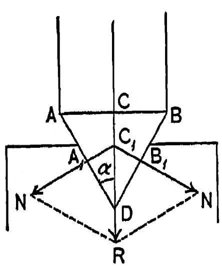
$$Cs = \frac{2}{3}\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}.$$

Hence the moment of the friction with reference to the axis is

$$M = \frac{2}{3}\mu N\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since for any point s there is a corresponding point s' for which the friction is equal and opposite, the moment of the friction is the moment of a couple and is therefore the same for any point in the plane of the base (page 185).

3. CONICAL PIVOT.—In the case of a conical pivot let R be the pressure along the axis, and let the half angle of convergence ADC be α.



If we divide the conical surface into a large number n of very small triangles with their vertices at the point D, each will sustain the vertical load $\frac{R}{n}$, and the normal pressure on each will be $\frac{R}{n \sin \alpha}$. If we denote the radius $C_1A_1 = C_1B_1$ of the pivot at the point of entrance by r_1, the resultant normal pressure upon each small elementary triangle acts at a distance $\frac{2}{3}r_1$ from the axis.

We have then for the total friction

$$F = \mu\frac{R}{\sin \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ is the coefficient of static sliding friction as given by the table page 224, and the moment of the friction with reference to the axis is

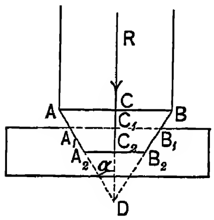
$$M = \frac{2}{3}\mu\frac{Rr_1}{\sin \alpha},$$

or, since $\frac{r_1}{\sin \alpha}$ = the side DA_1 of the cone of contact $= a$, we have

$$M = \frac{2}{3}\mu Ra. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is also the moment of a couple and hence the same for any point in the plane perpendicular to the axis at a distance above the point D equal to two thirds the height of the cone of contact.

4. PIVOT A TRUNCATED CONE.—Let R be the pressure along the axis, and let the half angle of convergence ADC be α.



Let R_2 be the pressure sustained by the flat base, and R_1 the pressure sustained by the conical surface.

Then

$$R_1 + R_2 = R.$$

Also, if r_1 is the radius C_1A_1 at the point of entrance, and r_2 the radius of the base,

$$R_2 : R :: \pi r_2^2 : \pi r_1^2, \quad \text{or} \quad R_2 = \frac{r_2^2}{r_1^2}R,$$

and hence

$$R_1 = R - R_2 = \frac{r_1^2 - r_2^2}{r_1^2}R.$$

We have then as in Case 1, page 224, for the flat pivot, the friction F_2 on the base

$$F_2 = \mu R_2 = \mu\frac{r_2^2}{r_1^2}R,$$

and its moment about the axis

$$M_2 = \frac{2}{3}\mu\frac{r_2^3}{r_1^2}R.$$

For the friction on the conical surface we have, as in Case 3, page 225, for the conical pivot

$$F_1 = \mu\frac{R_1}{\sin\alpha} = \mu \cdot \frac{r_1^2 - r_2^2}{r_1^2} \cdot \frac{R}{\sin\alpha},$$

and for its lever-arm, as in Case 2, page 225, for hollow pivot,

$$\frac{2}{3} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}.$$

Its moment, then, about the axis is

$$M_1 = \frac{2}{3}\mu \cdot \frac{r_1^3 - r_2^3}{r_1^2} \cdot \frac{R}{\sin\alpha}.$$

The total friction for the truncated pivot is then

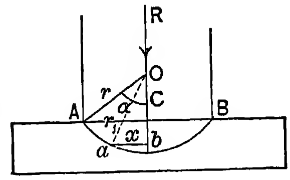
$$F = F_1 + F_2 = \frac{\mu R}{r_1^2}\left(r_2^2 + \frac{r_1^2 - r_2^2}{\sin\alpha}\right), \quad \ldots \ldots \quad (1)$$

and its total moment about the axis is

$$M = M_1 + M_2 = \frac{2}{3}\mu\frac{R}{r_1^2}\left(r_2^3 + \frac{r_1^3 - r_2^3}{\sin\alpha}\right), \quad \ldots \ldots \quad (2)$$

where μ is the coefficient of static sliding friction as given by the table page 224.

[**Pivot with Spherical End.**]—Let R be the pressure along the axis, denote the radius AO of the spherical surface by r, and the radius AC by r_1, and let the angle AOC be α,



Then the load per unit of area of horizontal projection is $\frac{R}{\pi r_1^2}$. Take any element of the surface at a, distant $ab = x$ from the axis, and let $Ob = y$. The horizontal projection of this element is $2\pi x dx$, and the load sustained by it is then $2\pi x dx \times \frac{R}{\pi r_1^2} = \frac{2Rxdx}{r_1^2}$.

The cosine of the angle aOb is $\cos aOb = \frac{y}{r} = \frac{\sqrt{r^2 - x^2}}{r}$. The normal pressure on the element at a is then

$$\frac{2Rxdx}{r_1^2} \cdot \frac{r}{\sqrt{r^2 - x^2}},$$

and the static friction is

$$\frac{2\mu Rr}{r_1^2} \cdot \frac{xdx}{\sqrt{r^2 - x^2}}.$$

Integrating between the limits of $x = 0$ and $x = r_1$, we have for the total friction

$$F = \frac{2\mu Rr}{r_1^2}\left(r - \sqrt{r^2 - r_1^2}\right),$$

or, since $\sqrt{r^2 - r_1^2} = r \cos \alpha$ and $r_1 = \sin \alpha$,

$$F = \frac{2\mu R}{\sin^2 \alpha}(1 - \cos \alpha) = \frac{2\mu R}{1 + \cos \alpha},$$

where μ is the coefficient of static sliding friction as given by the table, page 224.

For hemispherical end $\alpha = 90°$ and $F = 2\mu R$. For flat end $\alpha = 0$ and $F = \mu R$.

The moment about the axis of the friction on an element is

$$\frac{2\mu Rr}{r_1^2} \cdot \frac{x^2 dx}{\sqrt{r^2 - x^2}}.$$

Integrating between the limits $x = 0$ and $x = r_1$, we have for the total moment of the friction about the axis

$$M = \frac{2\mu Rr}{r_1^2}\left[\frac{r^2}{2} \sin^{-1}\frac{r_1}{r} - \frac{r_1}{2}\sqrt{r^2 - r_1^2}\right],$$

or, inserting the values of $\sqrt{r^2 - r_1^2} = r \cos \alpha$ and $r_1 = r \sin \alpha$ and reducing,

$$M = \mu Rr\left(\frac{\alpha}{\sin^2 \alpha} - \cot \alpha\right). \quad \text{. . . . . (2)}$$

For hemispherical end $\alpha = \frac{\pi}{2}$, $\sin \alpha = 1$, $\cot \alpha = 0$, and this becomes $M = \frac{\mu\pi Rr}{2}$.

Static Friction of Axles.—In all cases of the sliding of two surfaces, we denote the coefficient of static sliding friction by μ and take the value of μ as given by the table, page 224. We have then, *in all cases* of sliding friction, for the friction when *sliding is about to begin*

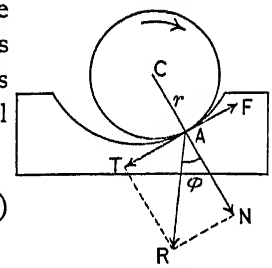
$$F = \mu N = N \tan \phi,$$

where N is the total normal pressure and ϕ is the angle of repose, and μ is given by the table, page 224.

The direction of the friction is always opposite to the direction of motion if motion were about to take place.

The application to axles is then simple.

1. Axle in Partially Worn Bearing.—Let the bearing be partially worn, then the axle at the moment when sliding begins touches the bearing at a point A, and the resultant pressure R at this point makes the angle of repose ϕ with the normal. We have then for the normal pressure $N = R \cos \phi$, and for the friction



$$F = N \tan \phi = R \sin \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is the angle of repose as given by the table, page 224.

Let r be the radius AC of the axle. Then the moment of the friction with reference to the axis is

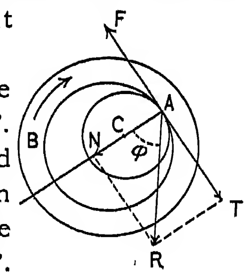
$$M = Rr \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the axle is well greased, the angle of repose ϕ is very small, and we may take $\mu = \tan \phi = \sin \phi$. In the practical case of a well-greased axle, then, we have

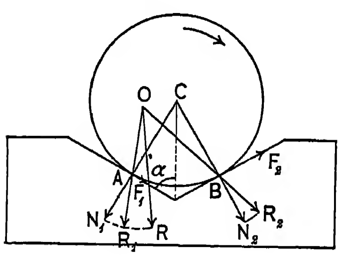
$$F = \mu R, \qquad M = \mu Rr,$$

where μ is given by the table, page 224.

If the wheel AB revolves, as shown, about a fixed axle AC, the friction is the same as before, but the lever-arm of the friction is not the radius of the axle, but the inner radius of the wheel.



2. Axle—Triangular Bearing.—If the bearing is triangular, the axle is supported at two points A and B. The resultant pressure R can be resolved into two components R_1 and R_2, and when sliding begins each of these makes the angle of repose ϕ with the normals at A and B. The normal pressure at A is then $N_1 = R_1 \cos \phi$, and the friction at A is



$$F_1 = N_1 \tan \phi = R_1 \sin \phi.$$

The friction at B is in like manner $F_2 = R_2 \sin \phi$. The total friction is then

$$F = (R_1 + R_2) \sin \phi.$$

Let the angle $ACB = 2\beta$. Then the angle $AOR = \beta - \phi$, and the angle $BOR = \beta + \phi$. We have then

$$R_1 : R :: \sin (\beta + \phi) : \sin 2\beta, \quad \text{or} \quad R_1 = \frac{\sin (\beta + \phi)}{\sin 2\beta} R,$$

and

$$R_2 : R :: \sin (\beta - \phi) : \sin 2\beta, \quad \text{or} \quad R_2 = \frac{\sin (\beta - \phi)}{\sin 2\beta} R.$$

Hence the total friction is

$$F = [\sin (\beta + \phi) + \sin (\beta - \phi)] \frac{R \sin \phi}{\sin 2\beta}.$$

But $\sin(\beta+\phi)+\sin(\beta-\phi) = 2\sin\beta\cos\phi$, and $\sin 2\theta = 2\sin\phi\cos\beta$. Hence we have

$$F = \frac{R\sin\phi\cos\phi}{\cos\beta} = \frac{R\sin 2\phi}{2\cos\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where ϕ is the angle of repose as given by the table, page 224.

The moment of friction with reference to the axis, if r is the radius of the axle, is

$$M = Fr = \frac{Rr\sin 2\phi}{2\cos\beta}.$$

If the axle is well greased, the angle of repose ϕ is very small, and we may take $\sin 2\phi = 2\sin\phi$, also $\mu = \tan\phi = \sin\phi$. In the practical case of a well-greased axle, then, we have

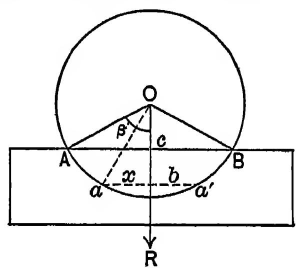
$$F = \mu\frac{R}{\cos\beta}, \qquad M = \mu\frac{Rr}{\cos\beta},$$

where μ is given by the table, page 224. If the angle β is small, $\cos\beta$ may be taken as unity, and F and M are then the same as in the preceding case,

$$F = \mu R, \qquad M = \mu Rr.$$

[3. AXLE—NEW BEARING.]—When the bearing is new and unworn, the axle touches it at all points.

Let R be the resultant vertical pressure acting at the centre O of the axle. Denote the radius AO of the axle by r, the distance AC by r_1, and let the angle AOC be β.



Then the load per unit of horizontal projection is $\frac{R}{2r_1}$. Take any element of the surface of the axle at a, distant $ab = x$ from R, and let $Ob = y$. The horizontal projection of this element is dx, and the load sustained by it is $\frac{Rdx}{2r_1}$. At a' we have a similar element.

The friction on these two elements is, from the preceding article,

$$\frac{\sin 2\phi \,.\, Rdx}{2r_1\cos aOb}.$$

But $\cos aOb = \frac{y}{r} = \frac{\sqrt{r^2-x^2}}{r}$, hence the friction for the two elements is

$$\frac{Rr\sin 2\phi}{2r_1}\;\frac{dx}{\sqrt{r^2-x^2}}.$$

Integrating between the limits $x = r_1$ and $x = 0$, we have for the entire friction

$$F = \frac{Rr\sin 2\phi}{2r_1}\sin^{-1}\frac{r_1}{r}.$$

Inserting the value of $r_1 = r\sin\beta$,

$$F = \frac{R\sin 2\phi}{2}\cdot\frac{\beta}{\sin\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where ϕ is the angle of repose as given by the table page 224.

The moment of the friction with reference to the axis is then

$$M = \frac{Rr\sin 2\phi}{2}\cdot\frac{\beta}{\sin\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If the axle is well greased, the angle of repose ϕ is very small, and we may take

$$\sin 2\phi = 2 \sin \phi, \quad \text{also} \quad \mu = \tan \phi = \sin \phi.$$

In the practical case of a well-greased axle, then, we have

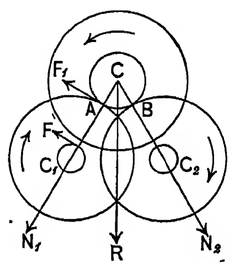
$$F = \mu R \cdot \frac{\beta}{\sin \beta}, \quad M = \mu Rr \frac{\beta}{\sin \beta},$$

where μ is given by the table page 224.

If the angle β is small, we may take $\beta = \sin \beta$, and then F and M are the same as in the two preceding cases,

$$F = \mu R, \quad M = \mu Rr.$$

4. FRICTION WHEELS.—By the use of friction wheels instead of bearing blocks, the friction of an axle can be greatly diminished.



Thus let the axle AC rest upon the circumferences of the friction wheels AC_1 and BC_2, touching them at the points A and B. The vertical pressure R on the axle C causes the pressures N_1, N_2 at A and B.

Let the angle $ACB = \beta$. Then

$$N_1 = N_2 = \frac{R}{2 \cos \beta}.$$

If the axles of the friction wheels are well greased, then, as we have seen, the least friction may be written

$$F = \mu(N_1 + N_2) = \frac{\mu R}{\cos \beta},$$

where μ is given by the table page 224.

If the radius of the axles of the friction wheels is r, the moment of the friction is

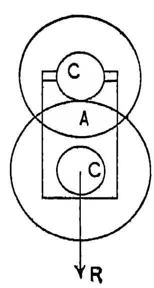
$$Fr = \frac{\mu Rr}{\cos \beta}.$$

The moment of the friction at the points A and B must be the same. If we call this F_1, we have, if the radius of the friction wheels is a,

$$F_1 a = Fr, \quad \text{or} \quad F_1 = \frac{r}{a} F = \frac{r}{a} \cdot \frac{\mu R}{\cos \beta}.$$

By making β small, we can take $\cos \beta = 1$, and have

$$F_1 = \frac{r}{a} \cdot \mu R.$$



By taking a large with respect to r, we may thus make the friction F_1 very small. If the axle C rests on bearings, its least friction is μR, as we have seen.

If we have a single friction wheel C_1A, then $\beta = 0$, and we have accurately

$$F_1 = \frac{r}{a} \mu R.$$

Static Friction of Cords and Chains.—Let a perfectly flexible cord stretched by a weight Q be laid over the edge C of a rigid body ABO, Fig. 1.

Let the force at the other end of the cord be P, and the angle of deviation $DCP = AOB = \alpha$.

FIG. 1.

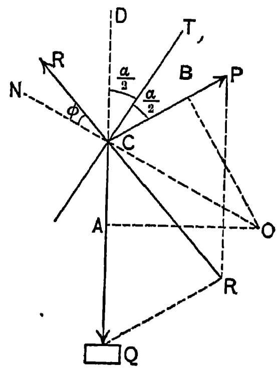
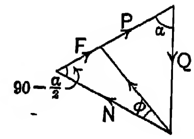


FIG. 2



Draw CT making the angle $TCP = \frac{\alpha}{2}$, and CN perpendicular to CT. Then, when motion is about to begin, the resultant R of P and Q makes the angle of repose ϕ with CN.

If the weight Q is about to sink, the friction F acts opposed to the motion, and we have

$$P + F = Q.$$

We have then, from Fig. 2,

$$F : 2Q \sin\frac{\alpha}{2} :: \sin\phi : \sin\left[90 - \left(\phi - \frac{\alpha}{2}\right)\right],$$

or

$$F = \frac{2Q\sin\frac{\alpha}{2}\sin\phi}{\cos\left(\phi - \frac{\alpha}{2}\right)} = \frac{2Q\sin\frac{\alpha}{2}\sin\phi}{\cos\phi\cos\frac{\alpha}{2} + \sin\phi\sin\frac{\alpha}{2}}.$$

Dividing numerator and denominator by $\cos\phi$, we have, since $\tan\phi = \mu =$ coefficient of static sliding friction, for the friction F_1 when the weight Q is *about to sink*

$$F_1 = \frac{2\mu Q\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} + \mu\sin\frac{\alpha}{2}} = \frac{2\mu Q\tan\frac{\alpha}{2}}{1 + \mu\tan\frac{\alpha}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the weight Q is *just about to rise* we have

$$P = Q + F, \quad \text{or} \quad Q = P - F,$$

and hence

$$F = \frac{2\mu Q\tan\frac{\alpha}{2}}{1 - \mu\tan\frac{\alpha}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

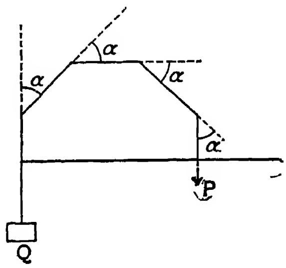
In the first case, then, when the weight Q is *about to sink*,

$$P_1 = Q - F_1 = \frac{Q\left(1 - \mu\tan\frac{\alpha}{2}\right)}{1 + \mu\tan\frac{\alpha}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and in the second case, when the weight Q is *about to rise*,

$$P = Q + F = \frac{Q\left(1 + \mu \tan \frac{\alpha}{2}\right)}{1 - \mu \tan \frac{\alpha}{2}}. \qquad (4)$$

If the cord passes over several edges, the force P_1 can be calculated by repeated application of these formulas.



Thus let the number of edges be n and the deviation at each edge be the same and equal to α. When the weight Q is *just about to sink*, the tension of the first portion of the cord is, from (3),

$$P_1 = \frac{Q\left(1 - \mu \tan \frac{\alpha}{2}\right)}{1 + \mu \tan \frac{\alpha}{2}}.$$

That of the second is

$$P_2 = \frac{P_1\left(1 - \mu \tan \frac{\alpha}{2}\right)}{1 + \mu \tan \frac{\alpha}{2}} = \frac{Q\left(1 - \mu \tan \frac{\alpha}{2}\right)^2}{\left(1 + \mu \tan \frac{\alpha}{2}\right)^2}.$$

That of the last is

$$P_n = \frac{Q\left(1 - \mu \tan \frac{\alpha}{2}\right)^n}{\left(1 + \mu \tan \frac{\alpha}{2}\right)^n}. \qquad (5)$$

If the weight Q is *just about to rise*, we have simply to interchange P and Q and we have

$$P_n = \frac{Q\left(1 + \mu \tan \frac{\alpha}{2}\right)^n}{\left(1 - \mu \tan \frac{\alpha}{2}\right)^n}. \qquad (6)$$

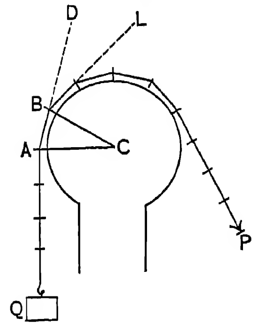
In the first case, when the weight is *about to sink*, we have for the friction

$$F_1 = Q - P_n = Q\left(1 - \frac{\left(1 - \mu \tan \frac{\alpha}{2}\right)^n}{\left(1 + \mu \tan \frac{\alpha}{2}\right)^n}\right). \qquad (7)$$

If the weight is *about to rise*,

$$F = P_n - Q = Q\left(\frac{\left(1 + \mu \tan \frac{\alpha}{2}\right)^n}{\left(1 - \mu \tan \frac{\alpha}{2}\right)^n} - 1\right). \qquad (8)$$

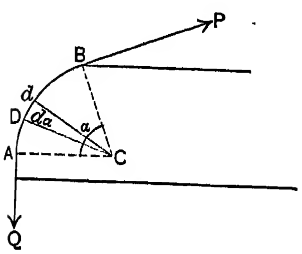
Formulas (5), (6), (7) and (8) are also applicable to the case of a chain composed of links which is passed round a cylindrical surface, where n is the number of links in contact. If the length of each link is $AB = l$, and the distance CA of the axis A of a link from the centre C is r, we have for the angle of deviation $DBL = ACB = \alpha$



$$\sin\frac{\alpha}{2} = \frac{l}{2r}, \quad \text{or} \quad \tan\frac{\alpha}{2} = \frac{l}{\sqrt{4r^2 - l^2}}.$$

[If a flexible cord lies in contact with a rough surface, let $ACB = \alpha$ be the arc of contact.

If T is the tension at any point of contact D for the indefinitely small portion of the cord Dd, the friction at this point is dT. Let the indefinitely small angle $DC\alpha$ be $d\alpha$. Then, from equation (1), page 231,



$$dT = \frac{2\mu T \tan\frac{d\alpha}{2}}{1 + \mu \tan\frac{\alpha}{2}}.$$

But since $d\alpha$ is indefinitely small, we may take the arc equal to the tangent and disregard $\mu \tan\frac{\alpha}{2}$ with reference to 1. We have then

$$\frac{dT}{T} = \mu d\alpha.$$

Integrating between the limits $\alpha = 0$ and α, we have, since for $\alpha = 0$, $T = Q$, and for $\alpha = \alpha$, $T = P$,

$$\text{logn } P = \mu\alpha + \text{logn } Q, \quad \text{or} \quad \text{logn } \frac{P}{Q} = \mu\alpha.$$

We have then, when motion in the direction of P just begins,

$$P = Q\epsilon^{\mu\alpha}, \quad \ldots \quad (9)$$

where $\epsilon = 2.3026$ = base of Naperian system of logarithms.

When motion in the direction of Q just begins, we have, by interchanging P and Q,

$$Q = P\epsilon^{-\mu\alpha}. \quad \ldots \quad (10)$$

Also, inversely,

$$\alpha = \frac{2.3026(\log P - \log Q)}{\mu}, \quad \ldots \quad (11)$$

where common logarithms are taken.

If the arc α of the cord is given in degrees instead of radians, we must substitute $\alpha = \frac{\alpha^\circ}{180^\circ}\pi$. If the surface is cylindrical and the number of coils n of the rope is given, we have $\alpha = 2\pi n$.

We see from (9) and (10) that the friction of a cord, $F = P - Q$ or $F = Q - P$, upon a surface *does not depend at all upon the radius of curvature*, but only upon the arc of contact α, or upon the number of coils, $2\pi n$, if the surface is cylindrical.

If we take $\mu = \frac{1}{3}$, we have for a cylindrical surface:

for $\frac{1}{4}$ coils, $P = 1.69Q$;

" $\frac{1}{2}$ " $P = 2.85Q$;

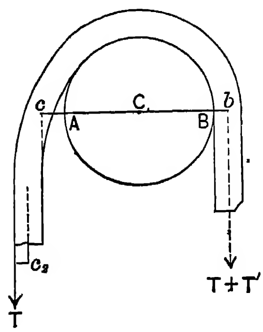
" 1 " $P = 8.12Q$;

" 2 " $P = 65.94Q$;

" 4 " $P = 4348\ 56Q$.

The friction can thus be increased to any amount by increasing the number of coils.]

Rigidity of Ropes.—When a rope is perfectly flexible it offers no resistance to bending. When a rope is not perfectly flexible it offers a resistance by reason of its rigidity when wound *on* to a drum, pulley or axle, though none is offered when it is wound *off*. Thus let a rope whose tension is T be on the point of being wound on to a pulley.



Let $a = \overline{AC} = \overline{BC}$ be the radius of the pulley, and t the thickness of the rope. Then the lever-arm of the axis of the rope on the *off* side is $\overline{Cb} = a + \frac{t}{2}$.

The distance $\overline{Ac}$ from the pulley to the rope on the *on* side will depend on the kind of rope and will be less as is greater. Thus for *hemp* ropes we can put

$$\overline{Ac} = \frac{c_1}{T},$$

where c_1 is a constant to be determined by experiment for the kind of rope; and for *wire ropes*

$$\overline{Ac} = \frac{c_1\left(a + \frac{t}{2}\right)}{T};$$

that is, $\overline{Ac}$ increases with the lever-arm $a + \frac{t}{2}$ and decreases as T increases.

It is also evident that those fibres farthest out on the *on* side are stretched more than those nearer the pulley. The resultant tension T will therefore act further from the pulley than the central axis of the rope. We denote the distance of T from the central axis by c_2.

Let the tension along the central axis on the *off* side be $T + T'$. Then we have for equilibrium, for *hemp ropes*,

$$T\left(a + \frac{t}{2} + \frac{c_1}{T} + c_2\right) = \left(T + T'\right)\left(a + \frac{t}{2}\right),$$

or

$$T' = \frac{c_1 + c_2 T}{a + \frac{t}{2}}; \qquad \text{(1)}$$

and for *wire ropes*,

$$T\left(a + \frac{t}{2} + \frac{c_1\left(r + \frac{t}{2}\right)}{T} + c_2\right) = \left(T + T'\right)\left(a + \frac{t}{2},\right)$$

or

$$T' = c_1 + \frac{c_2 T}{a + \frac{t}{2}}. \qquad \text{(2)}$$

We have then

$$T \times \overline{Cc} = (T + T')\overline{Cb}, \quad \text{or} \quad \overline{Cc} = \left(1 + \frac{T'}{T}\right)\overline{Cb}. \qquad \text{(3)}$$

The rope can be considered, then, as without rigidity if we increase the lever-arm of the tension on the on side by the amount $\frac{T'}{T}$.

Hemp Ropes.—For *tarred* hemp ropes experiment gives

$$T' = \frac{100 + 0.222\,T}{a + \frac{t}{2}} \text{ pounds,}$$

where T is to be taken in pounds, and a and t in inches.

For new hemp ropes, *untarred,*

$$T' = \frac{4 + 0.06457\,T}{a + \frac{t}{2}} \text{ pounds,}$$

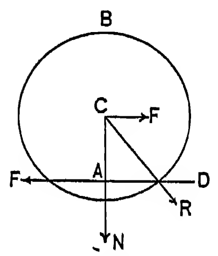
where T is to be taken in pounds, and a and t in inches.

Wire Ropes.—For *wire ropes* we have

$$T' = 1.08 + \frac{0.0937\,T}{a + \frac{t}{2}} \text{ pounds,}$$

where T is to be taken in pounds, and a and t in inches.

Static Rolling Friction.—Let ACB be a roller resting on a plane surface. By reason of the pressure N of the roller on the plane, the roller is compressed. Let a force F be applied at the centre C parallel to the plane. When the resultant R of F and N just passes through the edge D of the base, rolling begins and the force F is equal and opposite to the friction.



Let the distance $AD = d$. Then, when rolling is about to begin, the angle ACD is the angle of repose ϕ. Let r be the radius. Since the compression is small compared to the radius, we have $\tan \phi = \frac{d}{r} = \mu$ = coefficient of static rolling friction. Hence for equilibrium $Fr = Nd$, or

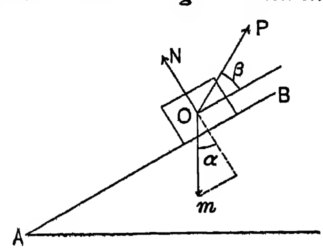
$$F = \mu N = \frac{d}{r} N.$$

The distance d depends on the materials in contact.

The theory of rolling friction is not yet well established and but few experiments upon it have been made.

In all practical cases of rolling we usually have to do with axle friction, which has already been discussed (page 228).

Examples.—(1) *A body of mass m rests upon a rough inclined plane which makes an angle α with the horizontal and is acted upon by a force P which makes the angle β with the plane. Find the conditions for equilibrium.* (For smooth plane see ex. (8), page 218.)



ANS. Consider the body as a particle at any point O. Let the angle $POB = \beta$ be positive when above the plane, and negative when below.

1. BODY ON THE POINT OF MOTION UP THE PLANE.—In this case we have the component of P parallel to the plane acting up the plane, and the friction μN acting down. We have, then, P, N, m and μN in equilibrium. If we put the algebraic sum of forces along the plane and perpendicular to the plane equal to zero, we have in gravitation measure

$$P\cos\beta - m\sin\alpha - \mu N = 0, \quad \text{or} \quad P = \frac{m\sin\alpha + \mu N}{\cos\beta},$$

$$P\sin\beta + N - m\cos\alpha = 0, \quad \text{or} \quad N = m\cos\alpha - P\sin\beta,$$

where μ is the coefficient of friction.

From these equations we have

$$P = \frac{\sin\alpha + \mu\cos\alpha}{\cos\beta + \mu\sin\beta}\cdot m = \frac{\sin(\alpha + \phi)}{\cos(\beta - \phi)} m, \quad \ldots \ldots \quad (1)$$

where ϕ is the angle of repose whose tangent is equal to μ. If $\mu = 0$, there is no friction and we have equation (2), example (8), page 219. If $\beta = 90° - \alpha$, $P = m$ and $N = 0$. For any greater value of positive β, N is negative and no equilibrium is possible. For negative β we must have β less than $90 - \phi$. If negative β is greater than this, we have N negative and no equilibrium is possible. Equation (1) holds good, then, for all values of β between $+(90 - \alpha)$ and $-(90 - \phi)$. The force P is a minimum when $\cos(\beta - \phi)$ is a maximum or when $\beta = \phi$. This minimum value of P is then

$$P = m\sin(\alpha + \varphi)$$

2. BODY ON THE POINT OF MOTION DOWN THE PLANE—α GREATER THAN ϕ.—In this case we have the friction μN acting up the plane and the component of P up the plane. Hence

$$P\cos\beta - m\sin\alpha + \mu N = 0, \quad \text{or} \quad P = \frac{m\sin\alpha - \mu N}{\cos\beta}.$$

$$P\sin\beta + N - m\cos\alpha = 0, \quad \text{or} \quad N = m\cos\alpha - P\sin\beta.$$

From these equations we have

$$P = \frac{\sin\alpha - \mu\cos\alpha}{\cos\beta - \mu\sin\beta}\cdot m = \frac{\sin(\alpha - \phi)}{\cos(\beta + \phi}\cdot m. \quad \ldots \ldots \quad (2)$$

Here, again, we see that equation (2) holds for all values of β between $+(90 - \alpha)$ and $-(90 - \phi)$. Outside of these limits no equilibrium is possible. The force P is a minimum when $\beta = -\phi$. This minimum value of P is then

$$P = m\sin(\alpha - \phi).$$

3 BODY ON THE POINT OF MOTION DOWN THE PLANE — α LESS THAN ϕ.—In this case we have the friction μN acting up the plane, and the component of P down the plane. Hence

$$-P\cos\beta - m\sin\alpha + \mu N = 0, \quad \text{or} \quad P = \frac{\mu N - m\sin\alpha}{\cos\beta},$$

$$P\sin\beta + N - m\cos\alpha = 0, \qquad N = m\cos\alpha - P\sin\beta.$$

From these equations we have

$$P = \frac{\mu\cos\alpha - \sin\alpha}{\cos\beta + \mu\sin\beta}\cdot m = \frac{\sin(\phi - \alpha)}{\cos(\beta - \phi)}\cdot m. \quad \ldots \ldots \quad (3)$$

We cannot have $+\beta$ greater than 90° or the component of P will not act down the plane. If $-\beta = 90 - \phi$, $P = m$ and $N = 0$. For any greater value of negative β, N is negative and no equilibrium is possible. Equation (3) holds for all values of β between $+90$ and $-(90 - \phi)$.

The force P is a minimum when $\beta = \phi$. This minimum value of P is then

$$P = m\sin(\phi - \alpha).$$

(2) *Find the conditions of equilibrium for a rough screw.* (For smooth screw see example (1), page 217)

ANS. If N is the normal pressure on the thread, we have, by resolving Q normally and horizontally, $N = \frac{Q}{\cos\alpha}$. The friction is then $\mu N = \frac{\mu Q}{\cos\alpha}$.

If P has a virtual displacement of θ radians, Q is raised a distance $\frac{p\theta}{2\pi}$, the friction moves through $\frac{r\theta}{\cos\alpha}$ and we have, by virtual work,

$$Pl\theta - \frac{Qp\theta}{2\pi} - \frac{\mu Qr\theta}{\cos^2\alpha} = 0.$$

Hence, since $\frac{p}{2\pi r} = \tan\alpha$, $\mu = \tan\phi$,

$$P = \frac{Q}{l}\left(\frac{p}{2\pi} + \frac{\mu r}{\cos^2\alpha}\right) = \frac{Qr}{l}\left(\tan\alpha + \frac{\tan\phi}{\cos^2\alpha}\right).$$

If we neglect friction we have $\mu = 0$, and $P = \frac{Qp}{2\pi l}$, which is the same result as in ex. (1), page 217.

(3) *Find the conditions for equilibrium for the differential screw given in example (2), page 217, taking friction into account.*

ANS. $P = \frac{Q}{l}\left[\frac{p_1 - p_2}{2\pi} + \frac{\mu(r_1 + r_2)}{\cos^2\alpha}\right]$.

(4) *Find the conditions of equilibrium for the wedge given in example (3), page 217, taking friction into account.*

ANS. $P = 2N\left(\sin\frac{\alpha}{2} \pm \mu\cos\frac{\alpha}{2}\right) = \frac{2N}{\cos\phi}\sin\left(\frac{\alpha}{2} \pm \phi\right)$,

where the (+) sign is taken for wedge on point of entering, and the (—) sign on point of sliding out.

If P is between these values, the wedge is neither on the point of entering nor sliding out. If $\frac{\alpha}{2} = \phi$, there is no force required to keep the wedge from sliding out. The angle of the wedge should not, then, exceed 2ϕ.

(5) *A rod rests with its ends against a rough vertical and horizontal plane. The mass of the rod is m, and its weight acts at its middle point. Find the conditions of equilibrium.*

ANS. Let θ be the angle with the horizontal, and N_1, N_2 the normal pressures on the horizontal and vertical planes respectively. Then

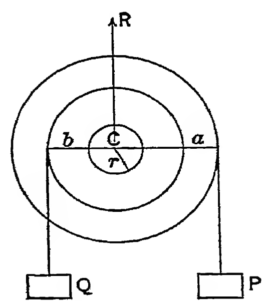
$$\tan\theta = \cot. 2\phi, \qquad N_1 = m\cos^2\phi, \qquad N_2 = m\sin\phi\cos\phi.$$

(6) *In a wheel and axle the radius of the wheel is a, and of the axle b. Find the conditions for equilibrium, taking into account friction and the rigidity of the rope, when a mass P hung from the wheel just balances a mass Q hung from the axle.* (Without friction and rigidity see example (13), page 214.)

ANS. We have seen (page 228) that for well-greased axle and small surface of contact we can take in all cases of axle friction the friction $F = \mu R = \mu(P + Q)$, where μ is the coefficient of static sliding friction.

Let the radius of the journal be r, and let t be the thickness of the rope.

Then when P is just about to fall, we have (page 234) for the lever-arm of Q, $\left(1 + \frac{T'}{Q}\right)\left(b + \frac{t}{2}\right)$, and hence for equilibrium



$$-P\left(a + \frac{t}{2}\right) + Q\left(1 + \frac{T'}{Q}\right)\left(b + \frac{t}{2}\right) + \mu r(P + Q) = 0,$$

or

$$P = \frac{\left(b + \frac{t}{2} + \mu r\right)Q + \left(b + \frac{t}{2}\right)T'}{a + \frac{t}{2} - \mu r},$$

where (page 235)

$$\textit{for hemp ropes } T' = \frac{c_1 + c_2 Q}{b + \frac{t}{2}};$$

$$\textit{for wire ropes } T' = c_1 + \frac{c_2 Q}{b + \frac{t}{2}};$$

the values of c_1 and c_2 being given on page 235.

When Q is just about to fall, we have (page 234) for the lever-arm of P, $\left(1+\frac{T'}{P}\right)\left(a+\frac{t}{2}\right)$, and hence

$$-P\left(1+\frac{T'}{P}\right)\left(a+\frac{t}{2}\right)+Q\left(b+\frac{t}{2}\right)-\mu r(P+Q)=0,$$

or

$$P=\frac{\left(b+\frac{t}{2}-\mu r\right)Q-\left(a+\frac{t}{2}\right)T'}{a+\frac{t}{2}+\mu r},$$

where (page 235)

$$\textit{for hemp ropes } T'=\frac{c_1+c_2P}{a+\frac{t}{2}};$$

$$\textit{for wire ropes } T'=c_1+\frac{c_2P}{a+\frac{t}{2}};$$

the values of c_1 and c_2 being given on page 235.

For values of P less than the first and greater than the second we have non-limiting equilibrium, and the wheel and axle is not upon the point of rotating in either direction.

If we neglect friction and rigidity, we have $P=\dfrac{b+\frac{t}{2}}{a+\frac{t}{2}}Q$, or, neglecting the thickness of the rope $P=\dfrac{b}{a}Q$, as in ex. (13), page 214.

If $b=a$, we have the case of the single pulley.

For *partially worn bearing* (page 228) we can put more accurately

$$\sin\phi \textit{ in place of } \mu,$$

where ϕ is the angle of repose.

For *triangular bearing* (page 228) we can put

$$\frac{\sin 2\phi}{2\cos\beta} \textit{ in place of } \mu,$$

where β is the half angle of the bearing.

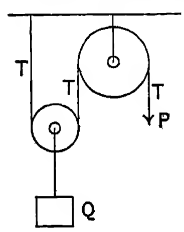
For *new bearing* (page 229) we can put

$$\frac{\beta\sin 2\phi}{2\sin\beta} \textit{ in place of } \mu,$$

where β is the half angle of contact.

(7) *In the single movable pulley find the relation between the force P and the mass Q for equilibrium, taking into account friction and the rigidity of the rope.* (Without friction and rigidity see ex. (5), page 218.)

ANS. Let r be the radius of the axle of each pulley, a the radius of each pulley, t the thickness or rope μ the coefficient of static sliding friction, and c_1, c_2 as given on page 235.



For convenience of notation let

$$u=a+\frac{t}{2}+\mu r+c_2,\quad w=a+\frac{t}{2}-\mu r.$$

Then from the preceding example, making $b=a$, we have, when P is just about to fall, for *hemp ropes*

$$P=\frac{uT_1+c_1}{w},$$

where T_1 is the tension in the first rope as shown in the figure.

We have in the same way

$$T_1=\frac{uT_2+c_1}{w}.$$

We have also

$$T_1 + T_2 = Q.$$

Eliminating T_1 and T_2, we have

$$P = \frac{u^2 Q + (w + 2u)c_1}{w(w + u)}.$$

In the same way we find when P is on the point of rising

$$P = \frac{(u - 2\mu r - c_2)^2 Q - c_1(w + 2u - 2\mu r - c_2)}{(w + u)(u - 2\mu r)}.$$

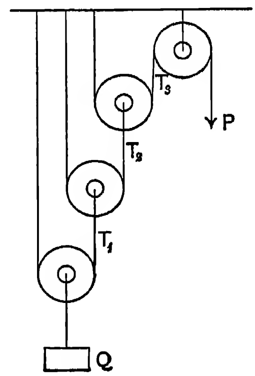
For values of P less than the first and greater than the second we have non-limiting equilibrium and P is not on the point of falling or rising.

For *wire ropes* we have only to substitute $c_1\left(a + \frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we can replace μ by the values given in the preceding example.

If we neglect friction and rigidity, we have $P = \frac{Q}{2}$ as in ex. (5), page 218.

(8) *In the system of pulleys shown find the relation between the force P and the mass Q for equilibrium, taking into account friction and rigidity of the rope.* (Without friction and rigidity see ex. (6), page 218.)



ANS. Let m be the mass of each movable pulley, and n the number of movable pulleys. Let r be the radius of the axle of each pulley, a the radius of each pulley, μ the coefficient of static sliding friction, t the thickness of the rope, and c_1 and c_2 as given on page 235.

For convenience of notation let

$$u = a + \frac{t}{2} + \mu r + c_2; \quad w = a + \frac{t}{2} - \mu r;$$

$$v = u + w = 2a + t + c_2.$$

Then, from the preceding example, we have, when P is just about to fall, for *hemp ropes*

$$T_1 = \frac{u(Q + m)}{v} + \frac{c_1}{v},$$

$$T_2 = \frac{u(T_1 + m)}{v} + \frac{c_1}{v},$$

$$T_3 = \frac{u(T_2 + m)}{v} + \frac{c_1}{v},$$

and so on. Inserting the values of T_1 and T_2, we have in general

$$T_n = \frac{u^n Q}{v^n} + \frac{(mu + c_1)(u^n - v^n)}{v^n(u - v)}.$$

But from the preceding example we have

$$P = \frac{uT_n}{w} + \frac{c_1}{w}.$$

Hence, since $u - v = -w$,

$$P = \frac{u}{wv^n}\left[u^n Q + \frac{(mu + c_1)(v^n - u^n)}{w}\right] + \frac{c_1}{w}.$$

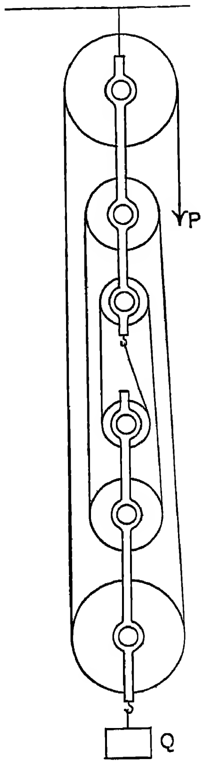
For *wire ropes* we have only to substitute $c_1\left(a+\frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we replace μ by the values given in ex. (6).

If we neglect friction and rigidity, we have $\frac{u}{w}=1$, $\frac{u}{v}=\frac{1}{2}$, $v=2\left(a+\frac{t}{2}\right)$, $u=a+\frac{t}{2}$ and $c_1=0$, and this reduces to $P=\frac{Q+(2^n-1)m}{2^n}$, which is the same result as given in ex. (6), page 218.

(9) *In the system of pulleys shown find the relation between the force P and the mass Q for equilibrium, taking into account friction and the rigidity of the ropes.* (Without friction and rigidity see ex. (7), page 218.)

ANS. Let m be the mass of the lower block, and n the number of ropes coming from the lower block.



Let r be the radius of the axle of each pulley, μ the coefficient of static sliding friction, t the thickness of the rope, and c_1 and c_2 as given on page 235.

Let a be the *mean radius of the pulleys.*

For convenience of notation let

$$u=a+\frac{t}{2}+\mu r+c_2,\qquad w=a+\frac{t}{2}-\mu r.$$

Then we have for *hemp ropes*, when P is about to descend,

$$P=\frac{u^n(u-w)}{w(u^n-w^n)}\left[(Q+m)+\frac{c_1}{u-w}\right]-\frac{c_1}{u-w}.$$

For *wire ropes* we have only to substitute $c_1\left(a+\frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we replace μ by the values given in ex. (6).

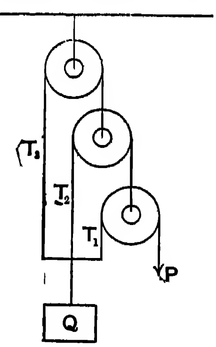
If we neglect friction and rigidity, we have $u=w$ and $c_1=0$. The value of P reduces then to $P=\frac{0}{0}$; but if we divide numerator and denominator by $u-w$ and then make $u=w$, we have

$$P=\frac{Q+m}{n},$$

which is the same result as given in ex. (7), page 218.

(10) *In the system of pulleys shown find the relation between the force P and the mass Q for equilibrium, taking into account friction and the rigidity of the ropes.*

ANS. Let m be the mass of each pulley and n the number of pulleys. Let r be the radius of the axle of each pulley, μ the coefficient of static sliding friction, t the thickness of the rope and c_1, c_2 as given on page 235.



Let a be the radius of each pulley, and for convenience of notation let

$$u=a+\frac{t}{2}+\mu r+c_2,\qquad w=a+\frac{t}{2}-\mu r.$$

Then we have, when P is about to descend, for *hemp ropes*

$$P=\frac{Q+nm-\frac{mu}{w}\left[\left(\frac{w}{u}+1\right)^n-1\right]+\frac{c_1}{w}\left[\left(\frac{w}{u}+1\right)^n-1\right]}{\left(\frac{w}{u}+1\right)^n-1}.$$

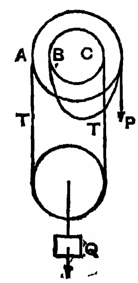
For *wire ropes* we have only to substitute $c_1\left(a+\frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we replace μ by the values given in ex. (6).

If we neglect friction and rigidity, we have $u=w$ and $c_1=0$, and

$$P=\frac{Q+nm-(2^n-1)m}{2^n-1}.$$

(11) *In the differential pulley shown in the figure an endless chain passes over a fixed pulley A, then under a movable pulley to which the mass Q is attached, and then over another fixed pulley B, a little smaller but co-axial with A. The two pulleys A and B are in one piece and obliged to turn together through the same angle. The two ends of the chain are joined so as to form a loop. The force P is applied to the right-hand portion of the loop. To prevent the chain from slipping, there are cavities in the circumferences of the upper pulleys into which the links of the chain fit. Find the relation of P to Q for equilibrium, taking into account friction.*



ANS. Let a be the radius of the pulley A, and b the radius of the pulley B, m the mass of each pulley above and below, r the radius of each axle, and μ the coefficient of friction. Since the pulley is worked by a chain, we can disregard rigidity and have only friction to take into account. Let T be the tension of the chain. Then for equilibrium

$$2T = Q + m, \quad \text{or} \quad T = \frac{Q + m}{2}.$$

Let F be the friction. Then taking moments about C, we have for equilibrium

$$-Pa + Ta - Tb + Fr = 0.$$

The pressure on each axle is $Q + 2m + P$. Therefore the friction is

$$F = 2\mu(Q + 2m + P).$$

Substituting this value of F and the value of T in the preceding equation, we have

$$P = \frac{(Q+m)\left(\frac{a-b}{2}\right) + 2\mu r(Q + 2m)}{a - 2\mu r}.$$

For partially worn bearing, or for new bearing, we can replace μ by the values given in ex. (6). If we neglect friction and the mass of the pulleys, we have $P = \frac{Q(a - b)}{2a}$.

KINETICS OF A PARTICLE.

CHAPTER I.

DEFLECTING FORCE.

Kinetics.—That portion of Dynamics which treats of forces with reference to the change of motion of bodies caused by them is called KINETICS.

Kinetics of a Particle.—We have seen (page 190) that when a rigid body is acted upon by any number of forces, the motion of the centre of mass is the same as if it were a particle of mass equal to that of the body, all the forces acting upon the body being transferred without change in direction or intensity to this particle.

When, therefore, we consider only the motion of translation of a body without reference to its rotation, we can always consider the body as a particle of equal mass concentrated at the centre of mass.

It will therefore be convenient to first consider the KINETICS OF A PARTICLE.

Impressed and Effective Force on a Particle.—Let F be the resultant of any number of forces F_1, F_2, F_3, etc., acting upon a particle. These forces we have called *impressed* forces (page 169), and the resultant F is the resultant impressed force.

Let m be the mass of the particle, and f its acceleration. Then f will be in the direction of F, and we have from equation (2), page 170,

$$F = mf. \qquad (1)$$

We call the quantity mf the *effective force* of the particle. We see from equation (1), then, that *the resultant impressed force is equal to the effective force.*

D'Alembert's Principle Applied to a Particle.—We can write equation (1)

$$F - mf = 0. \qquad (2)$$

That is, *if we reverse the direction* of the effective force it will hold the resultant impressed force in equilibrium.

Hence *the impressed forces acting on a particle and the reversed effective force of the particle constitute a system of concurring forces in equilibrium.*

This is D'Alembert's principle as applied to a particle. It reduces any kinetic problem to one of equilibrium between actual ("*impressed*") forces and a fictitious ("*reversed effective*") force.

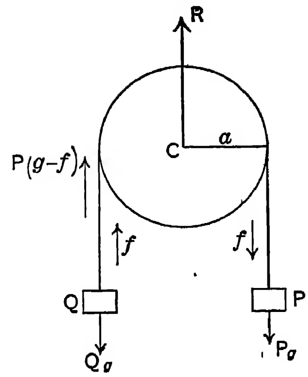
Examples.—(1) *Let a mass $\overline{m}$ be moved on a smooth horizontal plane by a rope which passes over the edge of the plane on a pulley and has a mass P hung at its end. Disregarding all friction and mass of pulley and rope and rigidity of rope, find the acceleration.*

ANS. This example has already been solved (see example (12), page 177). We solve it here by D'Alembert's principle.

The impressed force on $\overline{\mathrm{m}}$ is the tension of the rope $P(g-f)$. The effective force on $\overline{\mathrm{m}}$ is $\overline{\mathrm{m}}f$. Reversing this, we have, by D'Alembert's principle, for equilibrium

$$P(g-f) - \overline{\mathrm{m}}f = 0, \quad \text{or} \quad f = \frac{Pg}{P+\overline{\mathrm{m}}}.$$

(2) *Two masses P and Q, P being the greater, are hung by means of a rope over a pulley. Disregarding friction and the mass of the pulley and rope, find the acceleration.*



ANS. This example has already been solved (see example (13), page 178). We solve it here by D'Alembert's principle.

The impressed forces on Q are Qg downwards and the upward tension of the string $P(g-f)$. The effective force of Q is Qf upward. *Reversing* we have, by D'Alembert's principle, for equilibrium

$$P(g-f) - Qg - Qf = 0, \quad \text{or} \quad f = \frac{(P-Q)g}{P+Q}.$$

Deflecting Force.—We have seen (page 77) that when a particle moves in a curve whose radius of curvature is ρ with a velocity v at any instant, there must be a central acceleration f_ρ *always directed towards the centre of curvature* and given by

$$f_\rho = \frac{v^2}{\rho}.$$

This central acceleration causes no change of speed, but only change of direction of the velocity.

If m is the mass of the particle, we must then have *a central force always directed towards the centre of curvature* given by

$$mf_\rho = \frac{mv^2}{\rho}.$$

This, then, is the force which causes the particle to move in a curve. If this force did not act, the particle would move in a straight line. We therefore call this force the DEFLECTING FORCE, and denote it by F_ρ.

Since the centre of mass of a body moves as if the whole mass $\overline{\mathrm{m}}$ were concentrated at its centre of mass, we have, when the centre of mass of a body moves in a curve with velocity v, the deflecting force for the entire body

$$F_\rho = \frac{\mathrm{m}v^2}{\rho} = \overline{\mathrm{m}}\rho\omega^2 = \overline{\mathrm{m}}v\omega, \quad \ldots\ldots\ldots \quad (1)$$

where ρ is the distance from the centre of curvature *to the centre of mass*, and ω is the angular velocity, so that $\rho\omega = v$.

If the path is a circle, ρ is constant and equal to the radius r and we have

$$F_\rho = \frac{\overline{\mathrm{m}}v^2}{r} = \overline{\mathrm{m}}r\omega^2 = mv\omega. \quad \ldots\ldots\ldots \quad (2)$$

If there is no tangential acceleration f_t, there will be no change of speed and v will be constant in magnitude, changing only in direction.

In such case, if T is the time of revolution in the circle, we have

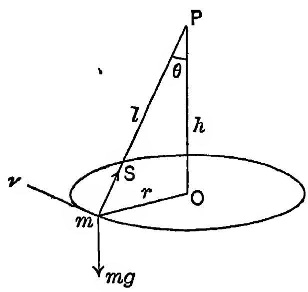
$$\omega = \frac{2\pi}{T}, \quad \text{or} \quad v = \frac{2\pi r}{T},$$

and

$$F_\rho = \frac{4\pi^2 \overline{m} r}{T^2} = \overline{m} r\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

All equations give the force in poundals. For gravitation units divide by g (page 171).

Simple Conical Pendulum.—The simple conical pendulum consists of a particle of mass m attached to a fixed point P by a massless inextensible string of length l, and moving with uniform speed v in a circular path about a vertical axis through the fixed point.



In this case the particle is acted upon by two forces, its weight mg vertically downwards and the stress S of the string directed towards the fixed point P. If the particle moves with uniform speed v in the circle whose radius is $r = l \sin \theta$, where θ is the inclination of the string to the vertical, the vertical component $S \cos \theta$ of the stress S must balance the weight mg, and the horizontal component $S \sin \theta$ of the stress S must be equal to the deflecting force $\frac{mv^2}{r}$ necessary to make the particle move in a circle with uniform speed. We have then

$$-mg + S\cos\theta = 0, \quad \text{or} \quad S\cos\theta = mg, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and from equation (2), page 243,

$$S\sin\theta = \frac{mv^2}{r} = \frac{mv^2}{l\sin\theta} = mr\omega^2 = ml\sin\theta.\omega^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ω is the angular speed.

We can find S from either (1) or (2). Squaring (1) and (2) and adding, we have also

$$S = m\sqrt{g^2 + \frac{v^4}{r^2}} = m\sqrt{g^2 + r^2\omega^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also dividing (2) by (1), we have

$$\tan\theta = \frac{v^2}{gr} = \frac{v^2}{gl\sin\theta} = \frac{r\omega^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For S in gravitation units we must divide (3) by g as usual.

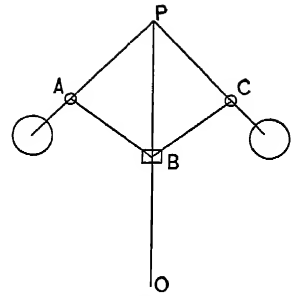
Let h be the distance of the fixed point P above the plane of motion, or the height of the pendulum. Then, since $h \tan\theta = r$, we have, from (4),

$$v^2 = \frac{r^2 g}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If, then, ω is the angular speed of the particle about the centre O, $r\omega = v$, and, from (5), $\omega = \sqrt{\frac{g}{h}}$. If t is the time of a revolution,

$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is the same as the time of oscillation of a simple pendulum of length h (page 138).



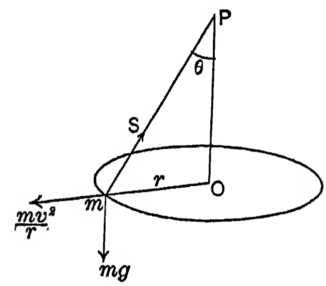
COR. 1.—If θ is indefinitely small, h and l are equal and

$$t = 2\pi\sqrt{\frac{l}{g}},$$

and we have the case of the simple pendulum.

COR. 2.—Since $\omega = \sqrt{\frac{g}{h}}$, we see that the greater the angular velocity the less h, and, as l is constant, the greater r. This fact is taken advantage of in the *steam-engine governor*. As the piston speed increases, the spindle PO revolves more rapidly, the balls separate and the slide at B rises and by means of levers acts upon the valves of the engine to diminish the supply of steam.

Centrifugal Force.—Let us now solve the preceding problem of the simple conical pendulum by applying D'Alembert's principle (page 242).



The impressed forces acting upon the particle are the weight mg and the stress S of the string. The effective force is the deflecting force $\frac{mv^2}{r}$ *acting towards* O. This is an actual force. If now we *reverse the direction* of this effective force, we have $\frac{mv^2}{r}$ *acting away from* O. *This is not an actual but a fictitious force.* There is in reality no force acting in this direction upon m. The actual force upon m is *towards* O. But by D'Alembert's principle the impressed and *reversed* effective forces form a system in equilibrium. We have then, by reversing the effective force,

$$-mg + S\cos\theta = 0,$$

$$S\sin\theta - \frac{mv^2}{r} = 0.$$

We thus evidently obtain the same results as in the preceding article, where we took $\frac{mv^2}{r}$ correctly as a force acting *towards* the centre O.

This reversed effective force $\frac{mv^2}{r}$ acting *away* from O is often called CENTRIFUGAL FORCE, and as thus used the term is allowable.

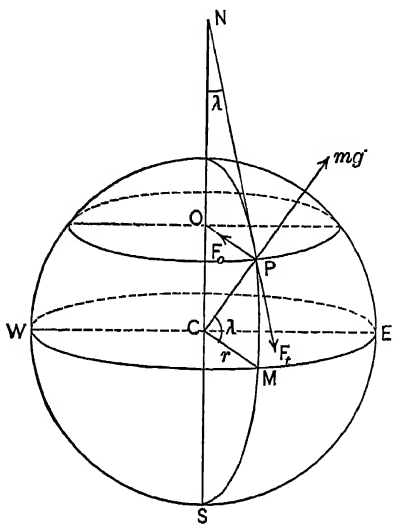
But it should be borne in mind that *it is a fictitious* and not an actual force. There is really no force acting on m away from the centre, and there really is a force acting on m towards the centre. If we really had a centrifugal force in equilibrium with S and mg, the particle *m would move in a straight line* and not in a circle. When, then, a particle moves in a curve, since its direction of motion changes, there must be an unbalanced force $\frac{mv^2}{\rho}$ acting *towards* the centre of curvature, that is, a deflecting force. If we consider this force reversed, we can apply D'Alembert's principle, and the term "centrifugal force" only means, then, this *reversed* force, which is a purely fictitious force not really existing.

A particle moving in a curve is often incorrectly represented, however, as possessing an inherent "centrifugal force" by virtue of which it "tends to fly away from the centre." Indeed it is sometimes represented that it is acted upon simultaneously by such a force, and

also by an equal and opposite deflecting force towards the centre.* Both views are erroneous and misleading.

The student would do well, therefore, to discard the term "centrifugal force," since it answers no real purpose. It is sufficient in all cases to take all forces, both impressed and effective, as they really are, and then to apply D'Alembert's principle.

Deflecting Force at the Earth's Surface.—Suppose the earth to be a homogeneous sphere of radius r and centre C. Let WME represent the equator, NS the axis, and let P be any point of the surface on the meridian PMS, so that the latitude is $\lambda = PCM$.



Let a particle of mass m rest on the surface at P. Let G be the actual acceleration of gravity acting towards the centre C along the radius PC.

Then mG is the actual force of attraction towards C acting upon the particle. Let g be the *observed* acceleration of gravity at P. Then the observed weight of the particle at P is mg. The pressure of the earth on the particle is then mg acting away from C.

The difference between mG and mg, or $mG - mg$, is that part of the earth's attraction necessary to keep the particle on the earth without pressure. This force $mG - mg$ acting towards C can be resolved into a component F_t tangent to the meridian at P, and a component F_0 in the direction PO. This latter component is the deflecting force which makes the particle move in the latitude circle whose radius is $PO = r\cos\lambda$ This, then, is the effective force, while $m(G - g)$ towards C and F_t are impressed forces.

If we reverse the effective force, we have, by D'Alembert's principle,

$$-m(G-g) + F_0\cos\lambda = 0,$$
$$-F_0\sin\lambda - F_t = 0.$$

But (page 244) since the angular velocity $\omega = \frac{2\pi}{T}$, where T is the time of rotation,

$$F_0 = mr\omega^2\cos\lambda = \frac{4\pi^2 mr}{T^2}\cos\lambda.$$

Hence

$$mg = mG - mr\omega^2\cos^2\lambda = mG - \frac{4\pi^2 mr}{T^2}\cos^2\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$F_t = -\frac{1}{2}mr\omega^2\sin 2\lambda = -\frac{2\pi^2 mr}{T^2}\sin 2\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

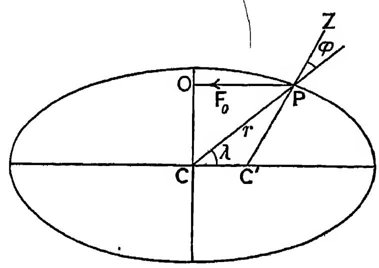
* "When I was about nine years old, I was taken to hear a course of lectures by an itinerant lecturer in a country town, to get as much as I could of the second half of a good, sound philosophical omniscience. . . .

"'You have heard what I have said of the wonderful centripetal force, by which Divine Wisdom has retained the planets in their orbits round the sun. But, ladies and gentlemen, it must also be clear to you that if there were no other force in action, this centripetal force would draw our earth and the other planets into the sun, and universal ruin would ensue. To prevent such a catastrophe, the same Wisdom has implanted a centrifugal force of the same amount, and directly opposite ' . . .

"I had never heard of Alfonso X. of Castile, but I ventured to think that if Divine Wisdom had just let the planets alone it would come to the same thing with equal and opposite troubles saved."—DE MORGAN, *Budget of Paradoxes.*

If the earth were a homogeneous sphere, the effect of this tangential component F_t upon liquid particles on the surface would be to force them towards the equator and thus increase the equatorial and diminish the polar diameter. The fact that the earth is not a sphere thus indicates that the now solid portions may once have existed in a plastic condition. The equatorial diameter is found to exceed the polar by about 26 miles. The ratio of this difference to the equatorial diameter, called the ELLIPTICITY of the earth, is about $\frac{1}{300}$.

The earth is considered, then, as an ellipsoid of revolution with this ellipticity, so that the direction of the observed force of gravity, or of the plumb-line ZP, is always normal to the surface and hence does not pass through C except at the equator and poles.



The force of attraction mG acting towards C is then resolved into two components, one normal to the surface along PC' and one F_0 along PO. This latter is the deflecting force necessary to keep the particle on the latitude circle whose radius is $PO = r\cos\lambda$. The former is balanced by the pressure of the earth upon the particle. There is then no tangential force F_t, and no tendency of the particle at P to move towards the equator.

The effective force is then F_0 as before acting towards O, and the impressed forces are mG acting towards C and mg acting towards Z. Let ϕ be the angle CPC'. If we reverse the effective force, we have, by D'Alembert's principle,

$$mg + F_0 \cos(\lambda + \phi) - mG\cos\phi = 0,$$

or, substituting for F_0 its value,

$$mg = mG\cos\phi - mr\omega^2 \cos\lambda\cos(\lambda+\phi) = mG\cos\phi - \frac{4\pi^2 mr}{T^2}\cos\lambda\cos(\lambda+\phi).$$

Since in the case of the earth the deviation from a sphere is small, the angle ϕ is very small and this equation reduces practically to (1). We can then treat the earth as a sphere of mean radius r and neglect the tangential component F_t.

COR. 1.—If we take the mean radius $r = 3960$ miles and $T = 86164$ seconds for a sidereal day, we have

$$r\omega^2 = \frac{4\pi^2 r}{T^2} = 0.111255 \text{ ft.-per-sec. per sec.} \qquad (3)$$

We have then, from (1), for the total acceleration of gravity G at any point P in latitude λ

$$G = g + 0.111255\cos^2\lambda. \qquad (4)$$

At the poles $\lambda = 90°$ and $G = g$, or the observed acceleration of gravity g at the poles is equal to the total acceleration of gravity G of the earth.

At the equator $\lambda = 0$ and here the observed value of g at sea-level is found to be about

$$g = 32.09022 \text{ ft.-per-sec. per sec.}$$

Hence from (4) we obtain, assuming the earth as a sphere,

$$G = 32.20148 \text{ ft.-per-sec. per sec.}$$

The resultant central force $mG - mg$ acting towards C is then, from (4),

$$mG\left(1 - \frac{g}{G}\right) = 0.111255m\cos^2\lambda = mr\omega^2\cos^2\lambda = \frac{4\pi^2 rm}{T^2}\cos^2\lambda.$$

At the equator $\lambda = 0$, and $1 - \frac{g}{G} = \frac{1}{289}$. Hence at the equator

$$\frac{1}{289}mG = 0.111255m = mr\omega^2 = \frac{4\pi^2 rm}{T^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

That is, *the deflecting force at the equator is about $\frac{1}{289}$ of the total force of gravity.*

COR. 2.—To find the time of rotation T_0 of the earth in order that a particle at any point P may have no observed weight, i.e., exert no pressure on the surface, we have from (1), by putting $mg = 0$ and $T = T_0$,

$$G = \frac{4\pi^2 r}{T_0^2}\cos^2\lambda, \quad \text{or} \quad T_0 = \sqrt{\frac{4\pi^2 r}{G}\cos^2\lambda}.$$

But, from (5), we have for the actual time of rotation T

$$\frac{1}{289}G = \frac{4\pi^2 r}{T^2}, \quad \text{or} \quad G = \frac{289 \times 4\pi^2 r}{T^2}.$$

Substituting this value of G, we obtain

$$T_0 = \frac{1}{17}T\cos\lambda.$$

At the equator we have $\lambda = 0$ and $T_0 = \frac{1}{17}T$. In order, then, that a body at the equator may have no weight, the earth should rotate in about one seventeenth of a day. If the earth were to rotate faster than this, bodies at the equator would not stay on the surface.

Examples.—(1) *A string just breaks with a weight of 20 pounds. It is fastened to a fixed point at one end and at the other to a mass of 5 lbs., which revolves round the point in a horizontal plane. Find the greatest number of complete revolutions in a minute before the string breaks, if the radius to the centre to mass is 5 feet.* ($g = 32$.)

ANS. To break the string requires a force of $20g$ poundals. Let n be the number of revolutions per minute. Then $v = \frac{2\pi rn}{60} = \frac{\pi n}{6}$ ft. per sec. The stress in the string is $\frac{mv^2}{r} = 20g$, or $v = \sqrt{20g} = \frac{\pi n}{6}$. Hence $n = \frac{6\sqrt{20g}}{\pi} = 48$ complete revolutions and a fraction over.

(2) *Suppose the mass revolves in a vertical plane?*

ANS. Then at the lowest point the stress is $\frac{mv^2}{r} + mg$ poundals and $\frac{mv^2}{r} + mg = 20g$, or $v = \sqrt{15g} = \frac{\pi n}{6}$. Hence $n = \frac{6\sqrt{15g}}{\pi} = 41$ complete revolutions and a fraction over.

(3) *What portion of their weight do bodies lose at the equator, taking the radius of the earth 4000 miles and $g = 32$ ft.-per-sec. per sec.*

ANS. About $\frac{1}{286}$

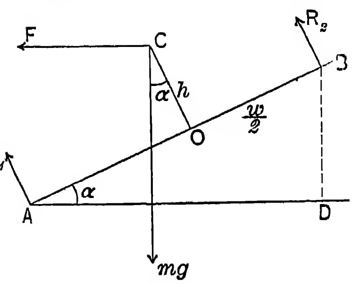
(4) *Find the length of day in order that a body in latitude 60° may possess no weight.*

ANS. One thirty-fourth of the present day.

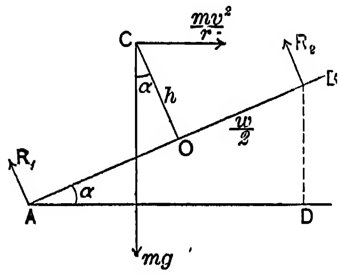
(5) *A skater moves in a curve of 100 feet radius with a speed of 40 feet per sec. Find his inclination to the ice.* ($g = 32$.)

ANS. 60 degrees.

(6) *Find the necessary elevation of the outer rail on a railroad-track on a curve of radius r, so that an engine weighing m lbs. moving with a speed v can pass without lateral pressure on the rails by the wheel-flanges. Also find the pressures on the rails.*



ANS. Let C be the centre of mass of the engine and $h = CO$ the distance of the centre of mass above the rails. We have acting at C the weight mg, downwards. Since there is no lateral pressure on the rails, the rail pressures are R_1 and R_2 at right angles to AB, as shown in the figure. The impressed forces are then mg, R_1 and R_2, and the effective force is $F = \frac{mv^2}{r}$ horizontal and acting towards the centre of curvature.



By D'Alembert's principle, if we reverse F we have a system of forces in equilibrium. Let α be the angle of elevation BAD, and w the width of track AB.

We have then, putting the algebraic sum of vertical components equal to zero,

$$R_1 \cos\alpha + R_2 \cos\alpha - mg = 0, \quad \text{or} \quad (R_1 + R_2)\cos\alpha = mg. \quad . \quad . \quad (1)$$

Putting the algebraic sum of the horizontal components equal to zero,

$$-R_1 \sin\alpha - R_2 \sin\alpha + \frac{mv^2}{r} = 0, \quad \text{or} \quad (R_1 + R_2)\sin\alpha = \frac{mv^2}{r}. \quad . \quad (2)$$

Taking moments about A and putting the algebraic sum of the moments equal to zero,

$$R_2 w - mg\left(\frac{w}{2}\cos\alpha - h\sin\alpha\right) - \frac{mv^2}{r}\left(h\cos\alpha + \frac{w}{2}\sin\alpha\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1) and (2) we have

$$\tan\alpha = \frac{v^2}{gr}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and substituting this in (3), we obtain

$$R_2 = \frac{mg}{2\cos\alpha}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and hence, from (1),

$$R_1 = \frac{mg}{2\cos\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equations (5) and (6) give R_2 and R_1 in poundals. For gravitation measure we divide by g and obtain

$$R_1 = R_2 = \frac{m}{2\cos\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (4) we obtain

$$\sin\alpha = \frac{v^2}{gr\sqrt{1 + \frac{v^4}{g^2r^2}}}, \qquad \cos\alpha = \frac{1}{\sqrt{1 + \frac{v^4}{g^2r^2}}}.$$

We have then for the elevation $DB = w\sin\alpha$

$$DB = \frac{wv^2}{gr\sqrt{1 + \frac{v^4}{g^2r^2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and for R_1 and R_2, in pounds,

$$R_1 = R_2 = \frac{m}{2\sqrt{1 + \frac{v^4}{g^2r^2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equations (8) and (9) are accurate. They admit of practical simplification. Thus if we take the maximum speed a mile per minute or 88 feet per second, and a ten-degree curve for which $r = 573.69$ ft., we have for $g = 32$, $\frac{v^2}{gr} = \frac{1}{26}$, or $\frac{v^4}{g^2r^2} = \frac{1}{676}$, and this value is far greater than it will ever actually be. We see, then, that $\frac{v^4}{g^2r^2}$ can be disregarded in (8) and (9), and we have practically

$$R_1 = R_2 = \frac{m}{2},$$

and for the elevation of the outer rail the practical formula

$$DB = \frac{wv^2}{gr};$$

or, taking $g = 32$ and $w = 4$ ft. $8\frac{1}{2}$ in.,

$$DB = \frac{7v^2}{4r} \textit{ inches}, \text{ nearly}, \quad \ldots \quad (10)$$

where r is to be taken *in ft.* and v *in ft. per sec.*

If r is taken *in ft.* and v in *miles per hour*, we have

$$DB = \frac{15v^2}{4r} \textit{ inches}, \text{ nearly}. \quad \ldots \quad (11)$$

Equations (10) and (11) are then practical formulas giving the elevation in inches for r in feet, and v in ft. per sec. or in miles per hour.

(7) *Find the elevation of outer rail for radius of 300 yards and speed of 45 miles per hour.*

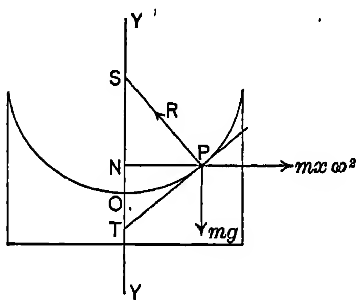
ANS. 8.43 inches.

(8) *Find the speed v of an engine on a curved level track of radius r and gauge w when it is just on the point of overturning, the centre of mass being h above the rails.*

ANS. $v = \sqrt{\frac{gwr}{2h}}$.

(9) *A vessel containing water revolves with uniform angular velocity ω about a vertical axis through the centre of mass. Find the curve of the surface of the water.*

ANS. Let a particle of mass m be at any point P of the surface whose co-ordinates are $ON = y$ and $NP = x$.



The impressed forces are the weight mg of the particle and the pressure R of the surface upon the particle. The surface at P must be at right angles to R. Hence the tangent PT is at right angles to R.

The effective force is $mx\omega^2$ acting towards N. If we *reverse* this force, we have, by D'Alembert's principle,

$$mx\omega^2 - R \sin NPT = 0,$$

$$R \cos NPT - mg = 0.$$

Hence we have

$$\tan NPT = \frac{x\omega^2}{g}.$$

Produce the direction of R to S, so that SP is the normal and SN is the subnormal at P. Then

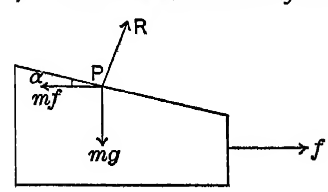
$$SN \times \tan NPT = x, \quad \text{or} \quad SN = \frac{g}{\omega^2}.$$

The property of a parabola is that the subnormal is constant and equal to the parameter.

Therefore the *curve is a parabola* and its equation is

$$y = \frac{\omega^2 x^2}{2g}.$$

(10) *A vessel containing water is moved horizontally with an acceleration f. Find the inclination of the water-surface.*



ANS. Let a particle of mass m be at any point P of the surface.

The impressed forces are the weight mg of the particle and the pressure R of the surface upon the particle. The surface must be at right angles to R. Let α be the angle of the surface with the horizontal.

The effective force is mf in the direction of f. If we *reverse* this force, we have, by D'Alembert's principle,

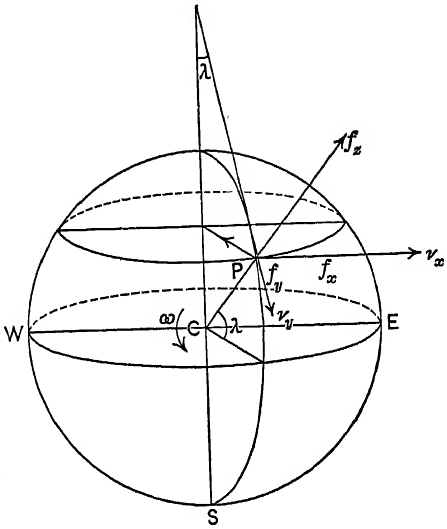
$$R \cos \alpha - mg = 0,$$

$$R \sin \alpha - mf = 0.$$

Hence

$$\tan \alpha = \frac{f}{g}.$$

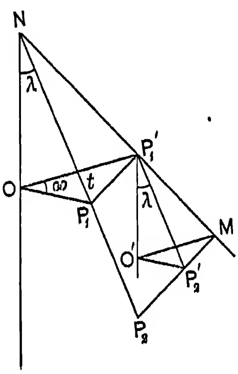
Particle Moving on the Earth's Surface.—Let the particle P, instead of being at rest on the surface of the earth, have a velocity v relative to the earth at any instant in any direction tangent to the earth's surface.



Take the point P as origin, the axis of X towards the east, the axis of Y towards the north, the axis of Z along the radius through P.

Let v_x and v_y be the components of v along the axes of X and Y, so that v_x is positive towards the east and negative towards the west, and v_y is positive towards the north and negative towards the south.

Let $P_1P_2 = v_yt$ be the distance south along the meridian described by P in north latitude, *in an indefinitely small time t.* If there were no rotation, P_1P_2 would coincide with the meridian through P_1. But owing to the rotation of the earth this meridian moves to $P_1'M$, while P_1 moves to P_2', so that if $P_1'O'$ is parallel to the axis NO, the angle $MO'P_2' = \omega t$, where ω is the angular velocity of rotation. The angle $O'P_1'P_2' = \lambda =$ the latitude of P_1. We have then $O'P_2' = v_yt \sin\lambda$ and



$$MP_2' = v_yt \sin\lambda \, . \, \omega t.$$

But if f_x is the acceleration due to rotation of P with reference to the meredian M, we have (page 92)

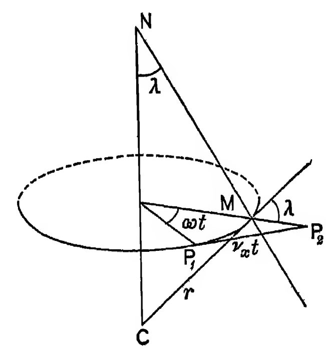
$$MP_2' = \frac{1}{2}f_xt^2 = v_y\omega t^2 \sin\lambda.$$

Hence we have for the acceleration of P *with reference to the meridian M,* due to rotation and the velocity v_y,

$$f_x = 2\omega v_y \sin\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) is general if we take v_y positive towards the north and negative towards the south, and λ positive or negative according as the point is north or south of the equator. Thus in the figure v_y is south or negative, and hence for north latitude f_x is negative or towards the west.

Again, let $P_1P_2 = v_xt$ be the distance east described by P in north latitude in an indefinitely small time t. The meridian moves to M while P_1 moves to P_2, so that the angle $MP_1P_2 = \omega t$. We have then



$$MP_2 = v_xt \, . \, \omega t,$$

and its projection on the meridian is $MP_2 \sin\lambda = \omega v_xt^2 \sin\lambda$ towards the south. If f_y is the acceleration of P with reference to the meridian, due to rotation and the velocity v_x, we have

$$\frac{1}{2}f_yt^2 = -\,\omega v_xt^2 \sin\lambda, \quad \text{or} \quad f_y = -\,2\omega v_x \sin\lambda. \quad . \quad (2)$$

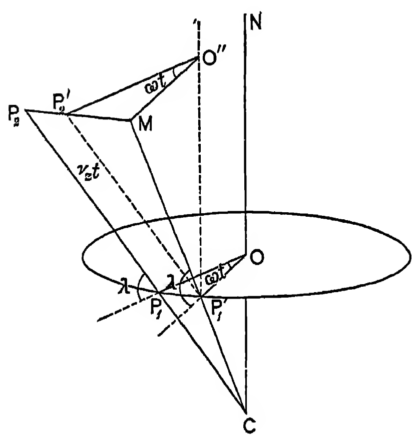
Equation (2) is general if we take v_x positive towards the east, negative towards the west, latitude north positive, south negative, and f_y positive towards the north, negative towards the south.

Again, we have in the preceding figure for the projection of MP_2 along the radius r, $MP_2 \cos\lambda = \omega v_x t^2 \cos\lambda$ upwards. If f_z is the radial acceleration due to rotation and the velocity v_x of P with reference to the meridian, we have

$$\frac{1}{2} f_z t^2 = \omega v_x t^2 \cos\lambda, \quad \text{or} \quad f_z = 2\omega v_x \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But we also have the acceleration of P with reference to the meridian when there is no rotation, $\frac{v^2}{r} = \frac{v_x^2 + v_y^2}{r}$ due to the velocity v, and also the downward acceleration g due to gravity. Hence the total radial acceleration of P *with reference to the meridian*, due to rotation and the velocities v_x and v_y, is

$$f_z = -g + \frac{v_x^2 + v_y^2}{r} + 2\omega v_x \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$



Equation (4) is general if we take v_x positive towards the east, negative towards the west, and v_y positive towards the north, negative towards the south, λ positive for north, negative for south latitude, and f_z positive upwards, negative downwards.

Deviation of a Falling Body by Reason of the Rotation of the Earth.—Let a particle be projected upwards along the radius of the earth with a relative velocity v_z, and let $P_1P_2 = v_z t$ be the distance described in an indefinitely small time t. If there were no rotation, P_1P_2 would coincide with the radius through P_1. But owing to the rotation of the earth the meridian moves to $P_1'M$ while P_1 moves to P_2', so that if $P_1'O''$ is parallel to the axis NO, the angle $MO''P_2' = \omega t$, where ω is the angular velocity of rotation. The angle $O''P_1'P_2' = 90 - \lambda$, where λ is the latitude of P_1. We have then $O''P_2' = v_z t \,.\, \cos\lambda$ and

$$MP_2' = -v_z t \cos\lambda \,.\, \omega t.$$

But if f_x is the acceleration due to rotation of P with reference to the meridian, we have

$$MP_2' = \frac{1}{2} f_x t^2 = -v_z t^2 \omega \cos\lambda.$$

Hence we have for the acceleration in longitude of P *with reference to the meridian*, due to rotation and the velocity v_z,

$$f_x = -2\omega v_z \cos\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) is general if we take v_z positive upwards and negative downwards, north latitude positive, south latitude negative, and f_x positive towards the east, negative towards the west.

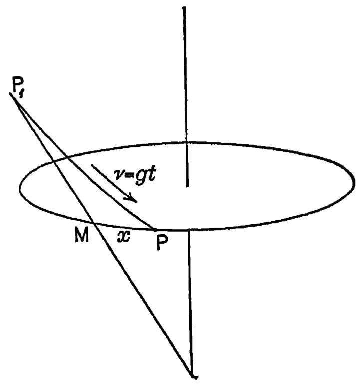
For a falling body, then, v_z is negative and we have f_x essentially positive or towards the east. Hence a falling body falls *to the east* of the point vertically beneath it at the start.

Let t be the time of fall. Then if the particle starts from rest, we have for the height of fall (page 92)

$$h = \frac{1}{2}gt^2, \quad \text{or} \quad t = \sqrt{\frac{2h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2$$

The resultant acceleration at any point of the path is then

$$\sqrt{g^2 + f_x^2} = \sqrt{g^2 + 4v^2\omega^2 \cos^2 \lambda}.$$



But $\omega = \dfrac{2\pi}{60 \times 60 \times 24}$. For ordinary falls, then, we may neglect $4v^2\omega^2 \cos^2 \lambda$, and we have practically the acceleration at any point of the path equal to g. The velocity at any point of the path is then practically $v = gt$, and, from (1),

$$f_x = 2\omega gt \cos \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The mean acceleration is then

$$\frac{1}{2}f_x = \omega gt \cos \lambda,$$

and the final velocity in longitude is then $\frac{1}{2}f_x t = \omega g t^2 \cos \lambda$. The velocity in longitude varies, then, as the square of the time, or as the ordinate to a parabola. The mean velocity in longitude is then $\frac{1}{3}\omega g t^2 \cos \lambda$, and hence the distance in longitude is

$$x = \frac{1}{3}\omega g t^3 \cos \lambda.$$

Inserting the value of t from (2), we have

$$x = \frac{1}{3}g\omega \cos \lambda \sqrt{\left(\frac{2h}{g}\right)^3} = \frac{2}{3} h\omega \cos \lambda \sqrt{\frac{2h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) gives the deviation in longitude of a falling body by reason of the earth's rotation. It is always towards the east or positive. For a body projected vertically upwards it is towards the west.

Examples.—(1) *An engine weighing 27 tons runs at the rate of 45 miles per hour on a straight track in latitude 30° north. Find the pressure on the rails* (*a*) *when it runs north;* (*b*) *south;* (*c*) *east;* (*d*) *west.* ($g = 32$.)

ANS. (*a*) 290 poundals or about 9 pounds on the east rail; (*b*) the same on the west rail; (*c*) the same on the south rail; (*d*) the same on the north rail.

(2) *In the preceding example let the latitude be 30° south.*

ANS. (*a*) 290 poundals or about 9 pounds on the west rail; (*b*) the same on the east rail; (*c*) the same on the north rail; (*d*) the same on the south rail.

(3) *In example* (*1*) *find the vertical pressure on the rails when the engine runs* (*a*) *north;* (*b*) *south;* (*c*) *east;* (*d*) *west.* ($r = 3960$ *miles.*)

ANS. (*a*) Less by 12.6 poundals or about 0.4 pounds; (*b*) the same; (*c*) less by 515 poundals or about 16 pounds; (*d*) increased by 490.2 poundals or about 15 3 pounds.

(4) *Find the velocity of a body in order that it may have no weight when it moves, in latitude 60°,* (*a*) *north;* (*b*) *south;* (*c*) *east;* (*d*) *west.* ($r = 3960$ *miles,* $g = 32$.)

ANS. (*a*) and (*b*) about 5 miles per sec; (*c*) about 4.85 miles per sec.; (*d*) about 5.14 miles per sec.

(5) *A particle in latitude 30° north has a velocity of 60 feet per sec. and moves on a perfectly smooth horizontal plane. Disregarding resistance of the air, find the acceleration and the distance described in*

latitude and longitude in 4 seconds (a) when the velocity is north; (b) south; (c) east; (d) west. ($r = 3960$ *miles.*)

ANS. (*a*) $f_x = 0.00436$ ft.-per-sec. per sec. east, distance 240 ft. north and 0.035 ft. east.

(*b*) $f_x = 0.00436$ ft.-per-sec. per sec. west, distance 240 ft. south and 0.015 ft. west.

(*c*) $f_y = 0.05236$ ft.-per-sec. per sec. south, distance 240 ft. east and 0.42 ft. south.

(*d*) $f_y = 0.05236$ ft.-per-sec. per sec. north, distance 240 ft. west and 0.42 ft. north.

(6) *A cannon-ball is fired in latitude 30° north with a velocity of 1440 feet per sec. Neglecting resistance of the air, find the acceleration and distance described in latitude and longitude in 4 seconds (a) when the velocity is north; (b) south; (c) east; (a) west.*

ANS. (*a*) $f_x = 0.10472$ ft.-per-sec. per sec. east, distance 5760 ft. north and 0.84 ft. east.

(*b*) $f_x = 0.10472$ ft.-per-sec. per sec. west, distance 5760 ft. south and 0.84 ft. west.

(*c*) $f_y = 0.15272$ ft.-per-sec. per sec. south, distance 5760 ft. east and 1.22 ft. south.

(*d*) $f_y = 0.15272$ ft.-per-sec. per sec. north, distance 5760 ft. west and 1.22 ft. north.

(7) *A particle in latitude 60° north falls from rest a distance of 1296 ft. to the ground. Find the deviation in latitude, disregarding resistance of the air.*

ANS. 0.2827 ft. towards the east.

(8) *In latitude 30° north, find the angular velocity of rotation of the plane of a pendulum.*

ANS. 0.0000363 radians per sec. in a direction clockwise to one facing the north. The plane rotates through 180° in 24 hours.

(9) *A locomotive weighing 32 tons runs at the rate of 45 miles per hour in latitude 30° north in a direction S. 30° E. on a curve of one mile radius in a counter-clockwise direction to one looking north. Find the pressure on the outer rail.*

ANS. 1868 pounds If we disregard rotation of the earth, the pressure would be 1848 pounds.

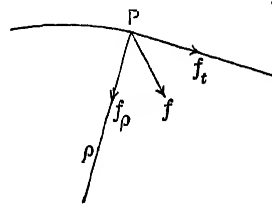
(10) *In the preceding example suppose the direction is clockwise to one looking north.*

ANS. 1825.6 pounds.

CHAPTER II.

TANGENTIAL FORCE. MOMENTUM. IMPULSE.

Tangential Force.—We have seen (page 77) that when a particle P moves in a curve whose radius of curvature is ρ, with a velocity v and an acceleration f at any instant, this acceleration f can be resolved into a central acceleration $f_\rho = \frac{v^2}{\rho}$ and a tangential acceleration $f_t = \frac{dv}{dt}$, where $\frac{dv}{dt}$ is the rate of change of speed.



If m is the mass of the particle, then the force F acting on it is $F = mf$ in the direction of f, and this force can be resolved into a central or deflecting force $F_\rho = mf_\rho = \frac{mv^2}{\rho}$, which causes change of direction of motion and a tangential force $F_t = mf_t = m\frac{dv}{dt}$, which causes change of speed.

The deflecting force F_ρ we have discussed in the preceding chapter.

Let us now consider the tangential force F_t.

Momentum.—Let the mass of a particle be m, and its velocity v. Then we call the quantity mv the MOMENTUM of the particle.

Hence *the momentum of a particle is the product of its mass and velocity.*

Momentum, then, has magnitude and direction, and we can represent it by a straight line, just like velocity, and all the principles which hold for velocity hold also for momentum.

We can therefore combine and resolve momentums, and we have the triangle and polygon of momentum just the same as for velocity or force.

Hence *the momentum of a particle in any direction is the product of its mass and velocity in that direction.*

We can also have the *moment of a momentum*, just the same as for a velocity, and the same principles apply.

The unit of momentum is, then, evidently the momentum of one unit of mass moving with one unit of velocity. We may call a unit of velocity, or one unit of length per unit of time, a "*velo.*"

The English unit of momentum is then the *pound-velo* (*lb.-velo*), or the momentum of a mass of one lb. moving with a velocity of one foot per second.

If, then, a particle of mass 40 lbs. has a velocity in *any given direction* at any instant of 8 ft. per sec., the momentum *in the direction of the velocity* is 320 *lb.-velos.*

In the C. G. S. system, we would have, then, the *gram-velo*, or *kilogram-velo*, that is, the momentum of a mass of one gram or one kilogram moving with a velocity of one centimeter per second.*

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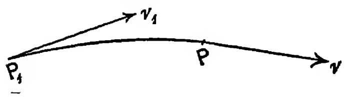
* A committee of the British Association have proposed for this the name "*bole.*" This has, however, never come into use.

Significance of Momentum.—Let a particle of mass m move in any path from the position P_1 to P in the time t. Let the velocity at P_1 be v_1, and at P be v,



We have (page 255) the tangential force F_t given by

$$F_t = m\frac{dv}{dt}.$$

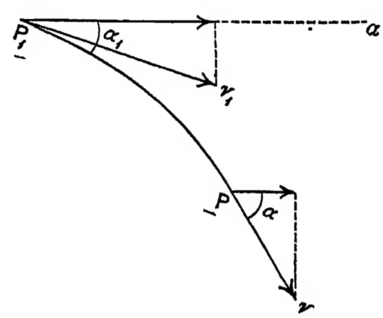
Now if F_t is constant in magnitude, the tangential acceleration $f_t = \frac{dv}{dt}$ is also constant in magnitude. In such case the instantaneous rate of change of speed $\frac{dv}{dt}$ is equal to the mean rate of change of speed $\frac{v - v_1}{t}$ for any interval of time t (page 74). We have then for F_t constant in magnitude

$$F_t = \frac{mv - mv_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now $mv - mv_1$ is the change of momentum *in the path* in the time t, $(v - v_1)$ is the change of speed in the time t, and $\frac{mv - mv_1}{t}$ is the time-rate of change of momentum *in the path.*

Hence, *whatever the path and however the actual tangential force may vary in magnitude during the time, the time-rate of change of momentum in the path gives a tangential force of constant magnitude which, acting for that time, would cause the change of speed in the path.*

Again, let us resolve v_1 and v into components in any given direction P_1a and at right angles to this direction. Let v_1 make the angle α_1, and v the angle α with the direction P_1a. Then the component velocities *in this direction* are $v_1 \cos \alpha_1$, $v \cos \alpha$, and the mean time-rate of change of speed in this direction is $\frac{v \cos \alpha - v_1 \cos \alpha_1}{t}$. If the force F in this direction were constant, the instantaneous rate of change of speed would be the same as the mean for any interval of time, and we should have



$$F_u = \frac{mv \cos \alpha - mv_1 \cos \alpha_1}{t}. \quad . \quad . \quad . \quad (2)$$

Hence *the time-rate of change of momentum in any direction gives the uniform force F_u in that direction which, acting for that time, would cause the change of velocity in that direction.*

No matter, then, how the actual force in the given direction may vary during the time t, we can find from (2) in any case that *equivalent uniform force F_u* which, acting for that time in that direction, would produce the change of velocity in that direction.

From equation (2) we see that as the time t decreases the force F_u increases for the same change of momentum. If t is zero, F_u becomes infinitely large. That is, *change of momentum requires time*, and the less the time the greater the force.

If the time is one second and the initial velocity $v_1 \cos \alpha_1$ in any direction s zero, we have, from (2),

$$F_u = \frac{mv \cos \alpha}{1 \text{ sec.}}.$$

If the time is one second and the final velocity $v \cos \alpha$ in any direction is zero, we have, from (2),

$$F_u = \frac{mv_1 \cos \alpha}{1 \text{ sec.}}.$$

That is, *the momentum in any direction is* NUMERICALLY *equal to that uniform force which, acting in the direction of the momentum, would give the particle starting from rest its velocity in that direction in one second; or which, acting opposite to the direction of momentum, would bring the particle to rest in one second.*

If we take mass in lbs. and velocity in feet per sec., the force in equations (1) and (2) is given in poundals. For gravitation measure divide by g (page 172).

If, then, a particle of mass 40 lbs. has a velocity in any given direction at any instant of 8 ft. per sec., the momentum in that direction is 320 *lb.-velos* (page 255), and a uniform force of 320 poundals or $\frac{320}{g}$ pounds acting in the direction of the velocity would give the particle starting from rest its given velocity *in one second.* Acting opposite to the velocity, it would bring the particle to rest *in one second.*

Impulse.—The product Ft of a *uniform force* F by its time of action t is called the IMPULSE of the force.

Hence *impulse is the product of a uniform force by its time of action, and it acts in the direction of the force.*

Impulse, then, has magnitude and direction, and we can represent it by a straight line just like force, and all the principles which hold good for force hold also for impulse. We can therefore combine and resolve impulses and have the triangle and polygon of impulse just the same as for force.

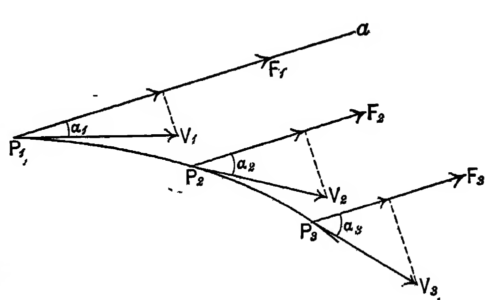
We can also have the *moment* of an impulse just the same as for force, and the same principles apply.

The unit of impulse is then, by definition, the impulse of one unit of force acting for one unit of time.

The English unit is, then, the poundal-sec. or the pound-sec. The C.G.S. unit is the dyne-sec.

If, then, a uniform force of 320 poundals acts for 3 sec., the impulse is 960 *poundal-sec.* or $\frac{960}{g}$ *pound-sec.*

Relation between Impulse and Momentum.—Let a particle of mass m move in any path through the positions P_1, P_2, P_3, etc. Let v_1, v_2, v_3, etc., be the corresponding velocities, t_1, t_2, t_3, etc., the corresponding times in passing from point to point, and F_1, F_2, F_3, etc., the corresponding forces in any given direction, as P_1a.



Let the points P_1, P_2, P_3 be consecutive, so that the times t_1, t_2, t_3 are indefinitely small. Then, however the force in the direction P_1a may vary, we can consider F_1 as constant for the time t_1. Its impulse is then F_1t_1. The impulse of F_2 is F_2t_2, and so on.

But from equation (2), page 256, we have

$$F_1t_1 = mv_2 \cos \alpha_2 - mv_1 \cos \alpha_1,$$
$$F_2t_2 = mv_3 \cos \alpha_3 - mv_2 \cos \alpha_2,$$
$$F_3t_3 = mv_4 \cos \alpha_4 - mv_3 \cos \alpha_3.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If we sum these equations and let the final velocity be v, making the angle α with the direction P_1a, we obtain

$$\Sigma(F_1t_1 + F_2t_2 + F_3t_3 + \text{etc.}) = mv \cos \alpha - mv_1 \cos \alpha_1. \quad . \quad . \quad . \quad . \quad (1)$$

Let t be the entire time, so that

$$t = t_1 + t_2 + t_3 + \text{etc.}$$

Let F_u be the *equivalent uniform force* in the direction P_1a, whose impulse F_ut for the entire time t is the same as the sum of all the actual impulses during that time, so that

$$F_ut = \Sigma(F_1t_1 + F_2t_2 + F_3t_3 + \text{etc.}).$$

We have then, from (1),

$$F_ut = \Sigma(F_1t_1 + F_2t_2 + F_3t_3 + \text{etc.}). = mv \cos \alpha - mv_1 \cos \alpha_1. \quad . \quad . \quad . \quad . \quad (2)$$

This is equation (2) of page 256.

Hence *the change of momentum in any direction for any time t gives the sum of the impulses in that direction during that time, no matter how the force in that direction may vary.*

Examples.—(1) *A ball-player catches a ball moving with a velocity of 50 ft. per sec. The mass of the ball is $5\frac{1}{2}$ oz. If the time of coming to rest is $\frac{1}{80}$ sec., find the equivalent uniform pressure.*

ANS. We have $m = \frac{11}{32}$ lb. The equivalent uniform pressure is then $F_u = \frac{mv}{t} = \frac{11 \times 50 \times 50}{32} = 859\frac{3}{8}$ poundals, or, taking $g = 32$ ft.-per-sec. per sec., 26.85 pounds.

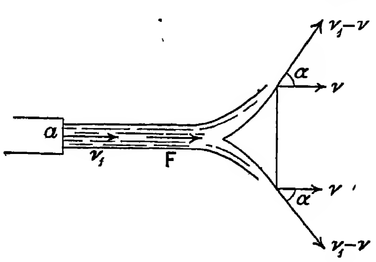
This is a uniform force which *would* stop the ball in the given time. The actual force acting at any instant we cannot tell without a full knowledge of the law of variation of the pressure with the time. If we assume the pressure at first contact to be zero and to increase directly with the time, then the final pressure would be twice as great as the equivalent uniform pressure.

(2) *An 80-ton gun fires a shot of 56 lbs. with a horizontal muzzle velocity v of 1800 ft. per sec. Find the velocity of recoil V.*

ANS. Mass of gun $M = 80 \times 2240 = 179200$ lbs. If the velocity of the shot is imparted in the time t, the sum of the impulses on the shot during that time is $mv = 56 \times 1800 = 100800$-lb. velos. Since action and reaction are equal at every instant, the sum of the impulses on the gun during the time is the same. Hence

$$MV = mv, \quad \text{or} \quad V = \frac{100800}{179200} = \frac{9}{16} \text{ ft. per sec.}$$

(3) *A horizontal stream of water whose cross-section is a and velocity v_1 meets a surface moving in the same direction with a velocity v. Find the pressure exerted on the surface.*



ANS. Let the water pass off the surface in a direction making the angle α with the direction of motion. The volume of water in any time t is av_1t. If γ is the density or mass of a unit of volume of water, the mass in the time t is $\gamma av_1t = m$.

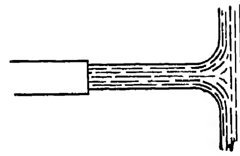
The velocity relative to the surface just before impact is $v_1 - v$. After impact the velocity relative to the surface, in the direction of motion is $(v_1 - v) \cos \alpha$.

We have then the impulse $Ft = mv_1 - m(v_1 - v) \cos \alpha$, or

$$F = \frac{m(v_1 - v) - m(v_1 - v) \cos \alpha}{t}.$$

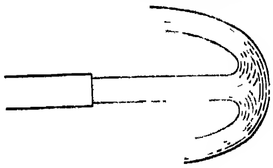
Inserting the value of $m = \gamma a v_1 t$,

$$F = \gamma a v_1 (v_1 - v)(1 - \cos \alpha).$$



If the surface is plane and at right angles to the stream, $\alpha = 90°$, $\cos \alpha = 0$ and

$$F = \gamma a v_1 (v_1 - v).$$



If the surface is curved so that the water is reversed in direction, $\alpha = 180°$, $\cos \alpha = -1$ and the pressure is twice as great, or

$$F = 2\gamma a v_1 (v_1 - v).$$

These values of F are given in poundals. For gravitation measure divide by g.

(4) *A rifle-bullet of one ounce mass is shot into a block of wood of 53 lbs. and gives the block a velocity of 2 ft. per sec. Find the velocity of the bullet.*

ANS. We have mv for the bullet equal to $(M + m)v$ for the combined mass of block and bullet, or

$$\frac{1}{16}v = \left(53 + \frac{1}{16}\right)2, \quad \text{or} \quad v = 1698 \text{ ft. per sec.}$$

(5) *A mass moving with a velocity of 3 ft. per sec. is brought to rest by a uniform opposing force of one pound in 2 sec. Assuming $g = 32$ ft.-per-sec. per sec., find the mass.*

ANS. A force of one pound is 1 lb. $\times g$, or g poundals. We have then $F = \frac{mv}{t}$, or $g = \frac{m \times 3}{2}$, or $m = \frac{2g}{3} = 21\frac{1}{3}$ lbs.

(6) *A uniform force of 10 pounds acts for 2 sec. upon a mass of 10 lbs. and then ceases. With what velocity will the mass continue to move in the direction of the force?*

ANS. A force of 10 pounds is $10g$ poundals. We have then $F = \frac{mv}{t}$, or $10g = \frac{10v}{2}$, or $v = 2g$ ft. per sec.

(7) *A particle of 10 lbs. mass has an initial velocity of 20 ft. per sec. north and is acted upon by two uniform forces, one of 3 pounds in a direction northeast and the other of the same magnitude in a direction northwest. Find its velocity after one minute.*

ANS. The resultant force is $3\sqrt{2}$ pounds, or $3\sqrt{2}.g$ poundals, north. We have then $Ft = m(v - v_1)$, or $3\sqrt{2}.g \times 60 = 10(v - 20)$. Hence $v = 18\sqrt{2}.g + 20$, or for $g = 32$, $v = 834.46$ ft. per sec. north.

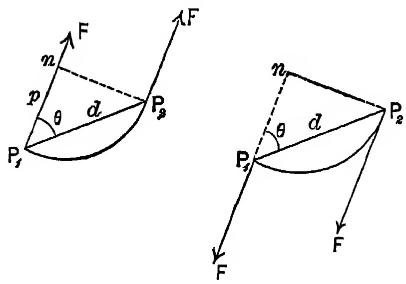
(8) *A particle of mass m is moving east with a velocity v. Find the uniform force necessary to make it move north with an equal velocity in t seconds.*

ANS. In the time t the velocity east is zero. The impulse *towards the west* is then mv. In the same time we must also have an impulse mv *towards the north*. The resultant impulse is then $Ft = mv.\sqrt{2}$ northwest, or $F = \frac{mv\sqrt{2}}{t}$ northwest.

CHAPTER III.

WORK. POWER.

Work.—Let a *uniform force* F act upon a particle which moves in any path from P_1 to P_2. Let the projection P_1n of the path *along the line of the force* be denoted by p. Then the product Fp is called WORK.



If the projection p of the path is in the direction of the force, work is done by the force and Fp is positive. If the projection p is opposite to the direction of the force, work is done against the force and Fp is negative.

Let the displacement $P_1P_2 = d$ make the angle θ with F. Then we have

$$p = d \cos \theta.$$

The work of F is then

$$W = + Fp = + Fd \cos \theta$$

if the projection $P_1n = p$ is in the direction of the force F, and

$$W = - Fp = - Fd \cos \theta$$

if the projection $P_1n = p$ is opposite to the direction of F.

We have then, in general,

$$W = \pm F \,.\, d \cos \theta = \pm d \,.\, F \cos \theta.$$

But $d \cos \theta$ is the displacement along the line of the force, and $F \cos \theta$ is the force along the displacement.

We can then define WORK generally as follows:

Work is the product of a uniform force by the component displacement along the line of the force ($W = \pm F \,.\, d \cos \theta$); *or, the product of the displacement by the component force along the line of the displacement.* ($W = \pm d \,.\, F \cos \theta$.)

COR. 1.—It is evident that if the displacement is at right angles to the force, the work is zero.

COR. 2.—We see that work is *independent of time*. A given uniform force and displacement give the same work *no matter what the time* occupied by the displacement.

COR. 3.—The work done in raising a body is equal to the weight of the body acting at the centre of mass, multiplied by the vertical displacement of the centre of mass. This work is done against the weight and is therefore negative.

Also, the work of lowering a body is equal to its weight multiplied by the vertical displacement of the centre of mass. This work is done in the direction of the weight and is positive.

If $\overline{m}$ is the mass of the body, then $\overline{m}g$ is its weight. If d is the vertical displacement, then

$$W = \pm \overline{m}gd,$$

and this is the same for the same weight and vertical displacement *whatever the time or whatever the path.*

Unit of Work.—The unit of work is then evidently the work of one unit of force with one unit of displacement.

The English unit is then the *foot-poundal,* or a uniform force of one poundal acting through one foot; or, in gravitation units, the *foot-pound,* or the weight of one lb. acting through one foot. This latter is of course variable, since the weight of one lb. varies at different localities.

If, then, we take d in feet and $\overline{m}$ in lbs.,

$$W = \pm \overline{m}gd$$

gives work in foot-poundals. For foot-pounds we divide by g and have

$$W = \pm \overline{m}d \text{ foot-pounds.}$$

The C. G. S. unit of work is the uniform force of one dyne acting through one centimeter. It is called an ERG.

A multiple of this equal to 10 000 000 ergs, or 10^7 ergs, is used in electrical measurements and called a JOULE, after Dr. James Prescott Joule.

Examples.—(1) *Find the work expended in raising 16000 lbs. through a distance of 20 feet.*

ANS. $320000g$ ft.-poundals, or, if $g = 32$, 10000 ft.-pounds, or the work of raising 10000 lbs. one foot at a locality where g is 32 ft.-per-sec. per sec., whatever the time or the path.

(2) *A body of 80 lbs. mass moves along a rough horizontal plane with a speed of 50 ft. per sec. If the retarding force of friction is constant and equal to 20 pounds, find the work done against friction in the first second and in coming to rest.*

ANS. For motion in a straight line with uniform acceleration, we have (page 92) for the distance described in any time t

$$s = v_1 t + \frac{1}{2} f_t t^2,$$

and for the distance described in coming to rest

$$s = -\frac{v_1^2}{2f_t}.$$

The force F_t in the path is

$$F_t = mf_t.$$

In the present case this force is a retarding force of 20 pounds, and therefore $F_t = -20g$ poundals. Since $m = 80$ lbs., we have

$$-20g = 80f_t, \quad \text{or} \quad f_t = -\frac{g}{4}.$$

The work, then, in any time t is

$$F_t s = -20g\left(v_1 t + \frac{1}{2} f_t t^2\right) \textit{ft.-poundals}, \quad \text{or} \quad -20\left(v_1 t - \frac{1}{2} f_t t^2\right) \textit{ft.-pounds},$$

and the work in coming to rest is

$$F_t s = -20g\frac{v_1^2}{2f_t} \textit{ft.-poundals}, \quad \text{or} \quad -20\frac{v_1^2}{2f_t} \textit{ft.-pounds}.$$

Taking $g = 32$ ft.-per-sec. per sec., $v_1 = 50$ ft. per sec. and f_t as found, we have for the work in one second

$$F_t s = -920 \textit{ ft.-pounds},$$

and for the work in coming to rest

$$F_t s = -3125 \textit{ ft.-pounds}.$$

(3) *A mass of 2 tons (2240 lbs.) is pulled up 100 feet of an incline which rises 1 foot in 25 ft. of length. Taking the resistance of friction at 150 pounds for each ton of weight, find the work done.*

ANS. Let α be the angle of inclination of the plane, and m the mass. The weight is then mg, and the component of the weight down the plane is $mg \sin \alpha$ poundals.

The friction is $300g$ poundals also down the plane. The total force down the plane is then

$$(mg \sin \alpha + 300g) \text{ poundals, or } (m \sin \alpha + 300) \text{ pounds.}$$

The work is then

$$F_t s = -(m \sin \alpha + 300)100 \textit{ ft.-pounds.}$$

In the present case $m = 4480$ lbs., $\sin \alpha = \frac{1}{25}$, and hence

$$F_t s = -47920 \textit{ ft.-pounds.}$$

(4) *Find the work done by a crane in lifting the materials for a stone wall 100 feet long, 10 feet high and 2 feet thick, the average density being 150 lbs. per cubic foot.*

ANS. 1 500 000 ft.-pounds.

Rate of Work—Power.—Work, as we have seen, is independent of time. If we raise one lb. one foot, we do the same work whether the time of raising is one second or one minute. But the rate at which the work is done is not the same.

If then we take time into consideration, *the time-rate of work* is called POWER. We exert *more power* when we raise one lb. one foot in one second than when we raise one lb. one foot in one minute, although we *perform the same work* in both cases.

The unit of power is then one unit of work per unit of time. The English absolute unit of power is then one ft.-poundal per sec., and the C. G. S. absolute unit is one erg per sec.

In gravitation units we have, in English measures one foot-pound per sec., and in French measures one meter-kilogram per sec.

These units are, however, inconveniently small in most cases.

The gravitation unit employed in English engineering calculations is therefore taken at 550 foot-pounds per sec. This is called a HORSE-POWER and is denoted by H. P. In French engineering calculations the gravitation unit employed is 75 meter-kilograms per sec. This is called FORCE DE CHEVAL.

In electrical measurements the unit adopted is 10^7 ergs per sec. This is called a WATT, after James Watt. The watt is therefore one joule per sec. (page 261).

Examples.—(1) *The area of the piston of a steam-engine is A sq. inches, the length of stroke L feet, the steam pressure in pounds per sq. inch is P, the number of strokes per minute N. Find the work per minute and the horse-power.*

ANS. Work per min. = P. L. A. N. $\quad$ H. P $= \dfrac{\text{P. L. A. N}}{33000}$.

(2) *Find the work done against gravity, neglecting friction, in pulling a car of 2.5 tons (2240 lbs.) loaded with 30 passengers weighing 154 lbs. each, up an incline the ends of which differ in level by 120 feet; also the horse-power if the time is half an hour.*

ANS. 1 226 400 ft.-pounds; 1.24 horse-power.

(3) *In the transmission of power by a belt, the wheel carrying the belt is 14 feet in diameter and makes 30 revolutions per minute, the tension of the belt being 100 pounds. Find the horse-power transmitted.*

ANS 4 horse-power.

(4) *Check this statement: Fifty-five pounds mean effective pressure at 600 ft. per sec. piston speed gives one H. P for each square foot of piston area.*

(5) *A train weighing 75 tons is running up an incline of 1 in 800 with a uniform speed of 40 miles an hour. Assuming friction to be equivalent to 6 pounds per ton, find the rate at which the engine is working.*

ANS. 70.4 H. P.

CHAPTER IV.

KINETIC FRICTION.

Kinetic Friction.—The friction which justs prevents motion, that is, the friction which exists when motion is just about to begin, we have called STATIC FRICTION, and we have fully discussed it already (page 220).

The friction which exists *after motion has taken place* is called KINETIC FRICTION.

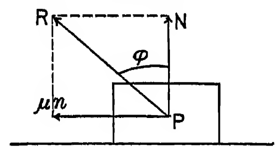
Coefficient of Kinetic Friction.—We have the same laws for kinetic as for static friction (page 222). The ratio of the total friction to the total normal pressure when motion is just about to begin we have called the coefficient of static friction. The same ratio after motion has taken place is the coefficient of kinetic friction.

We denote the coefficient of friction in general by μ. We have then, in all cases,

$$\mu = \frac{F}{N}, \quad \text{or} \quad F = \mu N,$$

where F is the total friction and N the total normal pressure.

Angle of Kinetic Friction.—If N is the normal pressure for a body moving on a rough surface, then μN is the friction. The resultant reaction of the surface is then R, making an angle ϕ with the normal given by



$$\tan \phi = \frac{\mu N}{N} = \mu.$$

We call the angle ϕ which the reaction R makes with the normal the ANGLE OF FRICTION. For static friction it is the ANGLE OF REPOSE (page 222). *Hence the coefficient μ of kinetic friction is equal to the tangent of the angle of friction ϕ.*

Kinetic Friction of Pivots, Axles, Ropes, etc.—The application of the equation

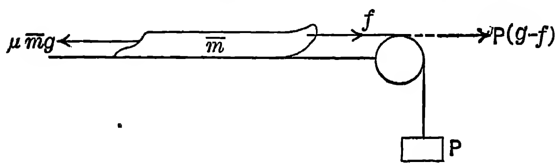
$$F = \mu N = N \tan \phi$$

to pivots, axles, ropes, etc., is then precisely the same as for static friction (page 224). We have only to let μ stand for the coefficient of kinetic instead of static friction.

With this change we have in each case the same value for the friction and moment of the friction as already given for static friction.

Experimental Determination of Coefficient of Kinetic Sliding Friction.—We may determine the coefficient of kinetic sliding friction by means of various contrivances, some of which we shall now describe.

1. BY SLED AND WEIGHT.—Let a sled rest upon a horizontal plane and be dragged along by means of a string passing over a pulley, to the end of which a weight is hung. In order to obtain coefficients for different substances, the runners and plane can be of the materials desired.



In such an apparatus the mass of string and pulley, and friction of string and pulley, as well as rigidity of the string, should all be insignificant, or else they must be allowed for.

Let us suppose them insignificant and let $\overline{m}$ be the mass of the sled, and P the suspended mass. The normal pressure is then $\overline{m}g$, the friction $\mu\overline{m}g$, and the weight of the suspended mass is Pg. Let the acceleration of the masses P and $\overline{m}$ be f. The impressed forces acting on $\overline{m}$ are then the friction $\mu\overline{m}g$, and the horizontal tension of the string $P(g-f)$. The effective force on $\overline{m}$ is $\overline{m}f$. By D'Alembert's principle (page 242), if we *reverse* the effective force we have

$$P(g-f) - \mu\overline{m}g - \overline{m}f = 0,$$

or

$$\mu = \frac{P}{\overline{m}} - \frac{(P+\overline{m})f}{\overline{m}g}. \qquad (1)$$

But for uniformly accelerated motion we have the distance described in any time t falling from rest (page 92)

$$s = \frac{1}{2}ft^2, \quad \text{or} \quad f = \frac{2s}{t^2}.$$

Substituting this in (1) we have

$$\mu = \frac{P}{\overline{m}} - \frac{2(P+\overline{m})s}{\overline{m}gt^2}. \qquad (2)$$

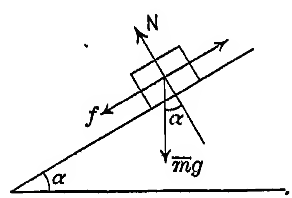
If then we observe the fall s of P starting from rest in any time t, we have μ, from equation (2), for kinetic friction.

If motion is *just about to begin*, $f = 0$, and we have, from (1), for *static* friction

$$\mu = \frac{P}{\overline{m}}.$$

We see, then, *that the coefficient for kinetic friction is less than the coefficient for static friction.*

2. BY SLED ON INCLINED PLANE.—If we place the sled on an inclined plane and the sled slides down with an acceleration f, we have the impressed forces $\overline{m}g \sin \alpha$ down the plane, the friction $\mu N = \mu\overline{m}g \cos \alpha$ up the plane, and the effective force $\overline{m}f$ down the plane. If we *reverse* the effective force, we have, by D'Alembert's principle (page 242),



$$\mu\overline{m}g \cos \alpha - \overline{m}g \sin \alpha + \overline{m}f = 0,$$

or

$$\mu = \tan \alpha - \frac{f}{g \cos \alpha}. \qquad (1)$$

The distance s described in the time t is, as before,

$$s = \frac{1}{2} ft^2, \quad \text{or} \quad f = \frac{2s}{t^2}.$$

Substituting in (1), we have

$$\mu = \tan \alpha - \frac{2s}{gt^2 \cos \alpha} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If then we observe the distance s described by the sled starting from rest in any time t, we have μ, from equation (2), for kinetic friction.

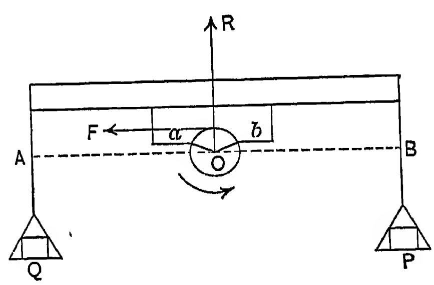
If motion is *just about to begin*, $f = 0$ and we have, from (1), for static friction

$$\mu = \tan \alpha = \frac{h}{b} = \tan \phi,$$

where h is the height and b the base of the plane, and ϕ the angle of repose (page 222).

Again, we see that the coefficient for kinetic friction is less than for static friction.

3. BY FRICTION BRAKE.—The friction brake consists of a lever AB of equal arms $AO = BO = l$, with a bearing-block at the centre. It is placed on an axle horizontally so that the entire weight of the lever with attached scale-pans, bearing-block, etc., acts at the centre O of the axle.



If the axle turns as indicated in the figure, the lever tends to turn in the same direction. If we put a mass Q in one pan and a mass P in the other and make *P just large enough to keep the lever horizontal while the axle is turning*, we have the weights Pg and Qg and the friction F in equilibrium.

If m is the mass of the apparatus, the pressure on the axle is $(P + Q + \mathrm{m})g = R$.

For a new bearing (page 229) the friction is

$$F = \mu R \frac{\beta}{\sin \beta} = \mu g(P + Q + \mathrm{m}) \frac{\beta}{\sin \beta},$$

where β is the bearing angle aOb.

Let r be the radius of the axle. Then we have for equilibrium

$$Qgl - Pgl + Fr = 0,$$

or

$$Qgl - Pgl + \mu g(P + Q + \mathrm{m}) \frac{\beta r}{\sin \beta} = 0.$$

Hence we have for the coefficient of kinetic friction

$$\mu = \frac{(P - Q)l \sin \beta}{(P + Q + \mathrm{m})r\beta} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Friction-brake Test.—The friction brake can be used for measuring the work done by an engine when working uniformly. Thus suppose the axle is driven by an engine, and that by means of a crank on the axle some machine, as, for instance, a pump, is worked.

We first count the number of revolutions n per minute while the pump is in action. If then we disconnect the pump, we shall find that the axle revolves much more rapidly than before, since the only work now done by the engine is against the friction of the axle-bearing. We now apply the brake and load it at the ends with P and Q until it rests horizontally, and the axle is slowed up to its former speed of n revolutions per minute. In this condition the work now done against the brake friction is equal to the work before consumed by the pump, provided the engine works uniformly.

But the friction F is given by

$$F = \frac{(P - Q)gl}{r},$$

or, in gravitation measure, by

$$F = \frac{(P - Q)l}{r}.$$

We have then for the work done in one revolution $2\pi rF$, and in n revolutions per minute the work per minute is $2\pi rnF$. Taking F in gravitation measure or in pounds, and r in feet, this is foot-pounds per minute. If we divide by 33000, we obtain (page 262) horse-power. Hence

$$\text{H.P.} = \frac{\pi rnF}{16500} = \frac{\pi n(P - Q)l}{16500},$$

where F, P and Q are in pounds, l and r in feet, and n is the number of revolutions per minute made while the pump was connected.

Work of Axle-friction.—The friction upon an axle in any case when μ is known is given on page 228. Thus for a *new bearing* we have (page 229)

$$F = \mu R \frac{\beta}{\sin \beta},$$

where R is the resultant pressure on the axle, and β is the bearing angle. If we substitute this in the place of F in the preceding article, we have the work per minute

$$2\pi rnF = 2\pi \mu Rrn \frac{\beta}{\sin \beta},$$

and for the horse-power

$$\text{H.P.} = \frac{\pi \mu Rrn\beta}{16500 \sin \beta},$$

where R is taken in pounds, r in feet, n in revolutions per minute. If the bearing angle is small, we have $\beta = \sin \beta$ nearly.

Coefficients of Kinetic Sliding Friction.—The following tables give a few values of the value of μ as determined by experiment for kinetic sliding friction and axle-friction.

COEFFICIENTS OF KINETIC SLIDING FRICTION, $\mu = \tan \phi$.

Substances in Contact.		Condition of Surfaces and Kind of Unguent.						
		Dry.	Wet.	Olive Oil.	Lard.	Tallow.	Dry Soap.	Polished and Greasy.
Wood on wood	Minimum	0.20	....	....	0.06	0.06	0 14	0 08
	Mean	0.36	0.25	....	0.07	0.07	0.15	0.12
	Maximum	0.48	....	....	0.07	0.08	0.16	0.15
Metal on metal	Minimum	0.18	....	0.06	0.07	0.07	....	0.11
	Mean	0.24	0.31	0.07	0.09	0.09	0.20	0 13
	Maximum	0.20	....	0.08	0.11	0.11	....	0.17
Wood on metal	Minimum	0.20	....	0.05	0.07	0.06	....	0 10
	Mean	0 42	0.24	0.06	0.07	0.08	0.26	0 14
	Maximum	0.62	....	0.08	0.08	0.10	....	0.16
Hemp ropes or plaits	On wood	0.45	0.33	....	....	... .		
	On iron	....	....	0.15	....	0.19		
Leather belts on wood or metal	Raw	0.54	0.36	0.16	....	0.20		
	Pounded	0 30	....	....	....	....		
	Greasy	....	0.25	....	....	....		
Same on edge for piston-packing	Dry	0.34	0.31	0.14	....	0.14		
	Greasy	....	0.24	....	....	....		

COEFFICIENTS OF AXLE-FRICTION.

	Dry or Slightly Greasy.	Oil, Tallow, or Lard.		Damp and Greasy.
		Ordinary Lubrication.	Thorough Lubrication.	
Bell-metal on bell-metal	... .	0.097	....	....
" " " cast iron	....	....	0.049	....
Wrought iron on bell-metal	0.251	0 075	0.054	0.189
" " " cast iron	....	0.075	0 054	....
Cast iron on cast iron	....	0.075	0.054	0.137
" " " bell-metal	0.194	0.075	0.054	0.161

Comparing the values in the tables just given with those in the table given on page 224, we see that the coefficient of kinetic *is always less than the coefficient of static sliding friction.*

We see also that for axle-friction in general we have for the coefficient of kinetic friction:

$$\textit{for ordinary lubrication } \mu = 0.070 \text{ to } 0.080;$$

$$\textit{for thorough lubrication } \mu = 0.054.$$

Efficiency—Mechanical Advantage.—In a machine we have in general a "moving force" F acting at some point called the "point of application," and at another point, called the "working point," we have a "useful resistance" F' overcome through a certain distance. If there is no friction, the rate of work of the moving force must always equal the rate of work of the resistance. Owing to friction it must always be greater.

The ratio of the rate of work of the useful resistance to the rate of work of the moving force is called the EFFICIENCY of the machine.

It must always be a fraction less than unity, and it approaches unity the more perfect

the machine and the less the friction. If we denote it by ϵ, and let v be the velocity of the moving force F, and v' the velocity of the resistance F', we have

$$\epsilon = \frac{F'v'}{Fv}, \quad \text{or} \quad F' = \epsilon\frac{v}{v'}F.$$

If there is no friction, $\epsilon = 1$ and $Fv' = F'v$. The ratio $\epsilon\frac{v}{v'}$ is called the MECHANICAL ADVANTAGE of the machine.

If F' is greater than F, v' must be less than v in nearly the same proportion, or, if friction is disregarded, in exactly the same proportion, that is, $\frac{F'}{F} = \frac{v}{v'}$.

Hence the familiar maxim that "What is gained in force is lost in speed."

Examples.—(1) *A body of 80 pounds mass is projected along a rough horizontal plane with a speed of 50 ft. per sec. It slides 155.28 ft. in coming to rest. Find the coefficient of kinetic sliding friction, the retarding force of friction, and the work done against friction in coming to rest.*

ANS. $\mu = \frac{v^2}{2gs}$, or, if $g = 32.2$ ft.-per-sec. per sec., $\mu = 0.25$. Retarding force of friction is 20 lbs.; work done, 3105.6 ft.-lbs.

(2) *A body of 80 pounds mass is dragged along a rough horizontal plane by means of a mass of 186 pounds attached to a string passing over a pulley* (page 264). *It is observed to slide 10 feet in the first second starting from rest. Disregarding rigidity of string and mass and friction of string and pulley, find the coefficient of kinetic sliding friction.* ($g = 32$.)

ANS. $\mu = 0.25$.

(3) *A body placed upon a rough inclined plane whose height is 1 foot and base 16 inches is observed to slide 6.4 inches in the first second, starting from rest. Find the coefficient of friction.* ($g = 32$.)

ANS. $\mu = \frac{25}{36}$ kinetic; $\mu = 0.75$ static.

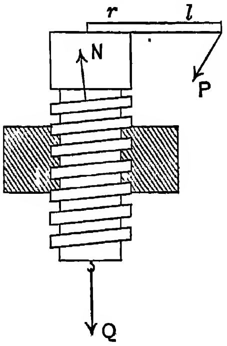
(4) *A friction brake of* $\mathfrak{m} = 15$ *lbs. mass, and length of 4 feet, is balanced on a rotating shaft of radius* $r = 6$ *inches, by masses of* $Q = 10$ *lbs. and* $P = 10$ *lbs. 10 oz. Find the coefficient of kinetic friction and the friction. Also, if the shaft makes 60 revolutions per minute, find the rate of work of the friction.*

ANS. $\mu = 0.07$, $F = 2.5$ lbs. Rate of work of friction $= 7.854$ ft.-lbs. per sec., or 0.01428 horse-power.

(5) *A screw of radius* $r = 1$ *inch is acted upon by a force of* $P = \frac{1}{2}$ *lb. with a constant lever-arm of* $l = 1$ *ft. and overcomes a resistance of* $Q = 5$ *lbs. If the angle of the thread is* $\alpha = 45°$, *find the coefficient of kinetic sliding friction if the number of revolutions per minute is 60. Also find the efficiency, and the acceleration of* P. *Disregard the mass of the screw, and take* $g = 32\frac{1}{6}$ *ft.-per-sec. per sec.*

ANS. Let P be the force applied at the end of the arm l, and let the radius of the screw be r, the pitch p, and the resistance Q.

If N is the sum of the normal pressures and α the inclination of the thread to the horizontal, we have



$N = \frac{Q}{\cos\alpha}$, and the friction $F = \mu N = \frac{\mu Q}{\cos\alpha}$, where μ is the coefficient of friction.

Let f be the acceleration of P. Then the moving force is $P(g - f)$ poundals. If s is the distance passed through by P in any time t, then the work of the moving force is

$$P(g - f)s \text{ ft.-poundals.}$$

The resistance Q is overcome through the distance $\frac{p'}{2\pi l}s$. The work of overcoming the resistance is then

$$-\frac{Qg}{l}\cdot\frac{p}{2\pi}s \text{ ft.-poundals.}$$

The friction is overcome through the distance $\frac{r}{l}\cdot\frac{s}{\cos\alpha}$. The work of overcoming the friction is then

$$-\frac{\mu Qg}{\cos\alpha}\cdot\frac{rs}{l\cos\alpha}.$$

The minus sign is used because work is done against friction and the resistance.

The work of $P(g-f)$ must be equal and opposite to the work done against friction and the resistance. Hence the algebraic sum must be zero, or

$$P(g-f)s - \frac{Qgps}{2\pi l} - \frac{\mu Qgrs}{l\cos^2\alpha} = 0.$$

From this we have, since $\frac{p}{2\pi r} = \tan\alpha$ and $f = \frac{2s}{t^2}$,

$$f = g\left[1 - \frac{Qr}{Pl\cos^2\alpha}(\sin\alpha\cos\alpha + \mu)\right] = \frac{2s}{t^2}, \quad . \quad . \quad . \quad . \quad (1)$$

and from (1), for the coefficient of kinetic friction,

$$\mu = \frac{Pl}{Qr}\cos^2\alpha - \sin\alpha\cos\alpha\left(1 + \frac{2sPl}{gt^2Qr\tan\alpha}\right). \quad . \quad . \quad . \quad . \quad (2)$$

For the efficiency we have

$$\epsilon = \frac{\frac{Qgps}{2\pi l}}{\frac{Qgps}{2\pi l} + \frac{\mu Qgrs}{l\cos^2\alpha}} = \frac{1}{1 + \frac{\mu}{\sin\alpha\cos\alpha}}. \quad . \quad . \quad . \quad . \quad (3)$$

If $f = 0$, *we have equilibrium*, and from (1) we have in this case

$$P = \frac{Qr}{l}\left(\tan\alpha + \frac{\mu}{\cos^2\alpha}\right),$$

or the same as already found, ex. (2), page 237.

In this case (2) becomes the coefficient of static friction,

$$\mu = \frac{Pl}{Qr}\cos^2\alpha - \sin\alpha\cos\alpha.$$

We see from (3) that the efficiency is a maximum when $\sin\alpha\cos\alpha$ is a maximum, or when $\sin\alpha = \cos\alpha$ or $\alpha = 45°$.

If n is the number of revolutions per minute, the distance s described in one minute is $2\pi ln$. We have then

$$\frac{2s}{gt^2} = \frac{4\pi ln}{60 \times 60 \times g} = \frac{\pi ln}{900g}. \quad . \quad . \quad . \quad . \quad (4)$$

Inserting in these equations the values $l = 1$ ft., $r = \frac{1}{12}$ ft., $P = \frac{1}{2}$ lb., $Q = 5$ lbs., $\alpha = 45°$, $n = 60$, $g = 32\frac{1}{6}$ ft.-per-sec. per sec., we have

$$\mu = 0.096, \quad \epsilon = 0.84, \quad f = 0.007, \quad g = 0.225 \text{ ft.-per-sec. per sec.}$$

(6) *A train runs on a horizontal track with the speed v_1, and by the application of brakes to the driving-wheels of the locomotive the speed is reduced to the speed v. Find the distance and time of running during the reduction of speed, disregarding all resistances other than those due to the action of the brakes.*

ANS. Let m be the mass of the train in pounds, v_1 the initial and v the final speed in feet per second, s the distance in feet, and t the corresponding time in seconds.

Let n be the number of driving-wheels braked, and R the pressure of each braked wheel on the rails in pounds.

Let μ_k be the coefficient of kinetic sliding friction, and μ_s the coefficient of static sliding friction.

1st. *Let the brakes be set so that the wheels do not turn.* In this case the retarding force due to friction is $n\mu_k R$ pounds, or $n\mu_k Rg$ poundals. We have then for the retardation f

$$mf = n\mu_k Rg, \quad \text{or} \quad f = \frac{n\mu_k Rg}{m}.$$

From page 92, for uniformly retarded motion the distance described is

$$s = \frac{v_1^2 - v^2}{2f} = \frac{m(v_1^2 - v^2)}{2n\mu_k Rg}. \quad . \quad . \quad . \quad . \quad (1)$$

2d. *Let the brakes be set so that the wheels are just on the point of slipping.* In this case the retarding force due to friction is $n\mu_s Rg$ poundals. We have then

$$mf = n\mu_s Rg, \quad \text{or} \quad f = \frac{n\mu_s Rg}{m}.$$

Hence the distance described is

$$s = \frac{m(v_1^2 - v^2)}{2n\mu^s Rg}. \qquad (2)$$

If the train is brought to rest, we have $v = 0$ in these equations.

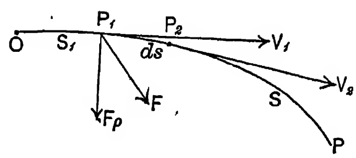
Now the coefficient μ_s for static sliding friction is *always less* than the coefficient μ_k for kinetic sliding friction (page 267).

We see, then, that the train *will be stopped in the least distance when the brakes are applied so that the wheels are just on the point of slipping but do not slip.*

CHAPTER V.

KINETIC AND POTENTIAL ENERGY. LAW OF ENERGY. CONSERVATION OF ENERGY. EQUILIBRIUM OF A PARTICLE.

Kinetic Energy.—Let a particle of mass m move in any path and have the initial velocity v_1 at P_1.



In the indefinitely small time τ let the particle move to P_2, so that P_1 and P_2 are consecutive points, and let the velocity at P_2 be v_2.

Then the tangential acceleration is

$$f_t = \frac{v_2 - v_1}{\tau},$$

and the tangential force is

$$F_t = mf_t = \frac{m(v_2 - v_1)}{\tau}.$$

The mean speed is $\frac{v_2 + v_1}{2}$, and hence the distance described is

$$ds = \frac{v_2 + v_1}{2} \cdot \tau.$$

Let F be the force on the particle. This force can be resolved into the tangential component F_t already found, and the force F_ρ in the direction of the radius of curvature. The work of the force F during the passage from P_1 to P_2 is equal to the sum of the works of the components (page 215). But since the path ds is at right angles to F_ρ, the work of F_ρ is zero, and hence the work of F is the same as the work of F_t.

We have then for the work done in passing from P_1 to P_2 when P_1 and P_2 are consecutive

$$W_1 = mf_t ds = \frac{m(v_2^2 - v_1^2)}{2}.$$

In the same way, in passing from P_2 to P_3 when P_2 and P_3 are consecutive

$$W_2 = \frac{m(v_3^2 - v_2^2)}{2}.$$

For the next two consecutive points

$$W_3 = \frac{m(v_4^2 - v_3^2)}{2},$$

and so on.

If we add together all these works and denote the final velocity at P by v we have for the entire work W, if s_1 and s are the distances of P_1 and P measured along the path from any point O of the path,

$$\Sigma_{s_1}^{s} mf_t ds = W = \frac{m}{2}(v^2 - v_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the final velocity v is greater than the initial velocity v_1, work is done on the particle in giving it increased velocity and is positive. If v_1 is greater than v, work is done against the particle and is negative. In the one case we have a tangential force F_t acting in the direction of motion. In the other case F_t is opposite to the direction of motion.

We see that equation (1) is independent of the time and path.

Hence, *whatever the time or path, and however the tangential force may vary, the work done in giving a particle of mass m an increase of velocity $(v - v_1)$ is equal to one half the product of the mass m and the difference of the squares of the final and initial velocities.*

If the initial velocity is zero, then $v_1 = 0$ and

$$W = \frac{1}{2} mv^2$$

is the work (positive) done in giving a particle of mass m the velocity v starting from rest, no matter what the time or path, or however the accelerating force may vary.

Conversely,

$$W = -\frac{1}{2} mv^2$$

is also the work (negative) done on the particle, or the work which a particle of mass m moving with a velocity v can do while being brought to rest by opposing force, no matter what the time or path or however the retarding force varies.

The work which a particle or body is capable of doing is called its ENERGY. Since $\frac{1}{2} mv^2$ is, then, the work which a particle of mass m and velocity v is capable of doing by reason of its velocity, we call it the KINETIC ENERGY of the particle.

We denote kinetic energy by the letter $\mathfrak{K}$. If then the initial velocity is v_1, the initial kinetic energy is

$$\mathfrak{K}_1 = \frac{1}{2} mv_1^2.$$

If the final velocity is v, the final kinetic energy is

$$\mathfrak{K} = \frac{1}{2} mv^2.$$

The work given by equation (1) can then be written

$$\Sigma_{s_1}^{s} mf_t ds = \mathfrak{K} - \mathfrak{K}_1 = \frac{1}{2} m(v^2 - v_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, *the work (positive) done on the particle is equal to the gain of kinetic energy, and inversely the work (negative) done by the particle is equal to the loss of kinetic energy.*

Kinetic Energy of a Rotating Body.*—Let a body have the angular velocity ω about any axis. Then the velocity of any particle at a distance r from the axis is $v = r\omega$. The kinetic energy of the particle is then $\frac{1}{2}mr^2\omega^2$. Since all the particles have the same angular velocity about the axis, we have the kinetic energy of the entire body,

$$\mathfrak{K} = \frac{1}{2}\omega^2\Sigma mr^2.$$

But Σmr^2 is the moment of inertia I' of the body relative to the axis (page 31). We have then for the kinetic energy of a rotating body

$$\mathfrak{K} = \frac{1}{2}I'\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the motion of a body at any instant consists in general (page 145) of translation and rotation about an axis through the centre of mass, we have for the kinetic energy of a rigid body in general

$$\mathfrak{K} = \frac{1}{2}\overline{m}\overline{v}^2 + \frac{1}{2}I\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In all equations $\mathfrak{K}$ is given in ft.-poundals if we take m in lbs. and v in ft. per sec. For foot-pounds divide by g.

Examples.—(1) *A fly-wheel has a mass of 30 tons and radius of 8 ft. What work can it do in coming to rest, considering it as a disc, if it is making 20 revolutions per minute?*

ANS. For a disc we have for a principal axis through the centre of mass (page 45) $I = \frac{m}{2} . r^2$. The angular velocity is $\omega = \frac{20 \times 2\pi}{60}$. The kinetic energy is then

$$\mathfrak{K} = \frac{mr^2}{4} . \frac{4\pi^2}{9} = \frac{30 \times 2240 \times 64}{4} . \frac{4 \times 9.87}{9} = 4716544 \text{ ft.-poundals,}$$

or, taking $g = 32$ ft.-per-sec. per sec.,

$$\mathfrak{K} = 147392 \text{ ft.-pounds.}$$

(2) *In the preceding example suppose the fly-wheel starts from rest and acquires its speed in one minute under the action of a constant force applied at the extremity of a crank 18 inches long. Find this force.*

ANS. The mean angular speed is 10 revolutions per minute. The distance s passed through by the constant force is then, since the length l of the crank is $1\frac{1}{2}$ ft.,

$$s = 10 \times 2\pi l = 30\pi \text{ ft.}$$

The kinetic energy is as before 147392 ft.-pounds. Hence the force is

$$\frac{147392}{10 \times 2\pi l} = \frac{147392}{30\pi} = 156 \text{ pounds.}$$

(3) *A ball-player catches a ball moving with a velocity of 50 ft. per sec. The mass of the ball is $5\frac{1}{2}$ oz. If the space in which the ball is brought to rest is 6 inches, find the average opposing pressure and the time of stoppage.*

ANS. We have $F_t s = \frac{1}{2}mv^2$, or $F_t = \frac{mv^2}{2s} = \frac{11 \times 50 \times 50 \times 2}{32 \times 2} = 859\frac{3}{8}$ poundals. Taking $g = 32$ ft.-per-sec. per sec., we have $F_t = 26.85$ pounds. The mean velocity is $\frac{v}{2} = 25$ ft. per sec. and $s = \frac{v}{2}t$, or $t = \frac{2s}{v} = \frac{1}{50}$ sec. (See example (1), page 258.)

(4) *If an ounce bullet with a velocity of 800 ft. per sec. strikes a man in "bullet-proof" armor and the bullet is brought to rest in a space of six inches, find the average pressure and time of stoppage.* ($g = 32$.)

ANS. Pressure = 1250 pounds; time = $\frac{1}{800}$ sec.

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* This article properly belongs to the discussion of the kinetics of a rigid body and not of a particle. It is, however, such a direct result of the preceding that we give it here.

(5) *Find the tension of a rope which pulls a car of 8 tons up a smooth incline of 1 in 5 and causes a uniform acceleration of 3 ft.-per-sec. per sec. If the rope breaks when the speed is 48 ft. per sec., how far will the car continue to move up the incline?* $(g = 32.)$

ANS. 5264 pounds; 180 feet.

(6) *A bullet of $2\frac{1}{2}$ oz. leaves a gun with a velocity of 1550 ft. per sec. The length of barrel is $2\frac{1}{2}$ ft. Find the average force of the powder.* $(g = 32.)$

ANS. 2346 pounds.

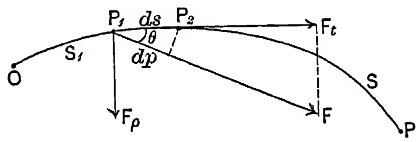
(7) *A train of 200 tons, starting from rest, acquires a speed of 40 miles an hour in three minutes. Find the effective moving force, assuming it uniform.*

ANS. 2.03 tons.

(8) *An engine exerts on a car weighing 20000 lbs. a net pull of 2 lbs. per ton of 2000 lbs. Find the energy of the car after going $2\frac{1}{2}$ miles. If shunted onto a level side track where friction is 10 pounds per ton, how far will it run? If this side track has a one per cent grade, how far will it run?*

ANS. 264000 ft.-lbs.; one-half mile; one sixth mile.

Potential Energy.—Let a particle be constrained to move in any path from P_1 to P. Let F be the resultant of all those forces acting upon the particle *which depend solely upon the position of the particle.*



For an indefinitely small displacement ds in the path F can be considered uniform, and its work is $F \cdot dp$, where dp is the component displacement along F.

We can resolve F into a tangential component F_t and a normal component F_ρ. In the displacement ds from P_1 to P_2, the work of F_ρ is zero. Since the work of the resultant is equal to the algebraic sum of the works of the components (page 215), we have

$$F \cdot dp = F_t \cdot ds.$$

This can also be proved as follows: If θ is the angle of F with the tangent, we have $dp = ds \cos\theta$ and $F \cdot dp = F \cdot ds \cos\theta$. But $F\cos\theta = F_t$. Hence

$$F \cdot dp = F_t \cdot ds.$$

The force F and hence F_t depends *solely* upon position. It is in general variable and changes in magnitude and direction as the particle changes its position. For any two consecutive points we have, however, F uniform.

If, then, s_1 is the distance of P_1, and s of P, measured along the path from any point O, we have for the entire work on the particle in passing from P_1 to P, due to change of position only,

$$\Sigma_{s_1}^{s} F \cdot dp = \Sigma_{s_1}^{s} F_t \cdot ds. \qquad (5)$$

The work which a particle is thus capable of doing by reason of *change of position only* is called its POTENTIAL ENERGY.

The potential energy of a particle or body, then, is the work it can do by reason of change of position or configuration under the action of forces which depend solely upon such change.

Thus a mass suspended at a distance above the earth can do work if released, by reason of the force of gravity, which is a force depending solely upon position. This work is potential energy. A bent spring can do work when released, by reason of the elastic forces between its particles, which depend solely upon configuration. This work is potential energy.

We denote potential energy by the letter $\mathfrak{P}$. The initial potential energy of the particle at P_1 in the figure, page 274, is then

$$\mathfrak{P}_1 = \Sigma_0^{s_1} F \,.\, dp = \Sigma_0^{s_1} F_t \,.\, ds.$$

The final potential energy at P is

$$\mathfrak{P} = \Sigma_0^{s} F \,.\, dp = \Sigma_0^{s} F_t \,.\, ds.$$

The work done on the particle due to change of position only is then

$$\mathfrak{P}_1 - \mathfrak{P} = \Sigma_{s_1}^{s} F \,.\, dp = \Sigma_{s_1}^{s} F_t \,.\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is, *the work (positive) done on the particle by reason of change of position only is equal to the loss of potential energy, or, inversely, the work (negative) done by the particle by reason of change of position only is equal to the gain of potential energy.*

Total Energy.—The total energy of the particle at any instant is the sum of its potential and kinetic energy at that instant. Let us denote the total energy by $\mathfrak{E}$. Then the initial energy is

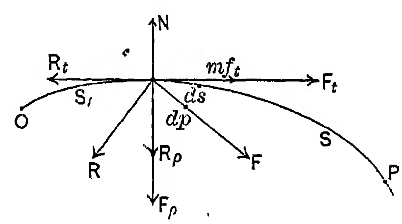
$$\mathfrak{E}_1 = \mathfrak{P}_1 + \frac{1}{2} m v_1^2 = \mathfrak{P}_1 + \mathfrak{K}_1,$$

and the final energy is

$$\mathfrak{E} = \mathfrak{P} + \frac{1}{2} m v^2 = \mathfrak{P} + \mathfrak{K}.$$

Law of Energy.—When a particle or body moves it generally encounters resistance due to friction, resistance of the air, etc. Such forces do not depend upon position solely, but upon velocity or upon the character of the surfaces in contact, upon pressure between these surfaces, etc. They always resist change of motion, and we call them therefore resistances.

Let, then, m be the mass of a particle constrained to move in any path from P_1 to P, and let F be the resultant of all those forces acting on the particle which depend solely upon its position, N the normal pressure of the path upon the particle and R the resultant of all the resistances. We can resolve F into F_t and F_p tangent and normal to the path. So, also, R can be resolved into R_t and R_p, and R_t is always opposite to the direction of motion. These are impressed forces. If the particle has the acceleration f, this can be resolved into f_t and f_p, and the components of the effective force mf are then mf_t and mf_p.



Now by D'Alembert's principle (page 242) the impressed and reversed effective forces can be treated as a system of forces in equilibrium, Hence

$$F_t - R_t - mf_t = 0.$$

Multiplying by ds, we have

$$F_t ds = mf_t \,.\, ds + R_t \,.\, ds.$$

If, then, s_1 is the distance of P_1, and s of P, measured along the path from any point O, we have for the entire work done by F from P_1 to P, whatever the path, and however the forces may vary in magnitude or direction,

$$\Sigma_{s_1}^{s} F \, . \, dp = \Sigma_{s_1}^{s} F_t \, . \, ds = \Sigma_{s_1}^{s} mf_t \, . \, ds + \Sigma_{s_1}^{s} R_t \, . \, ds. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But we have seen from equation (6) that

$$\Sigma_{s_1}^{s} F \, . \, dp = \Sigma_{s_1}^{s} F_t \, . \, ds = \mathfrak{P}_1 - \mathfrak{P},$$

the loss of potential energy.

We have also seen from equation (2), page 272, that

$$\Sigma_{s_1}^{s} mf_t \, . \, ds = \frac{1}{2} m(v^2 - v_1^2) = \mathfrak{K} - \mathfrak{K}_1,$$

or the gain of kinetic energy. We have then

$$\mathfrak{P}_1 - \mathfrak{P} = \mathfrak{K} - \mathfrak{K}_1 + \Sigma_{s_1}^{s} R_t \, . \, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

That is, *the loss of potential energy is equal to the gain of kinetic energy plus the work of overcoming all resistances.*

Since, as we have seen, we can write

$$\mathfrak{E}_1 = \mathfrak{P}_1 + \mathfrak{K}_1, \qquad \mathfrak{E} = \mathfrak{P} + \mathfrak{K}.$$

Equation (8) can be written

$$\mathfrak{E}_1 - \mathfrak{E} = \Sigma_{s_1}^{s} R_t \, . \, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

That is, *the loss of energy is equal to the work done in overcoming resistances.*

This is known as the law of energy.

Conservation of Energy.—If there are no resistances, equation (9) becomes

$$\mathfrak{E}_1 - \mathfrak{E} = 0, \quad \text{or} \quad \mathfrak{P}_1 - \mathfrak{P} = \mathfrak{K} - \mathfrak{K}_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

That is, if there are no resistances, but only forces which depend solely upon position, *there is no loss of energy*, or the loss of potential is equal to the gain of kinetic energy, and conversely, the gain of potential is equal to the loss of kinetic energy.

This is called the law of conservation of energy, and hence forces which depend solely upon position are called *conservative forces*, because for them the law of conservation holds. The force of gravity upon a particle depends solely upon the position of the particle and is therefore a conservative force. So is the elastic force of a spring which depends upon configuration only. So, also, are the forces of nature generally. For such we have the law of conservation of energy as given by equation (10).

Forces which do not depend upon position are called *non-conservative* forces. The resistance due to friction, and in general all resistances to motion, do not depend upon position and are therefore non-conservative. The resultant of such forces always acts opposite to the direction of motion. For such we have the law of energy as given by equation (8).

Equilibrium of a Particle.—If a particle is at rest under the action of forces, it is said to be in *static* equilibrium. If it moves with uniform speed in a straight line, that is, with uniform velocity, it is said to be in *molecular* equilibrium.

Let F_x, F_y, F_z be the components in any three rectangular directions of all the conservative forces acting upon a particle; let R_t be the resultant non-conservative force, and R_x, R_y, R_z its components.

Then for any possible indefinitely small displacement ds, real or virtual, we have the work of the conservative forces, or the loss of potential energy

$$\mathfrak{P}_1 - \mathfrak{P} = F_x dx + F_y dy + F_z dz,$$

and by the law of energy, equation (8),

$$\mathfrak{P}_1 - \mathfrak{P} = F_x dx + F_y dy + F_z dz = \frac{1}{2}m(v^2 - v_1^2) + R_t ds.$$

But $R_t ds = R_x dx + R_y dy + R_z dz$. Hence

$$\mathfrak{P}_1 - \mathfrak{P} - R_t ds = (F_x - R_x)dx + (F_y - R_y)dy + (F_z - R_z)dz = \frac{1}{2}m(v^2 - v_1^2).$$

For static equilibrium $v = 0$, $v_1 = 0$, and for molecular equilibrium $v = v_1$. Hence in all cases of equilibrium, static or molecular,

$$(F_x - R_x)dx + (F_y - R_y)dy + (F_z - R_z)dz = 0.$$

Since dx, dy, dz are not zero, we must have

$$F_x - R_y = 0, \qquad F_y - R_y = 0, \qquad F_z - R_z = 0.$$

That is, in all cases of equilibrium, static or molecular, the forces, conservative and non-conservative, acting upon the particle must constitute a system of forces in equilibrium.

Since in all cases of equilibrium, static or molecular, $\frac{1}{2}m(v^2 - v_1^2)$, or the change of kinetic energy during any indefinitely small displacement is zero, the kinetic energy is zero for static equilibrium, and either a maximum or a minimum when the particle is in molecular equilibrium.

Stable and Unstable Equilibrium of a Particle.—The equilibrium of a particle is said to be *stable* if, when at rest or when supposed to be at rest, it would return after any possible indefinitely small displacement to its original position. Now in thus returning work must be done by the conservative forces, and hence potential energy lost. In stable equilibrium, then, the potential energy *is a minimum*.

If the particle would not return to its original position, the equilibrium is said to be *unstable*. In such case work must be done against the conservative forces in order to bring it back, and potential energy would be gained. In unstable equilibrium, then, the potential energy *is a maximum*.

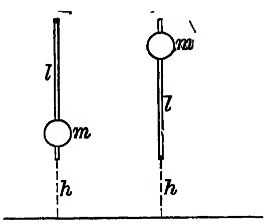
If potential energy is neither lost nor gained, the equilibrium is said to be *indifferent*, and the particle in its new position is still in equilibrium.

Now when potential energy is lost, kinetic energy is gained, and conversely.

Hence *if a particle is in equilibrium, static or molecular, all the acting forces must constitute a system of forces in equilibrium. If the potential energy is a minimum, the equilib-*

rium is stable and the kinetic energy, if any, a maximum. If the potential energy is a maximum, the equilibrium is unstable and the kinetic energy, if any, a minimum. If the potential energy does not change, the equilibrium is indifferent and the kinetic energy, if any, does not change.

Illustration.—Take the case of a pendulum. Let l be the length and m the mass of the bob. Disregard the mass of the string.

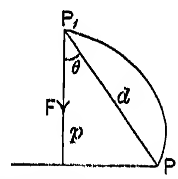


When the bob is at the lowest point the potential energy is $mg \,.\, h$. For every possible displacement h increases and the potential energy increases. The potential energy is then a minimum. This, then, is a position of stable equilibrium. If the bob is at rest, it is in stable static equilibrium. If the pendulum swings, when it arrives at its lowest point the kinetic energy is a maximum and the potential a minimum, and at the instant it is in stable kinetic equilibrium.

When the bob is at the highest point the potential energy is a maximum. This, then, is a position of unstable equilibrium. If the bob is at rest, it is in unstable static equilibrium. If in motion, when it arrives at the highest point the kinetic energy is a minimum and the potential a maximum, and at this instant it is in unstable kinetic equilibrium.

Change of Potential Energy.—In order to apply equations (8) or (10) it will be convenient to find the value of the change of potential energy $\mathfrak{P}_1 - \mathfrak{P}$ in special cases.

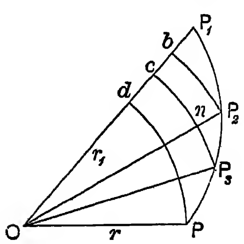
(*a*) FORCE UNIFORM.—If a particle is acted upon by a uniform conservative force F, the work done during a change of position from P_1 to P is, by equation (6), page 275,



$$\mathfrak{P}_1 - \mathfrak{P} = F \,.\, p = F \,.\, d \cos \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where d is the displacement and θ the angle of F with the displacement. If p and F are in the same direction, Fp is positive and we have loss of potential energy. If p and F are in different directions, we have gain of potential energy.

(*b*) CENTRAL FORCE —Let O be the centre of force.



Let $P_1, P_2, P_3 \ldots P$ be any path from P_1 to P, and let the positions P_1, P_2, P_3, etc., be consecutive. Draw OP_1, OP_2, OP_3, etc., and with O as a centre describe arcs of circles through P_2, P_3, etc., intersecting OP_1 at b, c, d, etc.

The force F_2 at P_2 acting towards O may be considered uniform for the indefinitely small displacement P_2 to P_3.

The work, then, from P_2 to P_3 is

$$F_2 \times P_2 n = F_2 \times bc.$$

Every element of the path may be treated in the same way.

Thus the work from P_1 to P_2 is

$$F_1 \times P_1 b;$$

from P_3 to P,

$$F_3 \times cd,$$

and so on. The entire work from P_1 to P is then

$$F_1 \times P_1 b + F_2 \times bc + F_3 \times cd + \text{etc.} = \Sigma F \,.\, dr,$$

where dr is the elementary distance along the radius vector OP_1 between any two arcs.

The work, then, necessary to move the particle from P_1 to P *by any path*, under the influence of a central force always directed towards O, is equal to that necessary to move it from P_1 to d in the straight line OP_1.

This work is *independent of the path* and depends only upon the initial and final positions and the magnitude of the force. If this force depends only upon position, this work is the loss of potential energy and we have in general

$$\mathfrak{P}_1 - \mathfrak{P} = \Sigma F dr. \quad \ldots \ldots \ldots \quad (2)$$

If the central force is opposite to dr, we have gain of potential energy, or

$$\mathfrak{P} - \mathfrak{P}_1 = \Sigma F \,.\, dr.$$

(*c*) CENTRAL FORCE CONSTANT.—If the magnitude of the central force F is constant, we have, from (2),

$$\mathfrak{P}_1 - \mathfrak{P} = F\Sigma dr = F(r_1 - r), \quad \ldots \ldots \ldots \quad (3)$$

where r_1 is the initial and r the final radius vector from O.

If r_1 is less than r, we have gain of potential energy, or

$$\mathfrak{P} - \mathfrak{P}_1 = F(r - r_1).$$

(*d*) CENTRAL FORCE PROPORTIONAL TO DISTANCE FROM CENTRE.—If the magnitude of the central force varies directly as the distance from the centre O, let F_0 be its known magnitude at a given distance r_0. Then the force at any other distance r is given by

$$F : F_0 :: r : r_0, \quad \text{or} \quad F = \frac{r}{r_0} F_0.$$

We have then at P_1 the force $F_1 = \frac{r_1}{r_0} F_0$, at P_2 the force $F_2 = \frac{r_2}{r_0} F_0$, at P_3 the force $\frac{r_3}{r_0} P_0$, and so on.

The average force between P_1 and P_2 is then $\frac{F_1 + F_2}{2} = \frac{F_0}{r_0} \cdot \frac{(r_1 + r_2)}{2}$, and the work from P_1 to P_2 is then, from (2),

$$\frac{F_0}{r_0} \cdot \frac{(r_1 + r_2)}{2} \cdot (r_1 - r_2) = \frac{F_0(r_1^2 - r_2^2)}{2r_0}.$$

In the same way we have for the work from P_2 to P_3

$$\frac{F_0(r_2^2 - r_3^2)}{2r_0},$$

and so on. The entire work from P_1 to P is then the loss of potential energy given by

$$\mathfrak{P}_1 - \mathfrak{P} = \frac{F_0}{2r_0}(r_1^2 - r^2), \quad \ldots \ldots \ldots \quad (4)$$

where r_1 is the initial and r_2 the final radius vector from O.

If r_1 is less than r, we have gain of potential energy, or

$$\mathfrak{T}_1 - \mathfrak{T} = \frac{F_0}{2r_0}(r^2 - r_1^2).$$

(*e*) CENTRAL FORCE INVERSELY PROPORTIONAL TO THE SQUARE OF THE DISTANCE FROM CERTRE.—If the magnitude of the central force varies inversely as the square of the distance from the centre, let F_0 be its known magnitude at a given distance r_0. Then the force F at any point is given by

$$F : F_0 : : r_0^2 : r^2; \quad \text{or} \quad F = \frac{r_0^2}{r^2}F_0.$$

We have then at P_1 the force $F_1 = \frac{r_0^2}{r_1^2}F_0$, and at P_2 the force $F_2 = \frac{r_0^2}{r_2^2}F_0$, and so on. For the indefinitely small displacement from P_1 to P_2, r_1 and r_2 are equal, and we have $r_1^2 = r_1 r_2$ and $r_2^2 = r_1 r_2$. Hence the force at P_1 can be written $F_1 = \frac{r_0^2 F_0}{r_1 r_2}$, and the force at P_2, $F_2 = \frac{r_0^2 F_0}{r_1 r_2}$.

The work, then, from (3), from F_1 to F_2 is

$$\frac{r_0^2 F_0}{r_1 r_2}(r_1 - r_2) = F_0 r_0^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

From P_2 to P_3 we have in the same way the work

$$\frac{r_0^2 F_0}{r_2 r_3}(r_2 - r_3) = F_0 r_0^2\left(\frac{1}{r_3} - \frac{1}{r_2}\right),$$

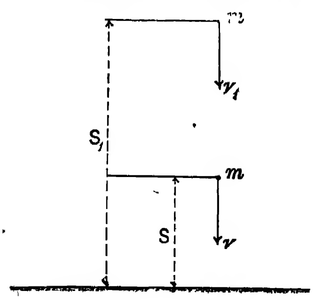
and so on. The entire work from P_1 to P is then the loss of potential energy and is given by

$$\mathfrak{T}_1 - \mathfrak{T} = F_0 r_0^2\left(\frac{1}{r} - \frac{1}{r_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If r_1 is less than r, we have gain of potential energy, or

$$\mathfrak{T} - \mathfrak{T}_1 = F_0 r_0^2\left(\frac{1}{r_1} - \frac{1}{r}\right).$$

Examples.—(1) *Apply the law of conservation of energy to a falling body.*



ANS. Let m be the mass of the body, v_1 its initial and v its final velocity. Then the gain of kinetic energy is

$$\mathfrak{K} - \mathfrak{K}_1 = \frac{1}{2}m(v^2 - v_1^2).$$

We have from equation (5), for central force varying inversely as the square of the distance, the loss of potential energy,

$$\mathfrak{T}_1 - \mathfrak{T} = F_0 r_0^2\left(\frac{1}{r} - \frac{1}{r_1}\right),$$

where r_1 is the initial distance and r the final distance from the centre of force, and F_0 is the known force at a given distance r_0. In the present case

the centre of force is the centre of the earth, r_0 is the radius of the earth, and F_0 at the earth's surface is mg. We have then

$$\mathfrak{P}_1 - \mathfrak{P} = \frac{mgr_0^2}{rr_1}(r_1 - r).$$

But $r_1 - r$ is the distance described, $(s_1 - s)$, if s_1 is the initial and s the final distance from the centre of the earth.

We have, by the principle of conservation of energy, gain of kinetic equals loss of potential energy, or

$$\mathfrak{K} - \mathfrak{K}_1 = \mathfrak{P}_1 - \mathfrak{P},$$

or

$$v^2 - v_1^2 = \frac{2gr_0^2}{rr_1}(s_1 - s).$$

This is the same as equation (5), page 113.

If we suppose the force to be uniform and equal to mg, as it practically is at the surface of the earth, we have, from equation (1), page 278,

$$\mathfrak{P}_1 - \mathfrak{P} = mg(s_1 - s),$$

and hence

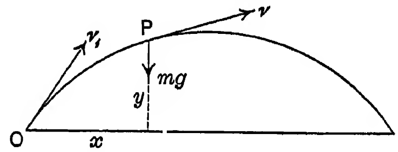
$$\frac{m}{2}(v^2 - v_1^2) = mg(s_1 - s),$$

or

$$v^2 = v_1^2 + 2g(s_1 - s),$$

which is equation (7), page 92.

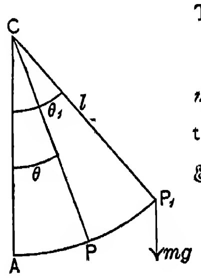
(2) *Apply the law of conservation of energy to a projectile.*



ANS. Let the force be vertical and equal to mg. The initial energy $\mathfrak{E}_1$ at O is $\frac{mv_1^2}{2}$. The energy $\mathfrak{E}$ at P is $\frac{m}{2}v^2$ kinetic and mgy potential. If there is no loss of energy, $\mathfrak{E}_1 = \mathfrak{E}$, or

$$\frac{mv_1^2}{2} = \frac{mv^2}{2} + mgy, \quad \text{or} \quad v^2 = v_1^2 - 2gy.$$

This is equation (11), page 104.



(3) *Apply the law of conservation of energy to the simple pendulum.*

ANS. Let the velocity at P_1 be zero. Then the initial energy at P_1 is $\mathfrak{E}_1 = mg(l - l\cos\theta_1)$ potential, relative to A. At any point P where the velocity is v we have the energy $\mathfrak{E}$, $\frac{mv^2}{2}$ kinetic and $mg(l - l\cos\theta)$ potential. If there is no loss of energy $\mathfrak{E}_1 = \mathfrak{E}$, or

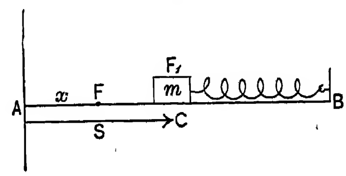
$$mg(l - l\cos\theta_1) = \frac{mv^2}{2} + mg(l - l\cos\theta).$$

Hence

$$v = \sqrt{2gl(\cos\theta - \cos\theta_1)}.$$

But $l(\cos\theta - \cos\theta_1)$ is the fall from P_1 to P. Hence the velocity is the same as for a particle falling through the vertical distance from P_1 to P.

(4) *Let a spring whose unstrained length is AB be fixed at the end B and compressed from A to C, where it presses against a body of mass m. Disregarding the mass of the spring, find the motion when released.*



ANS. Let the force at any distance x from A be F, and at the given distance $s = AC$ from A be F_1. Then we have

$$F : x :: F_1 : s, \quad \text{or} \quad F = \frac{F_1}{s}x.$$

The initial energy is $\mathfrak{E}_1 = F_1 s$, all potential. The energy $\mathfrak{E}$ at F is Fx potential and $\frac{mv^2}{2}$ kinetic. If there is no loss of energy,

$$\mathfrak{E}_1 = \mathfrak{E}, \quad \text{or} \quad F_1 s = Fx + \frac{mv^2}{2}.$$

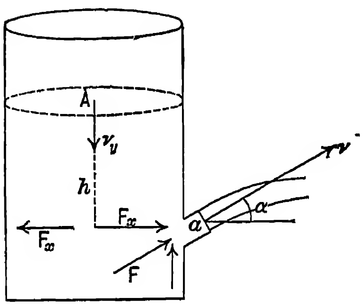
Substituting the value of F, we have

$$v^2 = \frac{2F_1}{ms}(s^2 - x^2).$$

We see then, from page 126, that the motion is simple harmonic.

(5) *A vessel containing water has a small orifice whose centre is at a distance h below the surface. The water flows in at top with a vertical velocity v_y, and the top area of cross-section is A. The water flows out of the orifice with a velocity v in any direction, and the area of the orifice at right angles to v is a. If water flows in as fast as it flows out, so that the water-level remains constant, find the theoretic velocity of efflux v, disregarding friction and all resistances.*

ANS. The quantity of water flowing in per second is Av_y, and the quantity flowing out as av. By the conditions, for constant level we must have



$$Av_y = av, \quad \text{or} \quad v_y = \frac{a}{A}v.$$

The initial energy $\mathfrak{E}_1$ of a particle of mass m at the top level is mgh potential and $\frac{mv_y^2}{2}$ kinetic, or

$$\mathfrak{E}_1 = mgh + \frac{mv_y^2}{2}.$$

When this particle reaches the orifice its energy $\mathfrak{E}$ is all kinetic and $\mathfrak{E} = \frac{mv^2}{2}$. If no energy is lost, $\mathfrak{E}_1 = \mathfrak{E}$, or

$$mgh + \frac{mv_y^2}{2} = \frac{mv^2}{2}.$$

Substituting the value of v_y, we have

$$v = \frac{\sqrt{2gh}}{\sqrt{1 - \frac{a^2}{A^2}}}. \quad \ldots \ldots \quad (1)$$

If a is very small compared to A, we can practically neglect the fraction $\frac{a^2}{A^2}$ relative to 1, and we then have

$$v = \sqrt{2gh}.$$

The theoretic velocity of efflux in this case is the same as for a particle falling freely through the distance h. This is known as *Torricelli's* principle.

If γ is the density or mass of a cubic unit of water, in a very small time τ the mass discharged is

$$m = \gamma av\tau.$$

The kinetic energy at efflux is then the work done in giving the mass m the velocity v, or

$$\text{work} = \frac{mv^2}{2} = \frac{\gamma av^3\tau}{2}.$$

If the mass m acquires the velocity v in a very short time τ from rest, the average velocity is $\frac{v}{2}$ and the distance is $\frac{v\tau}{2}$. If we divide the work by this distance, we have for the uniform force F during the short time τ *in the direction of* v

$$F = \gamma av^2, \quad \ldots \ldots \quad (2)$$

or, from (1),

$$F = \frac{2\gamma agh}{1 - \frac{a^2}{A^2}} \text{ poundals.}$$

For F in gravitation measure

$$F = \frac{2\gamma ah}{1 - \frac{a^2}{A^2}}. \quad \ldots \ldots \quad (3)$$

From (3), for a small relative to A

$$F = 2\gamma ah,$$

or *the weight of a column of water whose base is the area a of the orifice and whose height is twice the distance h.*

If α is the angle of v with the horizontal, we have for the horizontal component of F

$$F_x = 2\gamma ah \, . \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since action and reaction are equal, this is the horizontal pressure on the side of the vessel in a direction opposite to $v \cos \alpha$.

We can also obtain (2) directly by the principle of impulse (page 257). Thus

$$F\tau = mv, \quad \text{or} \quad F = \frac{mv}{\tau}.$$

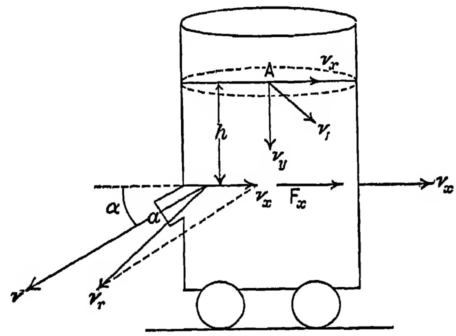
Substituting $m = \gamma av\tau$, we have at once

$$F = \gamma av^2.$$

(6) *In the preceding example let the vessel move horizontally with the uniform velocity v_x, while the water flows in at the top with a vertical velocity v_y, the top area being A, and is discharged with the velocity v, making an angle α with the horizontal, the area of orifice at right angles to v being a.*

ANS. We have for constant level, as before,

$$Av_y = av,$$



and, disregarding the motion of the vessel, we have, as in the preceding example,

$$\mathfrak{E}_1 = mgh + \frac{mv_y^2}{2}, \qquad \mathfrak{E} = \frac{mv^2}{2},$$

and therefore, just as in the preceding example, we have for $\mathfrak{E}_1 = \mathfrak{E}$

$$mgh + \frac{mv_y^2}{2} = \frac{mv^2}{2},$$

$$\text{or} \qquad h = \frac{v^2}{2g} - \frac{v_y^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for the velocity of efflux *relative to the vessel,*

$$v = \frac{\sqrt{2gh}}{\sqrt{1 - \frac{a^2}{A^2}}},$$

just as before.

Now the absolute velocity at A is given by

$$v_1^2 = v_x^2 + v_y^2.$$

Hence the total energy at A is

$$\mathfrak{E}_1 = \frac{mv_1^2}{2} + mgh = \frac{m}{2}(v_x^2 + v_y^2) + mgh. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The absolute velocity at a is given by

$$v_r^2 = v_x^2 + v^2 - 2vv_x \cos \alpha.$$

Hence the total energy at a is

$$\mathfrak{E} = \frac{mv_r^2}{2} = \frac{m}{2}(v_x^2 + v^2 - 2vv_x \cos \alpha). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (2) gives the work the mass m at A can do, and equation (3) the work this mass at a can do after leaving the vessel. The difference $\mathfrak{E}_1 - \mathfrak{E}$ is then the work *done upon the vessel.* If then we subtract (3) from (2) and reduce by (1), we obtain for the work done upon the vessel

$$\text{work} = mvv_x \cos \alpha.$$

In a very short time τ we have $m = \gamma av\tau$, where γ is the density or mass of a cubic unit of water. Hence

$$\text{work} = \gamma av^2 v_x \tau \cos \alpha. \quad \ldots \quad (4)$$

But the distance in the time τ is $v_x\tau$. Hence, dividing (4) by $v_x\tau$, we have for the horizontal force F_x on the vessel

$$F_x = \gamma av^2 \cos \alpha.$$

This is the same result as obtained in the preceding example. Thus, if we put $v = \sqrt{2gh}$, we have

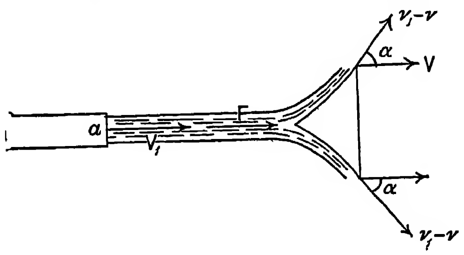
$$F_x = 2\gamma agh \,.\, \cos \alpha \text{ poundals,}$$

or, in gravitation measures,

$$F_x = 2\gamma ah \,.\, \cos \alpha,$$

just as before.

(7) *A horizontal stream of water whose cross-section is a and velocity v_1 meets a surface moving in the same direction with a velocity v. Find the pressure, disregarding friction and resistances.*



ANS. Let the water pass off the surface in a direction making the angle α with the direction of motion. In any small time τ the mass of water discharged is $m = \gamma av_1\tau$, where γ is the density or mass of a cubic unit of water. The initial kinetic energy of m is then

$$\mathcal{E}_1 = \frac{mv_1^2}{2} = \frac{\gamma av_1^3\tau}{2}.$$

The velocity of the water as it leaves the surface is $v_1 - v$ relative to the surface. The velocity of the surface is v. The resultant velocity is then given by

$$v_r^2 = (v_1 - v)^2 + v^2 + 2(v_1 - v)v \cos \alpha.$$

The final kinetic energy of the mass m is then

$$\mathcal{E} = \frac{m}{2} v_r^2 = \frac{\gamma av_1\tau}{2}[v_1^2 - 2vv_1 + 2v^2 + 2(v_1 - v)v \cos \alpha].$$

The difference $\mathcal{E}_1 - \mathcal{E}$ is the work done on the body,

$$\text{work} = \gamma av_1\tau(v_1 - v)v(1 - \cos \alpha). \quad \ldots \quad (1)$$

If we divide this by the distance $v\tau$, we have for the force on the surface

$$F = \gamma av_1(v_1 - v)(1 - \cos \alpha) \text{ poundals.} \quad \ldots \quad (2)$$

This is the same result as found in example (3), page 259, by the principle of impulse.

Equation (2) gives F in poundals. For gravitation measure we have

$$F = \frac{\gamma av_1}{g}(v_1 - v)(1 - \cos \alpha).$$

If the surface moves in the opposite direction, we have $v_1 + v$ in place of $v_1 - v$, and

$$F = \frac{\gamma av_1}{g}(v_1 + v)(1 - \cos \alpha).$$

If the surface is at rest, $v = 0$ and

$$F = \frac{\gamma av_1^2}{g}(1 - \cos \alpha).$$

If in the latter case $E = 90°$, this becomes

$$F = \frac{\gamma av_1^2}{g} = 2\gamma a \frac{v_1^2}{2g}.$$

Hence *the normal pressure of a jet of water against a plane surface at rest is equal to the weight of a column of water whose cross-section is that of the jet and whose height is twice that due to the velocity.*

If $\alpha = 180°$ and $v = 0$, we have

$$F = \frac{2\gamma a v_1^2}{g},$$

or twice as much as when $\alpha = 90°$.

The work done on the surface is, from (1), a maximum when $v = v_1 - v$ or $v = \frac{v_1}{2}$. That is, the work is a maximum when the velocity of the surface is half that of the jet. The maximum work is then, from (2),

$$\frac{\gamma a v_1^3 \tau(1 - \cos\alpha)}{4}.$$

If $\alpha = 180°$, this becomes $\frac{\gamma a v_1 \tau}{2} . v_1^2 = \frac{m v_1^2}{2}$, or all the kinetic energy of the water.

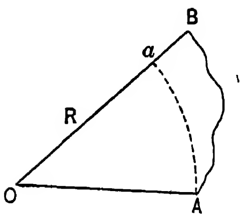
If $\alpha = 90°$, it becomes $\frac{\gamma a v_1 \tau}{4} . v_1^2 = \frac{m v_1^2}{4}$, or one-half the kinetic energy of the water.

Hence *the maximum work of a jet of water striking a plane surface at right angles, disregarding friction, is only one half the kinetic energy of the jet.*

CHAPTER VI.

THE POTENTIAL.

The Potential.—Let a particle at a fixed point O act either by attraction or repulsion upon a particle at B. Let BA be any path of the particle from B to A, the distance OB being R, and the distance OA being r. With $OA = r$ as a radius describe an arc of a circle Aa.



Then the force upon B is a central force, and we have proved, page 278, that the work done by or against the central force while the particle moves from B to A is *independent of the path* and equal to that necessary to move it from B to a, when $Oa = r$.

The fixed particle at O is then a centre of force, and the space surrounding this particle we call the FIELD OF FORCE.

If then we take any convenient point of reference, as C, the, work done in transferring a particle of unit mass from any point of the field to this point, or from this point to any point of the field, has a definite value for every point of the field, no matter what the path.

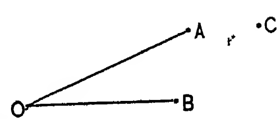
This definite work for any given point of the field *when the particle moved has a mass o unity* is called the POTENTIAL of the point.

The unit of potential is then the same as the unit of work, as one foot-poundal or one foot-pound or one erg.

The magnitude of the potential will depend upon the position of the point of reference. The sign will be plus or minus according as work is done by or against the force of the field. We denote the potential by the letter Π.

Principle of the Potential.—The application of the potential rests upon the following principle:

Let A and B be any two points in the field of force due to a particle at O, and let C be any point of reference. Then, since the work done during any displacement is independent of the path, the work done by or against the force of the field in transferring a unit mass from A to B is equal to the difference of the works done in transferring it from C to A and C to B.



If then Π_a and Π_b are the potentials of the points A and B, the difference is the work of moving unit mass from A to B or B to A. If F is the mean force in the direction AB, we have this work equal to $F \times AB$. Hence

$$F \times AB = \Pi_a \sim \Pi_b, \quad \text{or} \quad F = \frac{\Pi_a \sim \Pi_b}{AB}.$$

When A and B are indefinitely near, the mean force F becomes the instantaneous force in the direction AB, and $\frac{\Pi_a \sim \Pi_b}{AB}$ becomes the rate of change of the potential of the point A per unit of distance in the direction AB. Hence

The rate of change of the potential of any point per unit of distance in any direction is equal to the component force in that direction which acts upon a particle of unit mass placed at that point.

The particle possesses potential energy at whatever point of the field of force it may be placed. The excess of its potential energy at one point over its potential energy at another point is then the work done by or against the force of the field in moving from one point to the other. This is equal to the difference of potential. Hence the appropriateness of the term "potential."

The theory of the potential is of great use in magnetic and electrical investigations.

Equipotential Surface.—A surface at every point of which the potential has the same value is called an EQUIPOTENTIAL SURFACE.

If then a particle is moved from any point on such a surface to any other point on this surface, no work is done by or against the force of the field. There is then no component force in any direction tangential to such a surface, and hence no rate of change of potential per unit of distance in that direction. The resultant force at any point of such a surface is then normal to the surface. Thus the surface of water at rest forms an equipotential surface for which there is no rate of change of potential, and the resultant force for every particle on the surface is normal to the surface. The work done by or against gravity in moving a particle from one point to another of such a surface is zero.

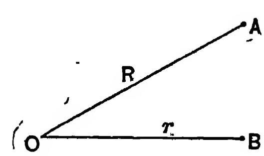
Lines of Force.—Any line so drawn in a field of force that its direction at every point is the direction of the resultant force at that point is called a LINE OF FORCE. As the resultant force at any point is normal to the equipotential surface passing through that point, lines of force are normal to the equipotential forces they meet.

Tubes of Force.—If from points in the boundary of any portion of an equipotential surface lines of force are drawn, the space thus marked off is called a TUBE OF FORCE.

Gravitational Potential.—The choice of the point of reference and of the mode of defining potential are matters of convenience and vary with the kind of field of force under consideration.

The potential in a field of force due to the attraction of gravity is called GRAVITATIONAL POTENTIAL. The point of reference is taken in this case *at an infinite distance*, and since it is convenient to have the potential for all points of a gravitational field positive, and the force of the field is always attractive, we define gravitational potential of a point as *the work done* BY *the force of the field in moving unit mass from a point at an infinite distance* TO *the given point*. Or, since there is thus a loss of potential energy, the work done by the force of the field must equal the gain of kinetic energy, and hence we may also define gravitational potential of a point as *the kinetic energy acquired by unit mass in falling from infinity under the attraction of a given mass to that point.*

The force of gravity varies inversely as the square of the distance, and we have seen (page 280) that the work of such a force when a particle moves from a distance R to a distance r from the centre of force is given by



$$W = F_0 r_0^2 \left(\frac{1}{r} - \frac{1}{r_1}\right),$$

where F_0 is the force at a given distance r_0.

If we take r_1 infinite, $\frac{1}{r_1}$ is zero and this becomes

$$W = F_0 r_0^2 \cdot \frac{1}{r}.$$

If the mass of the attracting particle at O is m and of the moving particle M, we have for the force of attraction at any distance r_0 (page 203)

$$F_0 = k\frac{Mm}{r_0^2},$$

where k (page 205) is given by

$$k = \frac{g r_0^2}{m_0},$$

where g is the acceleration of gravity, m_0 the mass of the earth and r_0 the radius of the earth. We have then for the work of moving a particle M from infinity to the distance r

$$W = kM\frac{m}{r}.$$

If we adopt the astronomical unit of mass (page 206), this becomes $W = M\frac{m}{r}$, and if we take the mass M as unity we have $W = [M]\frac{m}{r}$, where $[M]$ is the unit of mass, or the numerical equation $W = \frac{m}{r}$.

If the field of force is due to any number of particles of masses m_1, m_2, m_3, etc., at distances r_1, r_2, r_3, etc., from M, we have the numeric equation when M is unity

$$W = \Sigma\frac{m}{r}.$$

The expression $\Sigma\frac{m}{r}$ is the gravitational potential of the point at which the attracted particle of unit mass is placed. We have then

$$\Pi = \Sigma\frac{m}{r}, \quad \text{. . . . .} \quad (1)$$

or, mathematically defined, *the gravitational potential of any point due to the attraction of any mass is the sum of the quotients of all the elementary attracting masses divided by their distances from the point, provided we adopt the astronomical unit of mass.*

Since equation (1) gives the work done by the force of the field in moving *unit mass* from a point at an infinite distance to the given point, the work done in moving a mass M is

$$W = M\Sigma\frac{m}{r} = M\Pi \quad \text{. . . . .} \quad (2)$$

if we use the astronomical unit of mass (page 206), or

$$W = M\Sigma\frac{m}{r} = kM\Pi \quad \ldots \ldots \ldots \quad (3)$$

if we use the ordinary unit of mass, where k is given by

$$k = \frac{gr_0^2}{m_0}, \quad \ldots \ldots \ldots \quad (4)$$

where g is the acceleration of gravity, m_0 the mass and r_0 the radius of the earth.

[**Differential Equations.**—We have then for the gravitational potential of any point of a field of force due to the attraction of any number of particles m_1, m_2, m_3, etc., at distances r_1, r_2, r_3 from that point,

$$\Pi = \Sigma\frac{m}{r}. \quad \ldots \ldots \ldots \quad (1)$$

From the principle of the potential (page 286), if we take the point as an origin of co-ordinates, we have for the component force in the directions of the axes of X, Y, Z, for a *unit mass* at the point,

$$\left.\begin{aligned} F_x &= \frac{d\Pi}{dx}, \\ F_y &= \frac{d\Pi}{dy}, \\ F_z &= \frac{d\Pi}{dz}, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (2)$$

where the astronomical unit of mass (page 206) is to be used. For the ordinary unit of mass we multiply by

$$k = \frac{gr_0^2}{m_0}, \quad \ldots \ldots \ldots \quad (3)$$

where g is the acceleration of gravity, m_0 the mass and r_0 the radius of the earth (page 205).

For a mass M, then, at the point we multiply by kM.

For the resultant force on *unit of mass* in the direction of any radius vector from the point we have

$$R = \frac{d\Pi}{dr}, \quad \ldots \ldots \ldots \quad (4)$$

where the astronomical unit of mass (page 206) is to be used. For ordinary unit of mass we multiply by k, and for any mass M at the point by kM.

If ds is an element of the path of the attracted particle of unit mass, making an angle θ with r, then $ds = \frac{dr}{\cos\theta}$, and we have for the component of the force tangent to the path, upon *unit mass*,

$$F_t = \frac{d\Pi}{ds} = \frac{d\Pi}{dr}\cos\theta, \quad \ldots \ldots \ldots \quad (5)$$

where the astronomical unit of mass (page 206) is to be used.

For ordinary unit of mass we multiply by k, and for any mass M by kM.

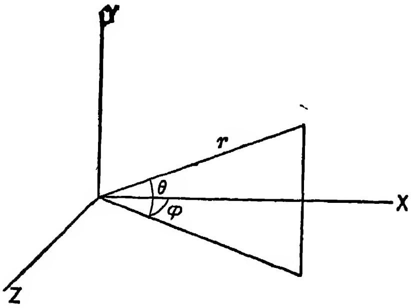
We have from (4), $\Pi = \int Rdr$; and since for an equipotential surface the potential has the same value at every point, the condition for an equipotential surface is

$$\Pi = \int Rdr = C, \quad \ldots \ldots \ldots \quad (6)$$

where C is a constant.

A surface which fulfils for each of its points this condition is an equipotential surface for the system of attractions. As any value can be given to C between its greatest and least values, there will be an indefinitely great number of equipotential surfaces corresponding to any given system of attractions.

From equation (5) we see that F_t is zero when $\theta = 90°$, and becomes equal to the resultant force R, equation (4), when $\theta = 0$. That is, the resultant attraction R is at right angles to the equipotential surface. The direction of R is then a line of force.



If dv is an element of volume, and δ its density, we have for its mass $m = \delta dv$. Hence for rectangular co-ordinates

$$\Pi = \int \frac{m}{r} = \int \frac{\delta dx\, dy\, dz}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we use polar co-ordinates, we have for the elementary volume $dv = r^2 dr \cos\theta\, d\theta\, d\phi$, and hence

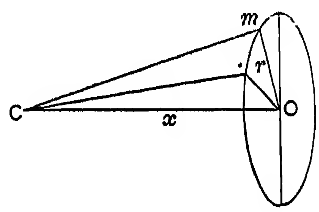
$$\Pi = \int\int\int \delta r\, dr \cos\theta\, d\theta\, d\phi. \quad . \quad . \quad . \quad (8)$$

Examples.—(1) *Particles of masses 3.928, 39.28 and 392.8 kilograms are situated at three of the corners of a square whose side is 1 metre. Find the potential at the fourth corner.*

ANS. $\Pi = \Sigma\frac{m}{r}$, and the astronomical unit of mass is 3928 grams (page 206). Hence $\Pi = 1.087$ ergs.

(2) *Find the potential and attraction of a homogeneous circular ring of radius r upon a point C on the perpendicular to its plane through its centre O.*

ANS. Let the distance of the point C from the centre O be x. Then the distance Cm for any particle of the ring is $\sqrt{r^2 + x^2}$. If the linear density of the ring is δ, the mass is $2\pi r\delta$, and therefore the potential $\Pi = \frac{2\pi r\delta}{\sqrt{r^2 + x^2}}$.

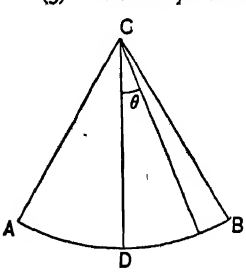


The attraction upon a unit mass at C parallel to the plane of the ring is then $\frac{d\Pi}{dr}$, taking the astronomical unit of mass (page 206). But r is constant and hence $\frac{d\Pi}{dr} = 0$. That is the sum of the component attractions of the elements of the ring in the plane of the ring is zero. The attraction in the direction CO upon a unit mass at C, taking the astronomical unit of mass, is $F_x = \frac{d\Pi}{dx} = -\frac{2\pi r\delta x}{(r^2 + x^2)^{\frac{3}{2}}}$, the minus sign denoting attraction or force towards the centre O.

If we multiply the value of Π and F_x by kM, where M is the mass of any particle at O, and k is $\frac{g r_0^2}{m_0}$ (page 205), we have the result for any mass M at C, using the ordinary unit of mass.

When $x = 0$, the potential at the centre of the ring is $\Pi = 2\pi\delta$.

(3) *Find the potential and attraction of a circular arc at its centre.*



ANS. Let θ be the angle subtended by any portion of the arc estimated from its middle point D.

The length of any element is $rd\theta$, its mass is $r\delta\, d\theta$, where δ is the linear density, and the potential is

$$\Pi = \int_{-\alpha}^{+\alpha} \frac{r\delta d\theta}{r} = 2\delta\alpha,$$

where α is the angle ACD.

This is independent of the radius r of the arc.

The attraction of any element whose mass is $r\delta d\theta$ for a unit mass at C, using the astronomical unit of mass (page 206), is $\frac{r\delta d\theta}{r^2}$. The component of this at right angles to CD is $\frac{r\delta d\theta}{r^2}\sin\theta$, and along CD, $-\frac{r\delta d\theta}{r^2}\cos\theta$

We have then for the resultant attraction at right angles to CD

$$F_x = \frac{\delta}{r}\int_{-\alpha}^{+\alpha} d\theta \sin\theta = 0,$$

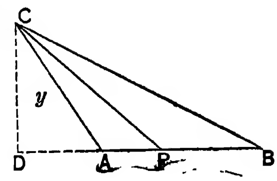
and for the resultant attraction along CD

$$F_y = -\frac{\delta}{r}\int_{-\alpha}^{+\alpha} d\theta \cos\theta = -\frac{2\delta}{r}\sin\alpha,$$

the minus sign denoting attraction.

For any mass M at C, using the ordinary unit of mass, we multiply by kM, where $k = \frac{gr_0^2}{m_0}$ (page 205).

(4) *Find the potential and attraction of a straight line upon an external point.*



ANS. Let AB be the line and C the point. Drop the perpendicular CD, take D as origin, and let $CD = y$. Then for any point P of the line distant $DP = x$ we have $CP = r = \sqrt{y^2 + x^2}$. Let δ be the linear density. Then the mass of any element is δdx, and the potential is

$$\Pi = \delta\int\frac{dx}{\sqrt{y^2 + x^2}} = \delta\log\frac{x + \sqrt{y^2 + x^2}}{y}.$$

Taking this between the limits of $x = DA = +a$ and $x = DB = +b$, we have

$$\Pi = \delta\log\frac{b + \sqrt{y^2 + b^2}}{a + \sqrt{y^2 + a^2}}.$$

The component attraction upon unit mass at C in the direction of the line is

$$F_x = \frac{d\Pi}{dx} = \frac{\delta}{\sqrt{y^2 + x^2}}.$$

Introducing the limits $+a$ and $+b$,

$$F_x = \delta\left(\frac{1}{\sqrt{y^2 + b^2}} - \frac{1}{\sqrt{y^2 + a^2}}\right) = \delta\left(\frac{1}{CB} - \frac{1}{CA}\right).$$

For the component attraction upon the unit mass at C perpendicular to the line we have

$$F_y = \frac{d\Pi}{dy} = \delta.\frac{d\left[\log\left(\frac{x}{y} + \sqrt{1 + \frac{x^2}{y^2}}\right)\right]}{dy}$$

$$= -\delta.\frac{\frac{x}{y^2} + \frac{yx^2}{y^4\sqrt{1 + \frac{x^2}{y^2}}}}{\frac{x}{y} + \sqrt{1 + \frac{x^2}{y^2}}} = -\frac{\delta x}{y\sqrt{y^2 + x^2}}.$$

Introducing the limits $+a$ and $+b$, we have

$$F_y = \frac{\delta}{y}\left(\frac{a}{\sqrt{y^2 + a^2}} - \frac{b}{\sqrt{y^2 + b^2}}\right) = \frac{\delta}{y}\left(\frac{a}{CA} - \frac{b}{CB}\right).$$

Let the angle $DCA = \alpha$, $DCB = \beta$, $ACB = \beta - \alpha = \gamma$. Then

$$\frac{1}{CB} = \frac{\cos\beta}{CD}, \quad \frac{1}{CA} = \frac{\cos\alpha}{CD}, \quad a = CA\sin\alpha, \quad b = CB\sin\beta,$$

and

$$F_x = \frac{\delta}{CD}\left(\cos\beta - \cos\alpha\right), \quad F_y = \frac{\delta}{CD}\left(\sin\alpha - \sin\beta\right).$$

The resultant force upon unit mass at C is then

$$R = \sqrt{F_x^2 + F_y^2} = \frac{\delta}{y}\sqrt{2 - 2\cos\gamma} = \frac{2\delta}{y}\sin\frac{1}{2}\gamma.$$

The tangent of the angle which this resultant force makes with the vertical is

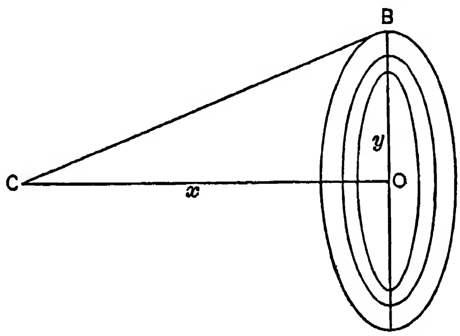
$$\frac{F_x}{F_y} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \tan\frac{\alpha + \beta}{2}.$$

Therefore the resultant attraction bisects the angle ACB.

The results are all for unit mass at C and astronomical unit of mass (page 206). For mass M at C and ordinary unit of mass we have only to multiply F_x, F_y, R by kM, where $k = \frac{gr_0^2}{m_0}$ (page 205).

(5) *Find the potential and attraction for a circular disc at a point on the perpendicular to its plane through its centre.*

ANS. Let y be the radius and dy the thickness of an elementary ring, and δ the surface density. Then the mass of the elementary ring is $2\pi\delta y\,dy$. If the distance OC is x, we have for the potential of the disc.



$$\Pi = 2\pi\delta\int\frac{y\,dy}{\sqrt{x^2 + y^2}}$$

$$= 2\pi\delta(\sqrt{x^2 + y^2} - x),$$

which for the limits $y = R$ = radius of disk, and $y = 0$ becomes

$$\Pi = 2\pi\delta(\sqrt{x^2 + R^2} - x).$$

For the centre of the disc this becomes $2\pi\delta R$.

The potential, then, is constant for x constant. The component force upon unit mass at C parallel to the disc is then $\frac{d\Pi}{dR} = 0$. For the component force along OC we have $F_x = \frac{d\Pi}{dx} = -2\pi\delta\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$, the minus sign denoting attraction.

For mass M at O and ordinary unit of mass we have only to multiply F_x by kM, where $k = \frac{gr_0^2}{m_0}$ (page 205).

(6) *Find the potential and attraction at the vertex for a right cone with circular base.*

ANS. Let the half angle at the vertex, OCB, of the preceding figure be θ. Then $\frac{x}{\sqrt{x^2 + R^2}} = \cos\theta$. Hence, from the preceding example, we see that the attraction of all circular elementary slices for a particle at C is the same, and equal to

$$-2\pi\delta\,dx(1 - \cos\theta).$$

The total attraction is then

$$F_x = \frac{d\Pi}{dx} = -2\pi\delta x(1 - \cos\theta),$$

which for the limits h and 0 becomes

$$F_x = -2\pi\delta h(1 - \cos\theta).$$

For mass M at C and ordinary units of mass we have only to multiply by kM, where $k = \frac{gr_0^2}{m_0}$ (page 25). We have then

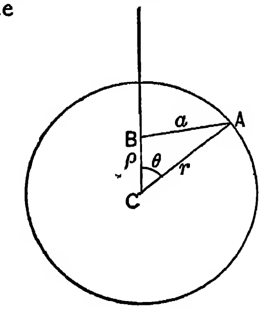
$$\Pi = -\pi\delta x^2(1 - \cos\theta),$$

or for limits 0 and h

$$\Pi = \pi\delta h^2(1 - \cos\theta).$$

(7) *Find the potential and attraction of a spherical shell at any point.*

ANS. Let r be the radius of the shell, t its thickness, ρ the distance of the point B from the centre C, $AB = a =$ the distance of any point of the shell from the given point B. Take the origin at C, and let BC coincide with the axis of Y.



Then the elementary volume is

$$dv = r^2 t \sin\theta \, d\theta \, d\phi,$$

and

$$a = \sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}.$$

Hence, if δ is the density,

$$\Pi = \delta t r^2 \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta \, d\theta \, d\phi}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}}.$$

Integrating first with respect to ϕ, we have

$$\Pi = 2\pi\delta t r^2 \int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}},$$

and then with respect to θ,

$$\Pi = \frac{2\pi\delta t r}{\rho}\left\{ (r^2 - 2r\rho\cos\theta + \rho^2)^{\frac{1}{2}} \right\}_0^{\pi}$$

$$= \frac{2\pi\delta t r}{\rho}\Big[(r^2 + 2r\rho + \rho^2)^{\frac{1}{2}} - (r^2 - 2r\rho + \rho^2)^{\frac{1}{2}}\Big].$$

When the point B is within the shell $\rho < r$, and when it is outside of the shell $\rho > r$.

In the first case, when B is within the shell, we have

$$\Pi = \frac{2\pi\delta t r}{\rho}[(r + \rho) - (r - \rho)] = 4\pi\delta t r = \frac{m}{r},$$

where m is the mass of the shell. The resultant force of attraction is then $R = \dfrac{d\Pi}{d\rho} = 0$. This is the same result as in example (1), page 207.

In the second case, when B is outside the shell, we have

$$\Pi = \frac{2\pi\delta t r}{\rho}[(r + \rho) - (\rho - r)] = \frac{4\pi\delta t r^2}{\rho} = \frac{m}{\rho},$$

where m is the mass of the shell. The resultant force of attraction then is $R = \dfrac{d\Pi}{d\rho} = -\dfrac{m}{\rho^2}$, where the minus sign denotes attraction.

If we take the mass M at B and use the ordinary unit of mass, we have $R = -k\dfrac{Mm}{\rho^2}$. This is the same result as already obtained, page 205.

(8) *Find the potential and attraction of a thick homogeneous spherical shell at any point.*

ANS. Let the external radius be r_1, and the internal radius r_2. Then in the preceding example we can put $t = dr$, and we have for the potential of that part of the shell outside of the spherical surface containing the point $\int_\rho^{r_1} 4\pi\delta r \, dr$, and for the potential of that part of the shell inside of the spherical surface containing the point $\int_{r_2}^{\rho} \dfrac{4\pi\delta r^2 dr}{\rho}$. Hence

$$\Pi = \frac{4\pi\delta}{\rho}\int_{r_2}^{\rho} r^2 dr + 4\pi\delta\int_{\rho}^{r_1} r\,dr$$

$$= \frac{4\pi\delta}{3\rho}(\rho^3 - r_2^3) + 2\pi\delta(r_1^2 - \rho^2).$$

The mass of the shell is $m = \frac{4\pi\delta}{3}(r_1^3 - r_2^3)$.

If the point is wholly within the shell,

$$\Pi = 2\pi\delta(r_1^2 - r_2^2);$$

and if the thickness is very small, $r_1 - r_2 = t$ and $r_1 + r_2 = 2r$, and $\Pi = 4\pi\delta tr$, as found in the preceding example. Also, the attraction is $R = \frac{d\Pi}{d\rho} = 0$, as found in the preceding example.

If the point is wholly without the shell,

$$\Pi = \frac{4\pi\delta}{3\rho}(r_1^3 - r_2^3) = \frac{m}{\rho},$$

and $R = -\frac{m}{\rho^2}$, as found in the preceding example.

If the shell becomes a sphere, $r_2 = 0$ and $r_1 = r$, and we have for an interior point

$$\Pi = 2\pi\delta r^2 - \frac{2\pi\delta\rho^2}{3},$$

$$R = -\frac{4\pi\delta\rho}{3} = -\frac{m}{r^3}\rho,$$

where $m = \frac{4}{3}\pi r^3\delta$. For an exterior point

$$\Pi = \frac{4\pi\delta r^3}{3\rho} = \frac{m}{\rho},$$

$$R = -\frac{m}{\rho^2}.$$

For $\rho = r$ we have in both cases

$$\Pi = \frac{4\pi\delta r^2}{3} = \frac{m}{r}; \qquad R = -\frac{m}{r^2}.$$

Hence we see that for a homogeneous sphere we may take the potential and attraction at any external point as though the whole mass were concentrated at the centre, while the attraction at an interior point is directly proportional to the distance from the centre. The first result has been proved, page 205; the second in example (2), page 207.

(9) *Find the potential and attraction for a cylinder of length l and radius R for a point on the axis at a distance d from the nearest end.*

ANS. We have found, example (5), for the component force along the axis of a circular disk $-2\pi\delta\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$; the component at right angles to the axis being zero. If the disk has a thickness dx, we have for a cylinder

$$\frac{d\Pi}{dx} = F_x = -2\pi\delta\int dx\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right) = -2\pi\delta(x - \sqrt{x^2 + R^2});$$

or, taking the limits $d+l$ and d,

$$F_x = -2\pi\delta(l - \sqrt{(d+l)^2+R^2} + \sqrt{d^2+R^2}).$$

Hence

$$\Pi = \int -2\pi\delta dx(x - \sqrt{x^2+R^2}) = -2\pi\delta\left[\frac{x^2}{2} - \frac{x}{2}\sqrt{x^2+R^2} - \frac{R^2}{2}\log\left(x + \sqrt{x^2+R^2}\right)\right].$$

The value of Π is obtained by taking the limits $d+l$ and d. For $d = 0$ we have for the attraction upon unit mass at the end surface

$$F_x = -2\pi\delta(l - \sqrt{l^2+R^2} + R),$$

and

$$\Pi = 2\pi\delta\left[\frac{l}{2}\sqrt{l^2+R^2} - \frac{l^2}{2} - \frac{R^2}{2}\log\left(l + \sqrt{l^2+r^2}\right)\right].$$

For a mass M, using the ordinary unit of mass, we multiply F_x by kM, where $k = \frac{gr_0^2}{m_0}$ (page 205).

(10) *If the radius of the earth is 4000 miles, find the potential for a point on the surface.*

ANS. From example (8) we have for astronomical unit of mass $\Pi = \frac{m_0}{r_0}$. For ordinary unit of mass we multiply by $\frac{gr_0^2}{m_0}$. Hence $\Pi = gr_0$ ft.-poundals, or r_0 ft.-pounds $= 4000 \times 5280 = 21\,120\,000$ ft.-pounds.

(11) *At the distance of the moon, 240 000 miles from the centre of the earth, find the shortest distance through which 1 lb. must be moved to do one ft.-pound of work.*

ANS. We have the work given by

$$W = mgr_0^2\left(\frac{1}{r} - \frac{1}{r_1}\right) \text{ ft.-poundals}$$

for a mass m at a distance r_1 moved to a distance r. In the present case $m = 1$ lb. Hence for work in ft.-pounds

$$W = r_0^2\left(\frac{1}{r} - \frac{1}{r_1}\right) = \frac{r_0^2}{rr_1}(r_1 - r).$$

If $W = 1$ ft.-pound,

$$r_1 - r = \frac{rr_1}{r_0^2}, \quad \text{or} \quad r_1 - r = \frac{r_1}{1 + \frac{r_0^2}{r_1}}.$$

Taking $r_0 = 4000$ miles and $r_1 = 240\,000$ miles, we have

$$r_1 - r = \frac{240\,000 \times 5280}{1 + \frac{(4000 \times 5280)^2}{240\,000 \times 5280}} = 3600 \text{ ft.}$$

Value of g above Sea-level.—Suppose a mountain of uniform density δ and *cylindrical in shape*, and a particle of mass m at the centre of its upper surface. Then, from example (9), we have for the force of attraction, using the astronomical unit of mass,

$$m \,.\, 2\pi\delta[l - \sqrt{l^2 + R^2} + R].$$

If we divide by m, we have for the acceleration due to the attraction of the mountain

$$2\pi\delta[l - \sqrt{l^2 + R^2} + R] = 2\pi\delta\left[l - R\sqrt{1 + \frac{l^2}{R^2}} + R\right],$$

where R is the radius of the cylinder and l its length, or the height of the mountain.

If R is so large compared to l that $\frac{l^2}{R^2}$ can be neglected, this reduces to $2\pi\delta l$. For the ordinary unit of mass we have, multiplying by $k = \frac{gr_0^2}{m_0}$ (page 205), for the acceleration due to the attraction of the mountain

$$2\pi\delta l \,.\, \frac{gr_0^2}{m_0}.$$

Let δ_0 be the mean density of the earth, so that $m_0 = \frac{4}{3}\pi\delta_0 r_0^3$. Then the acceleration due to the attraction of the mountain is

$$\frac{3}{2}\frac{\delta l}{\delta_0 r_0} \cdot g.$$

The acceleration due to the attraction of the earth is

$$\frac{r_0^2}{(r_0 + l)^2} \cdot g.$$

We have then for the acceleration at the distance l above sea-level

$$G = g\left[\frac{r_0^2}{(r_0 + l)^2} + \frac{3\delta l}{2\delta_0 r_0}\right].$$

The value of $\frac{\delta}{\delta_0}$ from what we know of the density of matter at the earth's surface, may be taken equal to $\frac{1}{2}$. Also we may write, approximately,

$$\frac{r_0^2}{(r_0 + l)^2} = 1 - \frac{2l}{r_0}.$$

Hence we have, approximately,

$$G = g\left(1 - \frac{2l}{r_0} + \frac{3l}{4r_0}\right) = g\left(1 - \frac{5l}{4r_0}\right),$$

where l is the height above sea-level, r_0 is the mean radius of the earth, and g the corresponding acceleration at sea-level.

The assumptions made in this determination are more applicable to elevated table-land than to a mountain. The equation obtained is the accepted formula for estimating the value of g at two places so far as dependent on the height above sea-level.

KINETICS OF A MATERIAL SYSTEM.

CHAPTER I.

GENERAL PRINCIPLES.

Material System.—A material system consists of an indefinitely large number of particles which act and react upon each other. Such a system may constitute a rigid body or a system of rigid bodies, an elastic body or a system of elastic bodies, a liquid or a gaseous mass.

Internal and External Forces.—The forces acting between the bodies or particles of a material system are internal forces or STRESSES (page 174). The forces exerted upon the system by other bodies or particles outside of the given system are external forces.

When one body or particle acts upon another with a certain force, it is itself acted upon by the other with an equal and opposite force (page 174). Hence the internal forces or stresses between any two bodies or particles of a material system consist of two equal and opposite forces in the same straight line.

It follows *that the internal forces or stresses existing between any system of bodies or particles must constitute a system of forces in equilibrium.*

Impressed and Effective Forces.—The external forces acting upon any material system we call IMPRESSED forces.

Any particle of the system of mass m has at any instant an acceleration f in a certain direction. By the equation of force (page 170) the force acting upon this particle is mf. This we call the EFFECTIVE force on the particle.

It is the force which, acting upon an isolated particle of the system at any instant, would make it move at that instant precisely as it really does move as part of the system.

D'Alembert's Principle.—Let m be the mass of a particle of any material system, and f its acceleration. Then its effective force as just defined is mf. Let F be the resultant impressed force on this particle, and S the resultant internal force or *stress* acting on the particle.

Then mf must be the resultant of F and S. Hence if mf be *reversed in direction*, we should have a system of forces F, S, and mf *reversed*, in equilibrium.

The same would hold for every other particle of the system. But we have just seen that for all the particles the internal forces S form by themselves a system in equilibrium.

Hence it follows that all the impressed forces F and *reversed effective* forces mf must also form a system of forces in equilibrium.

We have then the following general principle which *holds for any material system or body* whether solid, liquid or gaseous:

Consider all the effective forces as reversed in direction. Then the impressed forces acting upon the system, and the reversed effective forces, constitute a system of forces in equilibrium.

This is known as D'ALEMBERT'S PRINCIPLE. It reduces any dynamic problem to one of equilibrium between actual ("*impressed*") forces and fictitious ("*reversed effective*") forces.

Velocity of Centre of Mass—Momentum of a System.—Let m_1, m_2, m_3, etc., be the masses of the particles of any material system, s_1, s_2, s_3, etc., the distances of these particles from any given plane, $\bar{s}$ the distance of the centre of mass of the system from that plane, and $\overline{\mathrm{m}} = \Sigma m$ the mass of the system. Then we have (page 21)

$$\bar{s} = \frac{m_1 s_1 + m_2 s_2 + m_3 s_3 + \text{etc.}}{\overline{\mathrm{m}}}$$

At the end of an indefinitely small time τ, let the distances be s_1', s_2', s_3', etc. Then

$$\bar{s}' = \frac{m_1 s_1' + m_2 s_2' + m_3 s_3' + \text{etc.}}{\overline{\mathrm{m}}}$$

If we subtract the first equation from the second and divide by τ, we have

$$\frac{\bar{s}' - \bar{s}}{\tau} = \frac{\dfrac{m_1(s_1' - s_1)}{\tau} + \dfrac{m_2(s_2' - s_2)}{\tau} + \text{etc.}}{\overline{\mathrm{m}}}$$

But if τ is indefinitely small, $\dfrac{\bar{s}' - \bar{s}}{\tau}$ is the velocity $\bar{v}$ of the centre of mass in a direction perpendicular to the given plane, and $\dfrac{s_1' - s_1}{\tau}$, $\dfrac{s_2' - s_2}{\tau}$, etc., are the velocities v_1, v_2, etc. of the particles in this direction. Hence

$$\bar{v} = \frac{m_1 v_1 + m_2 v_2 + m_3 v_3 + \text{etc.}}{\overline{\mathrm{m}}} = \frac{\Sigma mv}{\overline{\mathrm{m}}},$$

or

$$\overline{\mathrm{m}}\bar{v} = \Sigma mv. \qquad (1)$$

But $m_1 v_1$, $m_2 v_2$, etc., are the momentums of the particles in any given direction, and Σmv is then the momentum of the system in that direction.

Hence, *for any material system, the momentum in any direction is the same as for a particle of mass equal to the mass of the system moving with the velocity of the centre of mass in that direction.*

Also, *the velocity of the centre of mass in any direction is equal to the momentum of the system in that direction divided by the mass of the system.*

From (1) we have

$$\Sigma mv - \bar{v}\Sigma m = 0 = \Sigma m(v - \bar{v}).$$

But $\Sigma m(v - \bar{v})$ is the momentum of the system in any direction *relative to the centre of mass.*

Hence *the momentum of a material system in any direction relative to the centre of mass is zero.*

Acceleration of the Centre of Mass.—We find the acceleration of the centre of mass in precisely the same way. Thus if $\bar{v}_1$ is the initial and $\bar{v}$ the final velocity of the centre of mass of any material system in any given direction during an indefinitely small time τ, we have, from (1),

$$\bar{v}_1 = \frac{\Sigma m v_1}{\overline{\text{m}}}, \qquad \bar{v} = \frac{\Sigma m v}{\overline{\text{m}}}.$$

Subtracting the first from the second and dividing by τ, we obtain

$$\frac{\overline{\text{m}}(\bar{v} - \bar{v}_1)}{\tau} = \frac{\Sigma m(v - v_1)}{\tau}.$$

If τ is indefinitely small, $\frac{\bar{v} - \bar{v}_1}{\tau}$ is the acceleration $\bar{f}$ of the centre of mass in the given direction, and $\frac{\Sigma m(v - v_1)}{\tau}$ is the sum Σmf of all the effective forces in that direction. Hence

$$\overline{\text{m}}\bar{f} = \Sigma mf, \quad \text{or} \quad \bar{f} = \frac{\Sigma mf}{\overline{\text{m}}}. \quad \ldots \ldots \ldots \quad (2)$$

That is, *for any material system the effective force in any direction is the same as for a particle of mass equal to that of the system moving with the acceleration of the centre of mass in that direction.*

Also, *the acceleration of the centre of mass in any direction is equal to the acceleration of the system in that direction divided by the mass of the system.*

From (2) we have

$$\Sigma mf - \bar{f}\Sigma m = 0 = \Sigma m(f - \bar{f}).$$

But $\Sigma m(f - \bar{f})$ is the effective force of the system relative to the centre of mass.

Hence *the effective force of a system in any direction relative to the centre of mass is zero.*

Motion of Centre of Mass.—Let F_1, F_2, F_3, etc., be the components in any direction of all the impressed or external forces acting upon any material system, and $m_1 f_1$, $m_2 f_2$, $m_3 f_3$, etc., the components in the same direction of all the effective forces.

Then, by D'Alembert's principle (page 297), we have

$$\Sigma F - \Sigma mf = 0, \quad \text{or} \quad \Sigma F = \Sigma mf = \overline{\text{m}}\bar{f}. \quad \ldots \ldots \quad (3)$$

Hence *the motion of the centre of mass of any material system is not affected by the internal forces or stresses between the particles of the system, but only by the external or impressed forces.*

Also, *the acceleration of the centre of mass is the same as for a particle of mass equal to the mass of the system, all the impressed forces being transferred to this particle without change of magnitude or direction.* This principle we have already deduced for a rigid body (page 175). We see that it holds for any material system, rigid or not.

Conservation of Centre of Mass.—If, then, there are no external or impressed forces, $\Sigma F = 0$, $\bar{f} = 0$, and the motion of the centre of mass cannot change. If at rest, it remains at rest. If in motion, it moves with uniform speed in a straight line, no matter what the internal forces or stresses may be.

That is, *no material system can of itself and without the action of external forces change the motion of its centre of mass.*

This is called the principle of *conservation of centre of mass.*

Conservation of Momentum.—If v_1 is the initial and v the final velocity of a particle in any direction during an indefinitely small time τ, the acceleration f in that direction is $\frac{v - v_1}{\tau}$ and the effective force mf in that direction is $\frac{m(v - v_1)}{\tau}$. We have then, from (3),

$$\Sigma F = \Sigma mf = \Sigma \frac{m(v - v_1)}{\tau}.$$

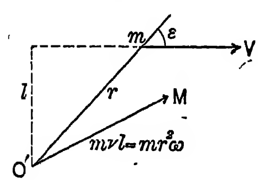
We see, then, that the change of momentum $\Sigma m(v - v_1)$ in any direction of a material system is not affected by the internal forces or stresses, but only by the impressed forces.

If then the impressed forces are zero, $\Sigma m(v - v_1)$ is zero, and the momentum of the system in any direction does not change.

That is, *no material system can of itself and without the action of external forces change its momentum in any direction.*

This is called the principle of *conservation of momentum.*

Moment of Momentum.—The *moment of momentum* of a particle relative to a given point or axis is the product of its mass by the moment of its velocity (page 89) relative to that point or that axis.



Thus let m be the mass of a particle, v its velocity, l the lever-arm or perpendicular from any point O' upon the direction of v. Then the moment of the velocity (page 89) is vl. Or if r is the radius vector of m, ϵ the angle of v with r, and ω the angular velocity relative to O', $r\omega = v \sin \epsilon$ and $v = \frac{r\omega}{\sin \epsilon}$. But $l = r \sin \epsilon$. Hence

$$vl = vr \sin \epsilon = r^2\omega.$$

The moment of momentum of the particle relative to O' is then

$$mvl = mr^2\omega.$$

Its line representative is a straight line $O'M$ at right angles to the plane of v and l, whose magnitude is $mvl = mr^2\omega$, and whose direction is such that looking along it in its direction, as shown by the arrow in the figure, the rotation is seen clockwise.

By "direction" of moment of momentum we always mean the direction of its line representative.

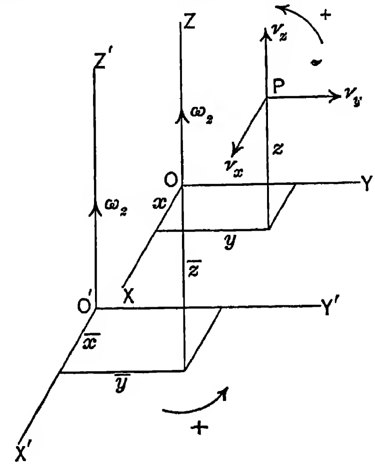
We see, then, that the moment of momentum of a particle relative to a point is represented in magnitude and direction by a straight line of definite magnitude and direction through that point, just like moment of velocity (page 89), and the same principles apply. The same holds for moment of momentum relative to an axis.

We can therefore combine and resolve moment of momentum just like moment of velocity, or just like velocity, acceleration or force, and can find the resultant for any number of concurring line representatives just as for force.

Moment of Momentum for a System.—For a material system each particle has its own velocity, and for any point O' its own moment of momentum of definite magnitude and direction, given by its line representative through O', as in the preceding figure. For the entire system we have then a number of line representatives concurring at O', and the resultant of all these is the moment of momentum of the system relative to O'.

The moment of momentum of a material system relative to any point reduces, then, to a resultant moment of momentum of definite magnitude about a definite axis through that point.

Let this axis be the axis of Z', and let $O'X'$, $O'Y'$ be the other two rectangular axes. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of mass O. With O as origin take parallel axes OX, OY, OZ, and let x, y, z be the co-ordinates of any particle P relative to O. Then the co-ordinates of P for origin O' are $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$.



Let $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ be the components of the velocity of the centre of mass O. Then the velocity components of P are

$$v_x = \bar{v}_x - y\omega_z, \qquad v_y = \bar{v}_y + x\omega_z,$$

where ω_z is the angular velocity relative to the axis OZ or $O'Z'$.

The moment of momentum for a particle of mass m at P is then given by

$$mv_y(\bar{x}+x) - mv_x(\bar{y}+y).$$

Substituting the values of v_x and v_y, we have for the entire system the moment of momentum

$$M = \Sigma m\bar{v}_y(\bar{x}+x) - \Sigma m\bar{v}_x(\bar{y}+y) + \Sigma m(x^2+y^2)\omega_z + \Sigma m(\bar{x}x+\bar{y}y)\omega_z.$$

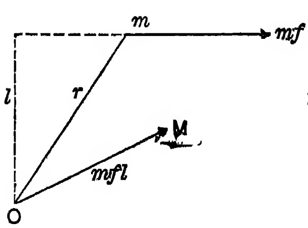
But $\Sigma m = \mathfrak{m}$ = mass of the system, $\bar{v}_x$ and $\bar{v}_y$ are constant, and $\Sigma mx = 0$, $\Sigma my = 0$, since O is the centre of mass. Hence

$$M = \mathfrak{m}\bar{v}_y\bar{x} - \mathfrak{m}\bar{v}_x\bar{y} + \Sigma m(x^2+y^2)\omega_z.$$

But $\Sigma m(x^2+y^2)\omega_z$ is the moment of momentum of the system about an axis parallel to the given axis, through the centre of mass O, and $\mathfrak{m}\bar{v}_y\bar{x} - \mathfrak{m}\bar{v}_x\bar{y}$ is the moment of momentum of a particle of mass $\mathfrak{m}$ at the centre of mass.

Hence *the moment of momentum of a material system about a given axis is equal to the moment of momentum about a parallel axis through the centre of mass plus the moment of momentum of a particle of mass equal to the mass of the system, situated at the centre of mass.*

Acceleration of Moment of Momentum.—If v_1 is the initial and v the final velocity of a particle of mass m, in any given direction, during an indefinitely small time τ, then $\frac{v - v_1}{\tau}$ is the acceleration f *in that direction*, and



$$mf = \frac{m(v - v_1)}{\tau}$$

is the effective force in that direction.

The moment of this force relative to any point O' is then

$$mfl = \frac{m(v - v_1)l}{\tau} = \frac{mr^2(\omega - \omega_1)}{\tau} = mr^2\alpha,$$

where r is the radius vector, ω_1 and ω are the initial and final angular velocities and α is the angular acceleration. The line representative is a straight line $O'M$ at right angles to the plane of f and l, whose magnitude is mfl and whose direction is such that, looking along it in its direction as shown by its arrow in the figure, the rotation is seen clockwise. By "direction" of moment of a force we always mean the direction of its line representative.

But $\frac{mvl - mv_1 l}{\tau}$ is the acceleration of the moment of momentum in the direction $O'M$ of the particle relative to O'.

For the entire system we have for the acceleration of moment of momentum in this direction

$$\Sigma mfl = \Sigma \frac{m(v - v_1)l}{\tau} = \Sigma mr^2\alpha.$$

But, by D'Alembert's principle (page 297), the external or impressed forces and the *reversed* effective forces form a system of forces in equilibrium. Hence

$$\Sigma Fl = \Sigma mfl,$$

or

$$\Sigma Fl = \Sigma \frac{m(v - v_1)l}{\tau} = \Sigma mr^2\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, *for a particle or for a material system of particles the acceleration of the moment of momentum in any direction relative to any point is equal to the moment in that direction of all the impressed forces relative to that point.*

Conservation of Moment of Momentum.—If now $\Sigma Fl = 0$, we have

$$\Sigma m(v - v_1)l = 0,$$

or the change of moment of momentum in any direction relative to any point is zero. But we have seen (page 301) that the moment of momentum of a particle or of a system relative to any point has a definite magnitude and direction. Its projection, then, in any direction does not change when $\Sigma Fl = 0$, and it is therefore invariable in magnitude and direction.

This is called the principle of *conservation of moment of momentum*, and it may be expressed generally as follows:

When the resultant moment ΣFl of all the impressed forces acting upon a particle or upon a material system, relative to any point or any axis, is always zero, the moment of momentum relative to that point or axis does not change either in magnitude or direction, and its line representative is then a straight line of constant magnitude and invariable direction, passing through that point or coinciding with that axis.

Now ΣFl is always zero for all points when there are no impressed forces; also when all the impressed forces are always in equilibrium. In both cases the principle of conservation of moment of momentum holds for any point, and we have an invariable axis for any point.

Also, ΣFl is always zero for any one given point when all the impressed forces always pass through that point or when they always reduce to a single resultant through that point. In this case the principle holds *for that point*, and we have an invariable axis *through that point.*

Also, ΣFl is always zero for any axis when all the impressed forces always pass through that axis or are always parallel to that axis. In this case the principle holds *for this axis.*

Invariable Axis and Plane.—In all these cases we have an invariable axis either through any point, or through a given point, or coinciding with a given axis. But from the principle of page 301 we see that in all cases we have a *parallel axis through the centre of mass*, whether the centre of mass is fixed or not.

This axis through the centre of mass, invariable in direction, is called the INVARIABLE AXIS of the system, and the plane through the centre of mass at right angles to it is the INVARIABLE PLANE of the system.

Thus, in the solar system, if there are no forces external to the system, we have an invariable axis through the centre of mass of the system and an invariable plane at right angles to this axis, through the centre of mass. Also, if for each planet and satellite the force of attraction on every particle always passes through the centre of mass of the system, we have for each planet and satellite an invariable axis through that point. The moment of momentum of each does not change in magnitude, and lies along its own invariable axis. The resultant of all is the moment of momentum of the system, and lies along its invariable axis.

If we take the axis of Z as the invariable axis of a system for which the moment of momentum is constant in magnitude, the projection on the axes of X and Y will be zero. *Hence, for the invariable axis, the moment of momentum of a system is a maximum.*

Conservation of Areas.—The moment vl of a velocity is equal to twice the areal velocity of the radius vector (page 89). The moment of momentum mvl of a particle is then proportional to twice the areal velocity of the radius vector.

Hence, from the principle of conservation of moment of momentum,

When the resultant moment ΣFl of all the impressed forces acting upon a particle, relative to any point or any axis, is always zero, the areal velocity of the radius vector of the particle does not change either in magnitude or direction.

For a system the areal velocity of any particle relative to the invariable axis does not change.

This is called the *principle of conservation of areas.* Thus, in the solar system, the areal velocity of the radius vector of any planet does not change. This is Kepler's second law (page 120).

Kinetic Energy of a System.—The kinetic energy of a material system is the sum of the kinetic energy of all its particles, or

$$\mathcal{K} = \frac{1}{2}\Sigma mv^2$$

Let $\overline{v}_x$, $\overline{v}_y$, $\overline{v}_z$ be the component velocities of the centre of mass O relative to the origin O', and v_x, v_y, v_z the component velocities of any particle relative to the centre of mass O. Then the component velocities of the particle relative to O' are

$$\overline{v}_x + v_x, \qquad \overline{v}_y + v_y, \qquad \overline{v}_z + v_z,$$

and the kinetic energy of the system is

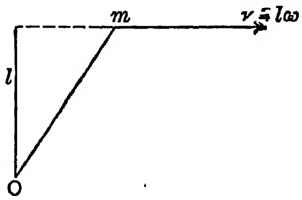
$$\mathcal{K} = \frac{1}{2}\Sigma m[(\overline{v}_x + v_x)^2 + (\overline{v}_y + v_y)^2 + (\overline{v}_z + v_z)^2].$$

Expanding, we can write

$$\mathcal{K} = \frac{1}{2}\Sigma m(\overline{v}_x^2 + \overline{v}_y^2 + \overline{v}_z^2) + \frac{1}{2}\Sigma m(v_x^2 + v_y^2 + v_z^2) + \overline{v}_x\Sigma mv_x + \overline{v}_y\Sigma mv_y + \overline{v}_z\Sigma mv_z.$$

But (page 298) $\Sigma mv_x = 0$, $\Sigma mv_y = 0$, $\Sigma mv_z = 0$. Also, $\frac{1}{2}\Sigma m(v_x^2 + v_y^2 + v_z^2)$ is the kinetic energy of all the particles moving with velocities equal to their velocities relative to the centre of mass O, and $\frac{1}{2}\Sigma m(\overline{v}_x^2 + \overline{v}_y^2 + \overline{v}_z^2) = \frac{1}{2}\mathfrak{m}(\overline{v}_x^2 + \overline{v}_y^2 + \overline{v}_z^2)$ is the kinetic energy of a particle of mass $\mathfrak{m}$ equal to the mass of the system moving with the velocity of the centre of mass.

Hence *the kinetic energy of a system is equal to the sum of the kinetic energy of the particles relative to the centre of mass plus the kinetic energy of a particle of mass equal to the mass of the system moving with the velocity of the centre of mass.*



If then l is the lever-arm of the velocity v relative to O, and ω is the corresponding angular velocity, we have $v = l\omega$, and the kinetic energy $\frac{1}{2}ml^2\omega^2$. The kinetic energy of all the particles relative to O is then $\frac{1}{2}\Sigma ml^2\omega^2$. If $\overline{v}$ is the velocity of the centre of mass, and $\overline{m}$ the mass of the system, we have then for the kinetic energy of the system

$$\mathcal{K} = \frac{1}{2}\overline{m}\overline{v}^2 + \frac{1}{2}\Sigma ml^2\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Potential Energy of a System.—The potential energy of a system is the sum of the potential energies of its particles. Since the potential energy of each particle depends upon its position (page 274), the potential energy of the system depends upon its configuration and position. If no external forces act upon the system, its potential energy will depend upon its configuration only.

Law of Energy.—Since the law of energy (page 275) holds for every particle of a system, it holds for the entire system.

Therefore, if $\mathcal{E}_1$ is the total initial and $\mathcal{E}$ the total final energy, potential and kinetic, of a system, and if W is the work done against non-conservative forces, we have

$$\mathcal{E}_1 - \mathcal{E} = W,$$

or the total loss of energy is equal to the work done against non-conservative forces. Such a system is called a non-conservative system.

Conservation of Energy.—If there are no non-conservative forces W is zero, and we have

$$\mathcal{E}_1 - \mathcal{E} = 0,$$

or the total energy does not change. Such a system is called a CONSERVATIVE system.

Perpetual Motion.—When a material system after a series of changes returns to its original position and configuration, its potential energy is not changed; and if the system is conservative, the kinetic energy is unchanged. Such a system would then go through its cycle of changes for ever if set in motion.

If the bodies of the solar system encounter no resistance, we have a system of this kind. If we could abolish friction and all other non-conservative forces, a machine could be constructed which, once started, would run forever, provided no work is done by it.

Since friction and other non-conservative forces cannot be thus abolished, such a machine is not possible.

Law of Conservation of Energy General.—The law of conservation of energy just enunciated

$$\mathcal{E}_1 - \mathcal{E} = 0$$

appears as a special case of the law of energy

$$\mathcal{E}_1 - \mathcal{E} - W$$

when W the work against non-conservative forces, such as friction, is zero. In general, then, when work is done against such forces we say that energy is "lost." Experiment shows that when energy is thus "lost" heat is always developed, and that heat is a form of energy dependent upon kinetic energy of particles. This heat is the exact equivalent of the energy "lost." If then we take into account heat energy, the law of conservation of energy becomes general, and we have in all cases, whether the forces are conservative or non-conservative,

$$\mathcal{E}_1 - \mathcal{E} = 0.$$

That is, *the entire energy of an isolated system, including kinetic, potential, and thermal, is constant.*

Examples.—(1) *What effect has the bursting of a bomb upon the motion of its centre of mass?*

ANS. Neglecting air resistance, none whatever. By the principle of conservation of centre of mass, internal forces cannot affect the motion of the centre of mass.

(2) *Show that the centre of mass of the universe must either be fixed in space or move in a straight line with uniform speed.*

ANS. By the principle of conservation of centre of mass no material system can of itself and without the action of external force change the motion of its centre of mass.

(3) *Two particles of mass m_1 and m_2 are moving in the same straight line with velocities v_1 and v_2. Find the velocity of their centre of mass.*

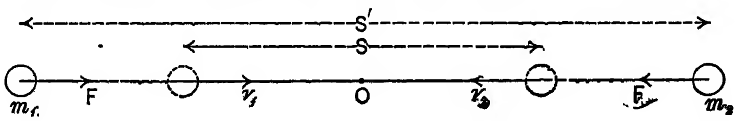
ANS. From equation (1), page 298, $\bar{v} = \frac{\Sigma mv}{\mathfrak{m}}$.

We have $\mathfrak{m} = m_1 + m_2$ and $\Sigma mv = m_1v_1 + m_2v_2$ if the velocities are in the same direction. Hence $\bar{v} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$ for velocities in the same direction and

$$\bar{v} = \frac{m_1v_1 - m_2v_2}{m_1 + m_2} \text{ if } v_2 \text{ is opposite to } v_1.$$

(4) *Two particles of mass m_1 and m_2 at a distance s' are at rest on a smooth horizontal plane and are drawn together by a uniform force F. After a time t the mass m_2 has a velocity v_2. Find the final velocity v_1 of the mass m_1, the internal force F, the distance s apart at the end of the time t, and the distance of each mass from the centre of mass.*

ANS. Let $\bar{v}$ be the velocity of the centre of mass O. Then, since v_1 and v_2 are opposite in direction, we have, by the principle of velocity of centre of mass, page 298,



$$\mathfrak{m}\bar{v} = \Sigma mv,$$

or $$(m_1 + m_2)\bar{v} = m_1v_1 - m_2v_2.$$

Hence $$\bar{v} = \frac{m_1v_1 - m_2v_2}{m_1 + m_2}.$$

But the centre of mass is initially at rest, or $\bar{v} = 0$, and by the principle of conservation of centre of mass (page 300) it must remain at rest. Hence

$$m_1v_1 - m_2v_2 = 0, \quad \text{or} \quad v_1 = \frac{m_2v_2}{m_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We also obtain this result directly from the principle of conservation of momentum (page 300).

Since the force F on m_1 is uniform, the acceleration is uniform and the distance passed over by m_1 in the time t, starting from rest, is $\frac{v_1}{2}t$. The distance passed over by m_2 is $\frac{v_2}{2}t$. The distance $s' - s$ is then given by

$$s' - s = \frac{(v_1 + v_2)}{2}t, \quad \text{or} \quad s = s' - \frac{(v_1 + v_2)}{2}t = s' - \frac{(m_1 + m_2)v_2t}{2m_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By the principle of impulse (page 257) the uniform force F on m_1 is $\frac{m_1 v_1}{t}$, and on m_2, $\frac{m_2 v_2}{t}$. Since these forces are equal and opposite, we have

$$F = \frac{m_1 v_1}{t} = \frac{m_2 v_2}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which we obtain again (1).

Again, the initial energy is $\mathfrak{E}_1 = Fs'$, potential. The final energy $\mathfrak{E}$ is Fs potential and $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$ kinetic, or

$$\mathfrak{E} = Fs + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2.$$

By the principle of conservation of energy, $\mathfrak{E}_1 = \mathfrak{E}$, or

$$Fs' = Fs + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2,$$

or

$$F = \frac{m_1 v_1^2 + m_2 v_2^2}{2(s' - s)}.$$

Inserting the value of $s' - s$ from (2), and of v_1 from (1), we obtain again

$$F = \frac{m_1 v_1}{t} = \frac{m_2 v_2}{t}.$$

The distance of m_1 from the centre of mass C at the start is

$$r_1' = \frac{m_2 s'}{m_1 + m_2},$$

and at the end of the time t

$$r_1 = \frac{m_2 s}{m_1 + m_2}.$$

The distance of m_2 from the centre of mass C at the start is

$$r_2' = \frac{m_1 s'}{m_1 + m_2},$$

and at the end of the time t

$$r_2 = \frac{m_1 s}{m_1 + m_2}.$$

If $m_1 = 50$ lbs., $m_2 = 100$ lbs., $v_2 = 10$ ft. per sec., $t = \frac{1}{20}$ sec., $s' = 3$ ft., we have

$$v_1 = 20 \text{ ft. per sec.}; \quad s = 2.25 \text{ ft.}; \quad F = 20000 \text{ poundals} = \frac{20000}{g} \text{ pounds};$$

$$r_1' = 2 \text{ ft.}; \quad r_1 = 1.5 \text{ ft.}; \quad r_2' = 1 \text{ ft.}; \quad r_2 = 0.75 \text{ ft.}$$

(5) *In the preceding example suppose the particles revolve about the centre of mass with the initial angular velocity ω'.*

ANS. Take the same notation as before and let ω_1 be the angular velocity of m_1 at the distance r_1, and ω_2 of m_2 at the distance r_2. We have the same values for v_2, $s' - s$, r_1', r_1, r_2' and r_2, as before. It is therefore only required to find F.

We have, by the principle of conservation of areas,

$$r_1'^2 \omega' = r_1^2 \omega_1 \quad \text{and} \quad r_2'^2 \omega' = r_2^2 \omega_2,$$

hence

$$\omega_1 = \frac{r_1'^2}{r_1^2} \omega' = \frac{s'^2}{s^2} \omega',$$

$$\omega_2 = \frac{r_2'^2}{r_2^2} \omega' = \frac{s'^2}{s^2} \omega',$$

or ω_1 and ω_2 are equal, as they should be, since, by the principle of conservation of centre of mass, the centre of mass is fixed.

The initial energy of the system is

$$\mathcal{E}_1 = Fs' + \frac{1}{2}m_1r_1'^2\omega'^2 + \frac{1}{2}m_2r_2'^2\omega'^2.$$

The final energy of the system is

$$\mathcal{E} = Fs + \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_1r_1^2\omega_1^2 + \frac{1}{2}m_2r_2^2\omega_2^2.$$

By the conservation of energy, $\mathcal{E}_1 = \mathcal{E}$, and inserting the values of v_2, ω_1, ω_2, r_1, r_2, r_1', r_2', we have

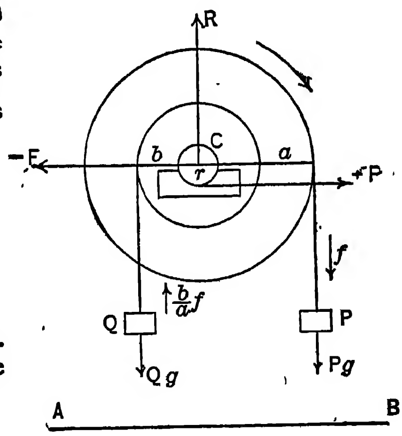
$$F(s' - s) = \frac{m_1(m_1 + m_2)v_1}{2m_2} + \frac{m_1m_2s'^2\omega'^2(s'^2 - s^2)}{2(m_1 + m_2)s^2}.$$

Hence

$$F = \frac{m_1v_1}{t} + \frac{m_1m_2s'^2\omega'^2(s' + s)}{2(m_1 + m_2)s^2}.$$

(6) *In a wheel and axle the radius of the wheel is $a = 3$ ft., of the axle $b = 2$ ft. Let $r = 1$ inch be the radius of the journal, and $\mu = 0.07$ be the coefficient of kinetic friction. Let the moving mass $P = 10$ lbs. and the mass lifted $Q = 5$ lbs. Let P start from rest and fall for a time $t = 5$ sec. Disregarding rigidity of the rope and mass of rope, wheel and axle, discuss the apparatus.* ($g = 32\frac{1}{6}$.)

ANS. The impressed forces are Pg and Qg down, the upward pressure of the axle R and the forces of friction which form a couple $+F, -F$ with lever-arm r as shown in the figure. The effective forces are Pf down and $Q\frac{b}{a}f$ up. *Reversing* these, we have, by D'Alembert's principle (page 297),



$$R - Pg - Qg + Pf - Q\frac{b}{a}f = 0,$$

or

$$R = (P + Q)g - \left(P - Q\frac{b}{a}\right)f \textit{ poundals},$$

if we take masses in lbs., distances in ft. and g, f in ft.-per-sec. per sec.

The friction for new bearing (page 229) is then, if β is the angle of bearing,

$$F = \frac{\mu\beta}{\sin\beta}R = \frac{\mu\beta}{\sin\beta}\left[(P + Q)g - \left(P - Q\frac{b}{a}\right)f\right].$$

Taking moments about the centre, we have, by D'Alembert's principle,

$$Qgb + Q\frac{b^2}{a}f - Pga + Pfa + Fr = 0,$$

or inserting the value for F just found and solving for f, we have

$$f = \frac{\left(P - Q\frac{b}{a}\right) - \frac{\mu r\beta}{a\sin\beta}(P + Q)}{\left(P + Q\frac{b^2}{a^2}\right) - \frac{\mu r\beta}{a\sin\beta}\left(P - Q\frac{b}{a}\right)}\cdot g.$$

If we disregard friction $\mu = 0$, and take $a = b$, we have the same value for f as already found in example (2), page 243.

If the angle of bearing β is small, $\sin\beta = \beta$, and we have

$$f = \frac{\left(P - Q\frac{b}{a}\right) - \frac{\mu r}{a}(P+Q)}{\left(P + Q\frac{b^2}{a^2}\right) - \frac{\mu r}{a}\left(P - Q\frac{b}{a}\right)} \cdot g = 0.544g = 17.49 \text{ ft.-per-sec. per sec.}$$

Again, let the centres of mass of P and Q be initially at the same distance h above any plane AB. The initial energy is then

$$\mathfrak{E}_1 = Pgh + Qgh.$$

At the end of the time t suppose P has fallen a distance s and Q risen a distance $\frac{bs}{a}$, while the mass P has the velocity v and the mass Q the velocity $\frac{b}{a}v$. Then the final energy of P is $Pg(h-s)$ potential and $\frac{P}{2}v^2$ kinetic, and of Q, $Qg\left(h + \frac{bs}{a}\right) + \frac{Qb^2}{2a^2}v^2$. The total final energy is then

$$\mathfrak{E} = Pg(h-s) + \frac{P}{2}v^2 + Qg\left(h + \frac{bs}{a}\right) + \frac{Qb^2}{2a^2}v^2.$$

The work consumed by friction is $F\frac{r}{a}s$. By the law of energy the loss of energy is equal to the work consumed by friction, or

$$\mathfrak{E}_1 - \mathfrak{E} = F\frac{r}{a}s.$$

Substituting the values of $\mathfrak{E}_1$, $\mathfrak{E}$ and F already found, we obtain

$$Pgs - Qg\frac{b}{a}s = \frac{P}{2}v^2 + \frac{Q}{2}\cdot\frac{b^2}{a^2}v^2 + \frac{r\mu\beta}{a\sin\beta}s\left[(P+Q)g - \left(P - Q\frac{b}{a}\right)f\right].$$

That is, the loss of potential energy, $Pgs - Q\frac{b}{a}s$, equals the gain of kinetic energy, $\frac{P}{2}v^2 + \frac{Q}{2}\cdot\frac{b^2}{a^2}v^2$, plus the work of overcoming friction. Also, the loss of potential energy of P or Pgs equals the gain of potential energy of Q or $Qg\frac{b}{a}s$, plus the gain of kinetic energy of the system, plus the work of overcoming friction.

If we substitute $s = \frac{1}{2}ft^2$, $v = ft$ and solve for f, we obtain the same value for f as already obtained.

The acceleration of Q is then

$$\frac{b}{a}f = 0.363g = 11.66 \text{ ft.-per-sec. per sec.}$$

The velocity of P at the end of $t = 5$ sec. is

$$v = ft = 87.44 \text{ ft. per sec.},$$

and the velocity of Q

$$\frac{b}{a}v = 58.29 \text{ ft. per sec.}$$

The distance passed through by P is

$$s = \frac{1}{2}ft^2 = 218.59 \text{ ft.},$$

and by Q

$$\frac{b}{a}s = 145.73 \text{ ft.}$$

Tension in string for P is $P(g-f) \quad = 4.56g$ poundals or 4.56 pounds;

" " " " Q " $Q\left(g+\frac{b}{a}f\right) = 6.82g$ " " 6.82 "

$$\text{work of } P = 4.56s \qquad = 996.77 \text{ ft.-lbs.} = Ps - \frac{Pv^2}{2g};$$

$$\text{" on } Q = 6.82 \times \frac{b}{a}s = 993.88 \quad \text{"} \quad = Q\frac{b}{a}s + \frac{Q}{2g}\cdot\frac{b^2}{a^2}v^2.$$

The difference of these works, or 2.89 ft.-lbs., is the work consumed by friction.

The power of P is $\frac{996.77}{5} = 199.35$ ft.-lbs. per sec., or $\frac{199.35}{550} \mp 0.36$ horse-power.

The efficiency is

$$e = \frac{993.88}{996.77} = 0.998.$$

CHAPTER II.

EQUILIBRIUM OF A MATERIAL SYSTEM.

Equilibrium of a Material System.—We have seen (page 299) that the motion of the centre of mass of a material system is not affected by the internal forces, but only by the external forces. Also, the acceleration of the centre of mass is the same as for a particle of mass equal to the mass of the system, all the external forces being transferred to this particle without change of magnitude or direction.

If all these forces constitute a system of forces in equilibrium, and every particle of the material system is at rest, it is said to be in *static* equilibrium.

If not at rest, since the external forces constitute a system of forces in equilibrium, the centre of mass has no acceleration, and the resultant moment ΣFl of all these forces relative to any point is zero. But we have seen (page 302) that when ΣFl is zero the moment of momentum does not change, and we have then uniform angular velocity of every particle about an invariable axis through the centre of mass, and uniform velocity of translation of this axis. A system in this condition is said to be in *molar* equilibrium.

If there is no rotation about an axis, but simply uniform translation in a straight line, then the forces acting upon every particle are in equilibrium, and the system is said to be in *molecular* equilibrium.

The necessary and sufficient conditions of molar equilibrium are then

$$F_x = 0, \qquad F_y = 0, \qquad F_z = 0,$$
$$M_x = 0, \qquad M_y = 0, \qquad M_z = 0.$$

That is, the external forces acting upon the system form a system of forces in equilibrium, or the algebraic sum of the components of all the external forces in any direction is zero, and the algebraic sum of the moments of these forces about any axis is zero.

For molecular equilibrium, we have, in addition to these conditions, the condition that all the forces external and internal *on every particle* form a system of forces in equilibrium.

For static equilibrium we have still the added condition that every particle is at rest.

In both molecular and static equilibrium, then, the work for an indefinitely small change of position or configuration must be zero. Let Σw be the work of all the external and $\Sigma w'$ of all the internal forces for an indefinitely small change of position or configuration. Then we have

$$\Sigma w + \Sigma w' = 0, \quad \text{or} \quad \Sigma w = -\Sigma w'.$$

If the system is rigid, $\Sigma w' = 0$, and hence $\Sigma w = 0$.

ILLUSTRATIONS.—Thus a billiard-ball at rest on a table is in static equilibrium. All the external forces acting upon it form a system of forces in equilibrium, and every particle is at rest. Hence the forces acting on every particle must form a system in equilibrium. For an indefinitely small change of position or configuration the work is zero.

If the ball rotates about a fixed vertical axis with uniform angular velocity, it is in molar equilibrium. All the external forces acting upon it form a system of forces in equilibrium, but the forces acting upon every particle do not. Since the moment of momentum cannot change, the centre of mass is at rest, and the ball must always rotate about the vertical axis with uniform angular velocity. For an indefinitely small change of position or configuration the work is not zero.

If the ball rolls so that the centre of mass moves with uniform speed in a straight line, it is also in molar equilibrium.

If the ball is projected through the air without rotation, so that every particle has uniform motion of translation only in a straight line, the ball is in molecular equilibrium. All the external forces form a system in equilibrium, and the internal forces on every particle also form a system in equilibrium. For any indefinitely small change of position or configuration the work is zero.

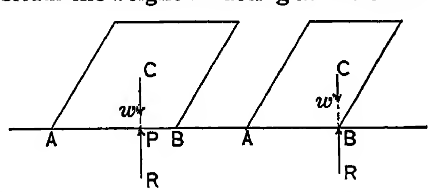
If the particles of a mass of water in a horizontal circular dish are rotating about a vertical axis through the centre of mass and are acted upon only by gravity the mass is in molar equilibrium. The centre of mass, is fixed, and no particle, disregarding friction, can change its moment of momentum about the vertical axis.

A beam or spring bent by a load and at rest is in static equilibrium. If every particle has uniform motion of translation only in a straight line, it is in molecular equilibrium. In both cases, for an indefinitely small change of configuration, the work of the load is equal and opposite to the work of the internal forces.

Stable Equilibrium.—The same conditions must hold for stable equilibrium of a material system as for each particle (page 277).

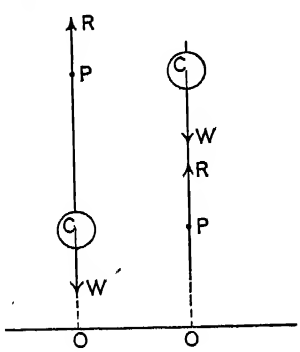
Hence for stable equilibrium the potential energy is a minimum and the kinetic energy, if any, is a maximum. If the potential energy is a maximum, the equilibrium is unstable and the kinetic energy, if any, a minimum. If the potential energy is neither a maximum nor a minimum, the equilibrium is stable for displacements for which the potential energy increases, and unstable for displacements for which the potential energy decreases. If the potential energy is constant, the equilibrium is neutral. If for all possible displacements, large or small, the potential energy is constant, the equilibrium is indifferent.

ILLUSTRATIONS.—Let a prism stand on a level base. For equilibrium the weight W acting at the centre of mass C, and the resultant upward pressure R on the base at P, must be equal and opposite and in the same straight line.



If the prism is so constrained that it can have displacement of translation only along a horizontal plane, the potential energy $W \times CP$ does not change for any displacement large or small, and the equilibrium is indifferent.

If it is possible to rotate the prism about an edge at B or A, the potential energy is increased, *so long as the weight W falls inside the base*, and the equilibrium is stable.



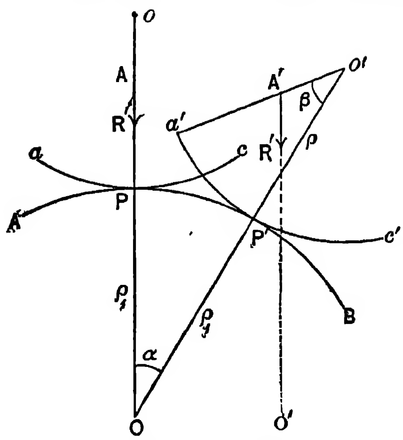
If, however, the weight W cuts the base at an edge B, then for rotation about A the potential energy increases and the equilibrium is stable, while for rotation about B the potential energy decreases and the equilibrium is unstable.

Again, if a pendulum hangs vertically with the bob below the point of suspension P, the potential energy $W \times CO$ is a minimum for all possible displacements and the equilibrium is stable. If the pendulum is reversed, the potential energy is a maximum and the equilibrium is unstable. If the pendulum swings, the kinetic energy is a maximum in the first case and a minimum in the second case.

Principle of Least Work.—The work done by external forces in changing the configuration of an elastic body can be given back by the body when the external forces are removed. It is therefore potential energy. If such a body is in static or molecular equilibrium, the forces external and internal form a system of forces in equilibrium. If the equilibrium *is stable*, the work done in causing the change of configuration *must be a minimum* consistent with the conditions of equilibrium of the forces.

This is called the "principle of least work" for elastic bodies.

Stability in Rolling Contact.—As an application of the preceding, let us investigate the equilibrium of a body with a curved surface resting and rolling upon a curved surface.



Let O be the centre of curvature of the fixed surface APB, and o the centre of curvature of the body aPc resting on it at P.

The reaction R at P is for equilibrium, equal and opposite to the resultant R' acting at A, of all the other forces, and in the same straight line.

Let aPc be displaced by rolling through an indefinitely small angle so that it comes into the position $a'P'c'$, A' and o' being the new positions of A and o. Let the radius of curvature $oP = \rho$ and $OP = \rho_1$, and the angle $a'o'P' = \beta$ and $POP' = \alpha$. Let the distance $PA = h$.

We have then $PP' = a'P'$, or

$$\rho_1 \alpha = \rho\beta.$$

Take a plane OX at right angles to R'. The equilibrium will be stable when the potential energy is a minimum, or when AO is less than $A'O'$, and unstable when the potential energy is a maximum, or when AO is greater than $A'O'$.

The distance $AO = \rho_1 + h$.

We have then for stable equilibrium

$$\rho_1 + h < A'O',$$

for unstable equilibrium

$$\rho_1 + h > A'O',$$

and for neutral equilibrium

$$\rho_1 + h = A'O'.$$

Now we have

$$A'O' = (\rho + \rho_1) \cos \alpha - (\rho - h) \cos (\alpha + \beta).$$

But, by Trigonometry,

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

and

$$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}, \qquad \cos \beta = 1 - 2 \sin^2 \frac{\beta}{2}.$$

Substituting, we have, after reduction,

$$A'O' = \rho_1 + h - 2\rho_1 \sin^2 \frac{\alpha}{2} + 2\rho \sin^2 \frac{\beta}{2}\left(1 - 2 \sin^2 \frac{\alpha}{2}\right) + \rho \sin \alpha \sin \beta$$
$$- h \sin \alpha \sin \beta - 2h \sin^2 \frac{\alpha}{2} - 2h \sin^2 \frac{\beta}{2}\left(1 - 2 \sin^2 \frac{\alpha}{2}\right).$$

Now when α is very small, $2 \sin^2 \frac{\alpha}{2}$ can be neglected relative to 1, and we can take the arc α in place of $\sin \alpha$. Hence, since $\beta = \frac{\rho_1}{\rho}\alpha$, we can take

$$\sin \alpha = \alpha, \qquad \sin \beta = \frac{\rho_1}{\rho}\alpha, \qquad \sin^2 \frac{\alpha}{2} = \frac{\alpha^2}{4}, \qquad \sin^2 \frac{\beta}{2} = \frac{\rho_1^2 \alpha^2}{4\rho^2}.$$

Making these substitutions, we have

$$A'O' = \rho_1 + h - \frac{\rho_1\alpha^2}{2} + \frac{\rho_1^2\alpha^2}{2\rho} + \rho_1\alpha^2 - \frac{h\rho_1\alpha^2}{\rho} - \frac{h\alpha^2}{2} - \frac{h\rho_1^2\alpha^2}{2\rho^2}.$$

We have then for stable equilibrium

$$\rho_1 + h < A'O', \quad \text{or} \quad h < \frac{\rho_1\rho}{\rho+\rho_1}, \quad \text{or} \quad \frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho};$$

for unstable equilibrium

$$\rho_1 + h > A'O', \quad \text{or} \quad h > \frac{\rho_1\rho}{\rho+\rho_1}, \quad \text{or} \quad \frac{1}{h} < \frac{1}{\rho_1} + \frac{1}{\rho};$$

while for neutral equilibrium we have

$$\rho_1 + h = A'O', \quad \text{or} \quad h = \frac{\rho_1\rho}{\rho+\rho_1}, \quad \text{or} \quad \frac{1}{h} = \frac{1}{\rho_1} + \frac{1}{\rho}.$$

If the concavity of either surface is turned the other way, we have the same result except that the sign of the corresponding radius will be changed. If either surface is plane, its radius is infinite.

The same results can be obtained more simply as follows: The equilibrium will be stable, unstable or neutral according as A' lies to the left, right or directly over P', that is, according as the horizontal distance of A' from P is less, greater than or equal to the horizontal distance of P' from P.

The horizontal distance of P' from P is $\rho_1 \sin\alpha$. The horizontal distance of A' from d is the same as from a', or $h \sin(\alpha+\beta)$.

Hence we have

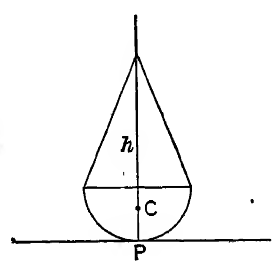
$$h \sin(\alpha+\beta) \overset{<}{\underset{>}{=}} \rho_1 \sin\alpha.$$

Inserting $\beta = \frac{\rho_1}{\rho}\alpha$, we have for α very small, so that we can take the arc for the sine, the same conditions as before.

Examples.—(1) *A body made up of a cone and a hemisphere having a common base rests with the axis vertical on a horizontal plane. Find the greatest height of the cone for stable equilibrium.*

ANS. Let h be the height of the cone, r the radius of the hemisphere, and C the centre of mass. The height required is that height for which $PC = r$.

The volume of the hemisphere is $\frac{2}{3}\pi r^3$. The volume of the cone is $\frac{1}{3}\pi r^2 h$.



The centre of mass of the hemisphere is at a distance above P equal to $\frac{5}{8}r$. The centre of mass of the cone is at a distance above P equal to $r + \frac{h}{4}$. We have then

$$PC = \frac{\frac{2}{3}\pi r^3 \times \frac{5}{8}r + \frac{\pi r^2 h}{3} \times \left(r + \frac{h}{4}\right)}{\frac{2}{3}\pi r^3 + \frac{\pi r^2 h}{3}} = r, \quad \text{or} \quad h = r\sqrt{3}.$$

(2) *A prolate spheroid rests with its axis horizontal on a rough horizontal plane. Show that for rolling displacement in its equatorial plane the equilibrium is indifferent, and for rolling displacement in the vertical plane through the axis it is stable.*

(3) *A right circular cylinder of radius r rests with its axis horizontal on a fixed sphere of radius R greater than r. Show that for rolling displacement the equilibrium is stable or unstable according as the plane of displacement makes an angle with the vertical plane through the axis of the cylinder whose sine is less or greater than* $\sqrt{1-\frac{r}{R}}$.

ANS. Let ρ be the radius of curvature of the rolling curve at the point of contact. Then the condition for stable equilibrium is

$$\frac{1}{r} > \frac{1}{R} + \frac{1}{\rho}.$$

Let the plane of displacement make the angle θ with the vertical plane through the axis of the cylinder. The rolling curve is then an ellipse whose semi-minor axis is r and whose semi-major axis is $\frac{r}{\sin\theta}$. The radius of curvature at the point of contact, that is, at the vertex of the minor axis, is

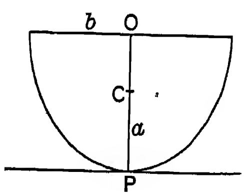
$$\rho = \frac{\left(\frac{r}{\sin\theta}\right)}{r} = \frac{r}{\sin^2\theta}.$$

Hence for stable equilibrium

$$\frac{1}{r} > \frac{1}{R} + \frac{\sin^2\theta}{r}, \quad \text{or} \quad \sin\theta < \sqrt{1-\frac{r}{R}}.$$

(4) *A prolate hemispheroid rests with its vertex on a horizontal plane. Show that for rolling displacement the equilibrium is stable or unstable according as the eccentricity of the generating ellipse is less or greater than* $\sqrt{\frac{3}{8}}$.

ANS. Let a be the semi-major and b the semi-minor axis. Then the distance OC to the centre of mass is



$$OC = \frac{3}{4}\,\frac{(2a-a)^2}{3a-a} = \frac{3}{8}a.$$

The distance PC then is $\frac{5}{8}a$. The radius of curvature at P is $\rho = \frac{b^2}{a}$. We have then for stable equilibrium

$$\frac{1}{PC} > \frac{1}{\rho}, \quad \text{or} \quad \frac{8}{5a} > \frac{a}{b^2}, \quad \text{or} \quad \frac{b^2}{a^2} > \frac{5}{8}.$$

But the eccentricity of the generating ellipse is

$$e = \sqrt{1-\frac{b^2}{a^2}}.$$

Hence for stable equilibrium $e < \sqrt{\frac{3}{8}}$.

CHAPTER III.

ROTATION ABOUT A FIXED AXIS.*

Rotation about a Fixed Axis—Effective Forces.—We have seen that D'Alembert's principle (page 297) reduces any kinetic problem to one of equilibrium between actual (*impressed*) forces and fictitious (*reversed effective*) forces. In order to apply it, then, we must be able to find in any given case the effective forces and the moments of the effective forces.

Let us consider first the case of rotation about a fixed axis. Let O be the centre of mass, and let us take the origin of co-ordinates O' at the point of intersection of the axis of rotation with a plane through the centre of mass O at right angles to this axis. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of mass O for this origin O' and any co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$.

Equations (1), page 158, give the components of the *tangential* acceleration for any particle of a rotating body. If we multiply each term by the mass m of the particle and sum up for all the particles, we shall have the components of the effective *tangential forces* of the body. These forces cause change of magnitude of the velocity of each particle in its plane of rotation. In summing up, since x, y, z are taken from the centre of mass O as origin, we have $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$.

We have then for the components of the effective tangential forces for all the particles of the body, since $\Sigma m = \mathfrak{m}$ the mass of the body,

$$\left.\begin{aligned}\Sigma mf_{tx} &= \mathfrak{m}\bar{z}\alpha_y - \mathfrak{m}\bar{y}\alpha_z,\\ \Sigma mf_{ty} &= \mathfrak{m}\bar{x}\alpha_z - \mathfrak{m}\bar{z}\alpha_x,\\ \Sigma mf_{tz} &= \mathfrak{m}\bar{y}\alpha_x - \mathfrak{m}\bar{x}\alpha_y,\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where α_x, α_y, α_z are the components of the angular acceleration α about the axis of rotation.

These forces cause change of magnitude of the velocity of each particle in its plane of rotation.

Equations (3), page 159, give the components of the *deflecting* acceleration for any particle of a rotating body. Here again, multiplying each term by the mass m of the particle and summing up for all the particles, we have the components of the effective *deflecting forces* for all the particles,

$$\left.\begin{aligned}\Sigma mf_{rx} &= -\mathfrak{m}\bar{x}\omega_y^2 - \mathfrak{m}\bar{x}\omega_z^2,\\ \Sigma mf_{ry} &= -\mathfrak{m}\bar{y}\omega_z^2 - \mathfrak{m}\bar{y}\omega_x^2,\\ \Sigma mf_{rz} &= -\mathfrak{m}\bar{z}\omega_x^2 - \mathfrak{m}\bar{z}\omega_y^2,\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

* Before reading this and the following chapters the student should be familiar with the principles of Chap. II, page 153, and Chap. III, page 31.

where ω_x, ω_y, ω_z are the components of the angular velocity ω about the axis of rotation. These forces cause change of direction of velocity of each particle in its plane of rotation.

For fixed axis the plane of rotation of each particle does not change. There are therefore no deviating accelerations.

Adding equations (1) and (2) we have then for the components of the effective forces for a body rotating about a fixed axis, for any co-ordinate axes we please.

$$\left.\begin{aligned}\Sigma mf_x &= -\mathrm{m}\bar{x}\omega_y^2 - \mathrm{m}\bar{x}\omega_z^2 + \mathrm{m}\bar{z}\alpha_y - \mathrm{m}\bar{y}\alpha_z,\\ \Sigma mf_y &= -\mathrm{m}\bar{y}\omega_z^2 - \mathrm{m}\bar{y}\omega_x^2 + \mathrm{m}\bar{x}\alpha_z - \mathrm{m}\bar{z}\alpha_x,\\ \Sigma mf_z &= -\mathrm{m}\bar{z}\omega_x^2 - \mathrm{m}\bar{z}\omega_y^2 + \mathrm{m}\bar{y}\alpha_x - \mathrm{m}\bar{x}\alpha_y.\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

If we take the fixed axis coinciding with one of the co-ordinate axes, as, for instance, with $O'X'$, we have $\omega_y = 0$, $\omega_z = 0$, $\alpha_y = 0$, $\alpha_z = 0$, and equations (3) become

$$\left.\begin{aligned}\Sigma mf_x &= 0,\\ \Sigma mf_y &= -\mathrm{m}\bar{y}\omega_x^2 - \mathrm{m}\bar{z}\alpha_x,\\ \Sigma mf_z &= -\mathrm{m}\bar{z}\omega_x^2 + \mathrm{m}\bar{y}\alpha_x.\end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

If we take distance in feet and mass in lbs., all these equations give force in poundals. For force in pounds we divide by g (page 171).

We see from these equations that the effective force in any direction is the same as for a particle of mass equal to that of the body moving with the acceleration of the centre of mass in that direction, as already proved on page 299.

If the axis of rotation passes through the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and hence $\Sigma mf_x = 0$, $\Sigma mf_y = 0$, $\Sigma mf_z = 0$.

That is, when a body rotates about an axis through the centre of mass the effective forces in any direction are zero.

Moments of the Effective Forces.—Equations (10), page 160, give the component moments of the acceleration for any particle of a rotating and translating body. For a fixed axis the deviating acceleration is zero, and therefore, as we see from equations (4), page 159, we have $\omega_x\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_z = 0$, $\omega_y\omega_x = 0$, $\omega_z\omega_x = 0$, $\omega_z\omega_y = 0$. We can also put x', y', z' in place of $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$, and for rotation only we have $\bar{f}_x = 0$, $\bar{f}_y = 0$, $\bar{f}_z = 0$.

If we make these changes in equations (10), page 160, multiply each term by the mass m of the particle and sum up for all the particles, we shall have the components of the moments of the effective forces for all the particles of the body. In summing up we have

$$\Sigma m(y'^2 + z'^2) = I'_x, \qquad \Sigma m(z'^2 + x'^2) = I'_y, \qquad \Sigma m(x'^2 + y'^2) = I'_z,$$

where I'_x, I'_y, I'_z are the moments of inertia of the body for the axes $O'X'$, $O'Y'$, $O'Z'$.

We have then, from equations (10), page 160, for the component moments of the effective forces,

$$\left.\begin{aligned}M'_{fx} &= -\alpha_y\Sigma my'x' - \alpha_z\Sigma mz'x' + (\omega_z^2 - \omega_y^2)\Sigma mz'y' + I'_x\alpha_x,\\ M'_{fy} &= -\alpha_z\Sigma mz'y' - \alpha_x\Sigma mx'y' + (\omega_x^2 - \omega_z^2)\Sigma mx'z' + I'_y\alpha_y,\\ M'_{fz} &= -\alpha_x\Sigma mx'z' - \alpha_y\Sigma my'z' + (\omega_y^2 - \omega_x^2)\Sigma my'x' + I'_z\alpha_z.\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

If the fixed axis coincides with one of the co-ordinate axes, as, for instance, $O'X'$, we have $\omega_y = 0$, $\omega_z = 0$, $\alpha_y = 0$, $\alpha_z = 0$, and hence

$$\left.\begin{aligned} M'_{fx} &= I'_x\alpha_x, \\ M'_{fy} &= \omega_x^2\Sigma mx'z' - \alpha_x\Sigma mx'y', \\ M'_{fz} &= -\omega_x^2\Sigma my'x' - \alpha_x\Sigma mx'z'. \end{aligned}\right\} \quad \ldots \ldots \quad (5)$$

Since we can take any co-ordinate axes we please, let the co-ordinate axes be principal axes at the point O'. Then (page 35) $\Sigma mx'y' = 0$, $\Sigma my'z' = 0$, $\Sigma mz'x' = 0$, and equations (4) become

$$M'_{fx} = I'_x\alpha_x, \qquad M'_{fy} = I'_y\alpha_y, \qquad M'_{fz} = I'_z\alpha_z. \quad \ldots \ldots \quad (6)$$

If the axis of rotation is a principal axis, let it coincide with $O'X'$, for instance. Then $\alpha_y = 0$, $\alpha_z = 0$, and from (5) we have $M_{fx} = I'_x\alpha_x$, $M_{fy} = 0$, $M_{fz} = 0$.

That is, when a body rotates about a principal axis with angular acceleration α, the moment of the effective forces relative to the axis of rotation is equal to the moment of inertia I' of the body relative to the axis multiplied by the angular acceleration, or

$$M'_f = I'\alpha.$$

If I_x, I_y, I_z are the moments of inertia for principal axes through the centre of mass. O, parallel to $O'X'$, $O'Y'$, $O'Z'$, we have (page 33)

$$I'_x = I_x + \overline{\text{m}}(\bar{y}^2 + \bar{z}^2), \qquad I'_y = I_y + \overline{\text{m}}(\bar{z}^2 + \bar{x}^2), \qquad I'_z = I_z + \overline{\text{m}}(\bar{x}^2 + \bar{y}^2),$$

where $\overline{\text{m}}$ is the mass of the body and $\bar{x}$, $\bar{y}$, $\bar{z}$ the co-ordinates of the centre of mass Inserting these values in (6), we see that the moment of the effective forces about any fixed axis is equal to the moment about a parallel axis through the centre of mass plus the moment of the effective force of a particle of mass equal to the mass of the body at the centre of mass.

If we take distance in feet and mass in lbs., these equations (4), (5), (6) give moments in poundal-feet. For pound-feet divide by g (page 171).

Origin of the Term "Moment of Inertia."—Let F be the resultant of all the impressed forces at right angles to the axis of rotation and p its lever-arm, so that Fp is the resultant moment of all the impressed forces relative to the axis of rotation. Then, by D'Alembert's principle, when the axis of rotation is a principal axis

$$Fp - M_f = 0, \quad \text{or} \quad Fp = I'\alpha.$$

The term "moment of inertia" is due to Euler. Euler used the term "inertia" as synonymous with what we call mass. Thus the equation of force,

$$F = mf,$$

would be read in the terminology of Euler

Force = *inertia* × linear acceleration.

In the equation

$$Fp = I'\alpha = \alpha\Sigma mr^2$$

Euler called the term Σmr^2 "moment of inertia" and thus obtained the analogous expression

moment of force = *moment* of inertia × angular acceleration.

The term "moment of inertia" in modern scientific terminology is an improper expression. Inertia (page 169) is a property of matter like color or hardness, and we cannot properly speak of moment of inertia any more than of moment of color or hardness. The term "second moment" of mass would more correctly describe the product Σmr^2, and has been used by some recent authors. The expression "moment of inertia," however, has become firmly established by long usage.

The student, while using it, should consider it simply as a name for the quantity Σmr^2, which occurs so frequently in dynamic problems that it is convenient to give it a special name.

Momentum of Rotating Body.—From equations (2), page 154, we have the component velocities for any particle of a rotating body

$$v_x = (\bar{z} + z)\omega_y - (\bar{y} + y)\omega_z, \quad v_y = (\bar{x} + x)\omega_z - (\bar{z} + z)\omega_x, \quad v_z = (\bar{y} + y)\omega_x - (\bar{x} + x)\omega_y.$$

If we multiply each term by the mass m of the particle and sum up for all the particles, we shall have the components of momentum for all the particles. In summing up, since x, y, z are taken from the centre of mass, we have $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$.

We have then for the components of the momentum of a rotating body, since $\Sigma m = \overline{\mathrm{m}}$ the mass of the body,

$$\left.\begin{aligned} \Sigma mv_x &= \overline{\mathrm{m}}\bar{z}\omega_y - \overline{\mathrm{m}}\bar{y}\omega_z, \\ \Sigma mv_y &= \overline{\mathrm{m}}\bar{x}\omega_z - \overline{\mathrm{m}}\bar{z}\omega_x, \\ \Sigma mv_z &= \overline{\mathrm{m}}\bar{y}\omega_x - \overline{\mathrm{m}}\bar{x}\omega_y. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (7)$$

For axis of rotation through the centre of mass we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and hence $\Sigma mv_x = 0$, $\Sigma mv_y = 0$, $\Sigma mv_z = 0$.

Hence *the momentum of a body rotating about an axis is the same as for a particle of mass equal to the mass of the body at the centre of mass.*

Moment of Momentum—Rotating Body.—Equations (10), page 155, give the component moments of the velocity for any particle of a rotating and translating body. For rotation only we have $\bar{v}_x = 0$, $\bar{v}_y = 0$, $\bar{v}_z = 0$. If we make these changes in equations (10), page 155, multiply by the mass m of the particle and sum up for all the particles, we shall have the components of the moment of momentum for the body, for any co-ordinate axes we please. Let us take these axes principal axes at O'. Then we have $\Sigma mx'y' = 0$, $\Sigma my'z' = 0$, $\Sigma mz'x' = 0$. We have also $\Sigma m(y'^2 + z'^2) = I'_x$, $\Sigma m(z'^2 + x'^2) = I'_y$, $\Sigma m(x'^2 + y'^2) = I'_z$, and $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$.

We have then from equations (10), page 155, the component moments of momentum

$$M'_{vx} = I'_x\omega_x, \qquad M'_{vy} = I'_y\omega_y, \qquad M'_{vz} = I'_z\omega'_z. \quad \ldots \ldots \quad (8)$$

If the axis of rotation coincides with a principal axis, as, for instance, $O'X'$, we have $\omega_y = 0$, $\omega_z = 0$, and from (8) we have $M'_{vx} = I'_x\omega_x$, $M_{vy} = 0$, $M'_{vz} = 0$.

That is, when a body rotates about a principal axis with angular velocity ω, the moment of momentum relative to the axis of rotation is equal to the moment of inertia I' of the body relative to that axis, multiplied by the angular velocity, or

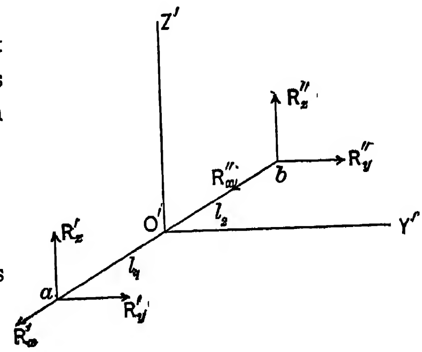
$$M'_v = I'\omega.$$

If I_x, I_y, I_z are the moments of inertia for principal axes through the centre of mass O, we have

$$I'_x = I_x + \overline{\mathrm{m}}(\bar{y}^2 + \bar{z}^2), \qquad I'_y = I_y + \overline{\mathrm{m}}(\bar{z}^2 + \bar{x}^2), \qquad I_z = I_z + \overline{\mathrm{m}}(\bar{x}^2 + \bar{y}^2).$$

Inserting these values in (8), we see that the moment of momentum for a body rotating about an axis is equal to the moment of momentum about a parallel axis through the centre of mass, plus the moment of momentum for a particle of mass equal to the mass of the body at the centre of mass.

Pressures on Fixed Axis.—Equations (4) give the component effective forces for a body rotating about any axis $O'X'$, and equations (5) give the component moments of the effective forces for a body rotating about any axis $O'X'$.



Let the axis be supported at the ends a and b so that it cannot change its direction, and let the distances $O'a = l_1$, $O'b = l_2$, the plane $Y'Z'$ being a plane through the centre of mass O.

Let the component pressures at a and b be

$$R'_x,\ R'_y,\ R'_z \quad \text{and} \quad R''_x,\ R''_y,\ R''_z.$$

Let the components of all the other impressed forces be

$$\Sigma F_x, \quad \Sigma F_y, \quad \Sigma_z F,$$

and the component moments of these forces about the co-ordinate axes be

$$\text{about } O'X' = \Sigma F_z y - \Sigma F_y z,$$

$$\text{“} \quad O'Y' = \Sigma F_x z - \Sigma F_z x,$$

$$\text{“} \quad O'Z' = \Sigma F_y x - \Sigma F_x y.$$

We have then from equations (4), by D'Alembert's principle, for any fixed axis $O'X'$

$$\left.\begin{aligned} R'_z + R''_z + \Sigma F_z &= -\overline{m}\bar{z}\omega^2_x - \overline{m}\bar{y}\alpha_x, \\ R'_y + R''_y + \Sigma F_y &= -\overline{m}\bar{y}\omega^2_x - \overline{m}\bar{z}\alpha_x, \\ R'_x + R''_x + \Sigma F_x &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $\overline{m}$ is the mass of the body; $\bar{x}$, $\bar{y}$, $\bar{z}$, the co-ordinates of the centre of mass.

We have also from equations (5), by D'Alembert's principle, for any axis $O'X'$

$$\left.\begin{aligned} R_y' l_1 - R_y'' l_2 + \Sigma F_y x - \Sigma F_x y &= -\omega_x^2 \Sigma m y' x' - \alpha_x \Sigma m x' z', \\ -R_z' l_1 + R_z'' l_2 + \Sigma F_x z - \Sigma F_z x &= \omega_x^2 \Sigma m x' z' - \alpha_x \Sigma m x' y', \\ \Sigma F_z y - \Sigma F_y z &= I_x' \alpha_x. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

From the last of equations (10) we can find α_x, and then from the first two and the first two of equations (9) we can find R_y', R_y'', R_z', R_z''. This leaves R_x' and R_x'' indeterminate, but their sum is given by the last of equations (9).

If we take distance in feet and mass in lbs., the pressures R will be in poundals. For pounds divide by g (page 171).

In equations (9) the terms $-\overline{m}\bar{z}\omega_x^2$ and $-\overline{m}\bar{y}\omega_x^2$ are the sums of the components parallel to Z' and Y' of the effective deflecting forces (page 315) of all the particles, and the terms $-\overline{m}\bar{y}\alpha_x$ and $-\overline{m}\bar{z}\alpha_x$ are the sums of the components parallel to Z' and Y' of the effective tangential forces (page 315) of all the particles.

In equations (10) the terms $-\omega_x^2\Sigma my'x'$ and $\omega_x^2\Sigma mx'z'$ are the moments about Z' and Y' of the effective deflecting forces, and $-\alpha_x\Sigma mx'z'$, $-\alpha_x\Sigma mx'y'$ are the moments about Z' and Y' of the effective tangential forces, and $I_x'\alpha_x$ is the moment about the axis of rotation X' of the effective tangential forces.

If the axis of rotation passes through the centre of mass we have $\bar{y} = 0$, $\bar{z} = 0$, and we see from equations (9) that the sums of the components of the effective deflecting and tangential forces are zero, or these forces reduce to a couple.

If the axis of rotation is a principal axis, then taking the other two axes as principal axes we have $\Sigma my'x' = 0$, $\Sigma mx'z' = 0$, and we see from equations (10) that the moments of the effective deflecting and tangential forces about Y' and Z' are zero. There is then no tendency of the axis of rotation $O'X'$ to turn about the other two co-ordinate axes $O'Z'$ or $O'Y'$.

If, then, the fixed axis of rotation is a principal axis through the centre of mass, there will be no pressure on this axis due to rotation, and in such case we have, from equations (9) and (10),

$$R_x' + R_x'' + \Sigma F_x = 0, \qquad R_y' + R_y'' + \Sigma F_y = 0, \qquad R_z' + R_z'' + \Sigma F_z = 0,$$

$$-R_y'l_1 - R_y''l_2 + \Sigma F_y x - \Sigma F_x y = 0, \qquad -R_z'l_1 + R_z''l_2 + \Sigma F_x z - \Sigma F_z x = 0,$$

$$\Sigma F_z y - \Sigma F_y z = I_x\alpha_x.$$

That is, *for a body rotating about a fixed principal axis through the centre of mass the pressures on the axis are the same as if the body did not rotate, and are given by the conditions of equilibrium of the impressed forces only.*

Conservation of Moment of Momentum—Fixed Axis.—We have, from equations (8), for the component moments of momentum for a body rotating about a fixed axis, taking the co-ordinate axes as principal axes at O',

$$M_{vx}' = I_x'\omega_x, \qquad M_{vy}' = I_y'\omega_y, \qquad M_{vz}' = I_z'\omega_z, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and, from equations (6), for the component moments of the effective forces

$$M_{fx}' = I_x'\alpha_x, \qquad M_{fy}' = I_y'\alpha_y, \qquad M_{fz}' = I_z'\alpha_z. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If, in equations (b), we have $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, we shall evidently have ω_x, ω_y, ω_z constant in equations (a). But when $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, we have $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$.

Since, by D'Alembert's principle, the moment of the effective forces is equal to the moment of the impressed forces, we have the moment of the impressed forces zero.

Hence, *if the moment of the impressed forces about a fixed axis is always zero, the moment of momentum about that axis is constant.*

Kinetic Energy—Fixed Axis.—The kinetic energy of a particle of mass m and velocity v is $\frac{1}{2}mv^2$ (page 272). If a particle has the component velocities v_x, v_y, v_z, we have

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

and hence

$$\frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2).$$

From page 154 we have for the component velocities for any particle of a rotating body

$$v_x = z'\omega_y - y'\omega_z,$$
$$v_y = x'\omega_z - z'\omega_x,$$
$$v_z = y'\omega_x - x'\omega_y,$$

where x', y', z' are taken from the origin O', the intersection with the axis of rotation of a plane through the centre of mass O at right angles to the axis of rotation. If we square these component velocities, multiply each term by $\frac{1}{2}m$, sum up for all the particles and add, we shall have the kinetic energy for a rotating body for any co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$, we please. Let us take these co-ordinate axes as principal axes of the body at the origin O'. Then we have

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0.$$

We thus obtain for the kinetic energy of a rotating body

$$K = \frac{1}{2}\omega_x^2\Sigma m(y'^2 + z'^2) + \frac{1}{2}\omega_y^2\Sigma m(z'^2 + x'^2) + \frac{1}{2}\omega_z^2\Sigma m(x'^2 + y'^2).$$

But

$$\Sigma m(y'^2 + z'^2) = I'_x, \qquad \Sigma m(z'^2 + x'^2) = I'_y, \qquad \Sigma m(x'^2 + y'^2) = I'_z.$$

Hence

$$K = \frac{1}{2}I'_x\omega_x^2 + \frac{1}{2}I'_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2. \qquad (11)$$

But $\omega_x = \omega^2 \cos^2 \alpha$, $\omega_y^2 = \omega^2 \cos^2 \beta$, $\omega_z^2 = \omega^2 \cos^2 \gamma$, where ω is the angular velocity about the axis of rotation and α, β, γ are the direction angles of this axis. Therefore equation (11) becomes (page 35)

$$K = \frac{1}{2}I'\omega^2, \qquad (12)$$

where I' is the moment of inertia relative to the axis of rotation.

Hence *the kinetic energy for a body rotating about an axis is equal to one half the product of the moment of inertia of the body relative to that axis and the square of the angular velociy about that axis.*

Analogy between the Equations for Rotation and Translation.—The student should not fail to note the analogy between the equations for rotation and translation.

Thus for translation

$$F = \overline{m}\overline{f} = \text{force},$$

while for rotation (principal axis)

$$Fp = I'\alpha = \textit{moment} \text{ of force}.$$

For translation

$$\overline{m}\overline{v} = \text{momentum},$$

while for rotation (principal axis)

$$I'\omega = \textit{moment} \text{ of momentum}.$$

For translation

$$\frac{\overline{m}(\overline{v} - \overline{v}_1)}{t} = \text{uniform force},$$

while for rotation (principal axis)

$$\frac{I'(\omega - \omega_1)}{t} = \textit{moment} \text{ of uniform force.}$$

For translation

$$\frac{1}{2}\,\overline{\mathrm{m}}\bar{v}^2 = \text{kinetic energy,}$$

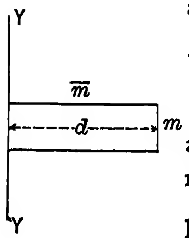
while for rotation (any axis)

$$\frac{1}{2}I'\omega^2 = \text{kinetic energy.}$$

We see that if *linear* acceleration and velocity are replaced by *angular* acceleration, and velocity and *mass by moment of inertia*, force and momentum become moment of force and momentum, and kinetic energy is given in both cases.

Reduction of Mass.—It is often desirable to be able to reduce a rotating mass to a particle of equivalent mass at any desired distance from the axis.

Thus let a body of mass $\overline{\mathrm{m}}$ rotate about a principal axis YY with angular velocity ω and angular acceleration α. Then the moment of momentum is $I'\omega$ and the moment of the force is $I'\alpha$, and the kinetic energy is $\frac{1}{2}I'\omega^2$.



Now suppose a particle of mass m at any desired distance d from the axis to have the same angular velocity and acceleration. Then its moment of momentum is $md^2\omega$, the moment of the force is $md^2\alpha$, and the kinetic energy is $\frac{1}{2}md^2\omega^2$.

If then we have

$$md^2 = I', \quad \text{or} \quad m = \frac{I'}{d^2},$$

the particle m at d will have the same moment of momentum, moment of force and kinetic energy as the body itself. It is therefore equivalent to the body.

Hence to reduce a rotating mass to a particle of equivalent mass at any desired distance from the axis we *divide the moment of inertia by the square of the distance.*

Examples.—(1) *A driving-wheel of a locomotive has a crank-pin and cross-head, etc., of mass $m_1 = 2600$ lbs., the centre of mass being at a distance $r_1 = 6$ inches from centre of wheel. If the wheel has a radius of $r = 3$ feet and is running 60 miles an hour, find the pressure on the rail when the crank-pin has its lowest and highest position.* (Take $g = 32$ ft.-per-sec. per sec.)

ANS. Let $\overline{\mathrm{m}}$ be the mass of the wheel alone. A speed of 60 miles an hour is 88 feet per sec. The angular speed is then given by $r\omega = 88$, or $\omega = \frac{88}{3}$ radians per sec.

The impressed forces are the upward pressure R of the rail and the weight $(\overline{\mathrm{m}}+m_1)g$ of the wheel and crank-pin and cross-head. The effective force is $m_1r_1\omega^2$ acting towards the centre, for the crank. The sum of the effective forces for the wheel is zero (page 316).

By D'Alembert's principle, for crank at its highest position we have

$$R - (\overline{\mathrm{m}} + m_1)g + m_1r_1\omega^2 = 0, \quad \text{or} \quad R = (\overline{\mathrm{m}} + m_1)g - m_1r_1\omega^2 \text{ poundals.}$$

Inserting numerical values and dividing by g, we have

$$R = (\overline{\mathrm{m}} - 32355) \text{ pounds.}$$

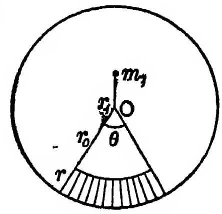
For crank at lowest position we have

$$R - (\overline{\mathrm{m}} + m_1)g - m_1r_1\omega^2 = 0, \quad \text{or} \quad R = (\overline{\mathrm{m}} + 37555) \text{ pounds.}$$

The effect is to cause a blow upon the rail every time the crank passes the lowest point.

(2) *In the preceding example let the thickness of the wheel be $t = 2$ inches, and density $\delta = 480$ lbs. per cubic foot. It is required to balance the crank by filling in between the spokes. Find the inner radius r_0 for any given angle $\theta = 60°$.*

ANS. Let m_2 be the mass of the ring, and $\bar{y}$ the distance of its centre of mass. Then we should have



$$m_1 r_1 \omega^2 = m_2 \bar{y} \omega^2, \quad \text{or} \quad m_1 r_1 = m_2 \bar{y}.$$

We have (page 29) $m_2 = \frac{\delta t \theta}{2}(r^2 - r_0^2)$ and $\bar{y} = \frac{4 \sin\frac{\theta}{2}}{3\theta}\left(\frac{r^3 - r_0^3}{r^2 - r_0^2}\right)$.

Substituting and solving, we have for r_0

$$r_0 = \sqrt[3]{r^3 - \frac{3 m_1 r_1}{2\delta t \sin\frac{\theta}{2}}}.$$

Inserting numerical values, we have $r_0 = 2.79$ feet.

3. *A sphere of mass $\overline{\text{m}}$ and radius r_1 has an angular velocity ω_1, and contracts until its radius $\frac{1}{n}r_1$. Find the final angular velocity ω if there are no external forces.*

ANS. The moment of momentum cannot change (page 320). Hence $I_1\omega_1 = I\omega$, or $\omega = \frac{I_1\omega_1}{I}$.

We have (page 49) $I_1 = \frac{2}{5}\overline{\text{m}} r_1^2$ and $I = \frac{2}{5}\overline{\text{m}}\frac{r_1^2}{n^2}$. Hence

$$\omega = n^2 \omega_1.$$

The angular velocity increases as the sphere contracts.

Find the gain of kinetic energy.—The initial kinetic energy is (page 321)

$$\mathcal{E}_1 = \frac{1}{2} I_1 \omega_1^2,$$

and the final kinetic energy is

$$\mathcal{E} = \frac{1}{2} I \omega^2.$$

The gain of kinetic energy is then

$$\mathcal{E} - \mathcal{E}_1 = \frac{1}{5}\overline{\text{m}} r_1^2 (n^2 - 1)\omega_1^2.$$

This gain of kinetic energy must be at the expense of potential energy (page 304).

(4) *In the preceding example find the loss of potential energy due to contraction.*

ANS. Let m be the mass of a particle on the surface of the sphere. The attraction between the sphere and this particle is (page 203)

$$\frac{k\overline{\text{m}}m}{r_1^2} = \frac{\overline{\text{m}} m r_0^2 g}{m_0 r_1^2},$$

where r_0 is the radius of the earth and m_0 the mass of the earth, and g the acceleration of gravity at the earth's surface.

The attraction for any point within the sphere varies directly as the distance from the centre. Hence at a distance ρ_1 from the centre the attraction is

$$\frac{\overline{\text{m}} m r_0^2 g}{m_0 r_1^2} \cdot \frac{\rho_1}{r_1}.$$

During contraction the attraction is inversely as the square of the distance. Hence the attraction at a distance x of a particle originally at a distance ρ_1 is

$$\frac{\overline{\text{m}} m r_0^2 g \rho_1}{m_0 r_1^3} \cdot \frac{\rho_1^2}{x^2}.$$

The loss of potential energy of the particle is then

$$\frac{\overline{\mathrm{m}}mr_0^2 g\rho_1^3}{m_0 r_1^3}\int_{\frac{\rho_1}{n}}^{\rho_1}\frac{dx}{x^2} = \frac{\overline{\mathrm{m}}mr_0^2 g\rho_1^2}{m_0 r_1^3}(n-1).$$

The mass of a unit of volume of the sphere is $\dfrac{\overline{\mathrm{m}}}{\frac{4}{3}\pi r_1^3}$. The volume of a spherical shell of radius ρ_1 is $4\pi\rho_1^2 d\rho_1$. Hence the mass of an elementary shell is

$$\frac{\overline{\mathrm{m}}}{\frac{4}{3}\pi r_1^3}\cdot 4\pi\rho_1^2 d\rho_1 = \frac{3\overline{\mathrm{m}}\rho_1^2 d\rho_1}{r_1^3}.$$

Substituting this for m, we have for the loss of potential energy of an elementary shell

$$\frac{3\overline{\mathrm{m}}^2 r_0^2 g(n-1)}{m_0 r_1^6}\rho_1^4 d\rho_1.$$

The total loss of potential energy is then

$$\frac{3\overline{\mathrm{m}}^2 r_0^2 g(n-1)}{m_0 r_1^6}\int_0^{r_1}\rho_1^4 d\rho_1 = \frac{3\overline{\mathrm{m}}^2 r_0^2 g(n-1)}{5m_0 r_1}. \qquad (1)$$

This loss of potential energy must be converted into kinetic energy (page 304).
We have just seen that the gain of kinetic energy is

$$\frac{1}{5}\overline{\mathrm{m}}r_1^2(n^2-1)\omega_1^2. \qquad (2)$$

Hence if (1) is greater than (2) the difference must be converted into heat. The energy converted into heat is then

$$\frac{\overline{\mathrm{m}}(n-1)}{5m_0 r_1}[3\overline{\mathrm{m}}r_0^2 g - m_0 r_1^3(n+1)\omega_1^2].$$

If we divide by g we have this energy in ft.-lbs. If we then divide by the "mechanical equivalent of heat" J, we obtain the number of heat-units. We have then for the number of heat-units generated

$$\text{No. of heat-units} = \frac{\overline{\mathrm{m}}(n-1)}{5m_0 r_1 J}[3\overline{\mathrm{m}}r_0^2 g - m_0 r_1^3(n+1)\omega_1^2].$$

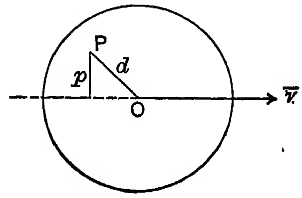
If δ is the density of the sphere and γ is the density of water, the mass of a volume of water equal to the sphere is $\dfrac{\delta\overline{\mathrm{m}}}{\gamma}$. If σ is the specific heat of the sphere and T the number of degrees rise of temperature, we have

$$\text{No. of heat-units} = \frac{\sigma\delta}{\gamma}\overline{\mathrm{m}}T.$$

Hence

$$T = \frac{\gamma(n-1)}{5\sigma\delta m_0 r_1 J}[3\overline{\mathrm{m}}r_0^2 g - m_0 r_1^3(n+1)\omega_1^2].$$

(5) *A disc of mass $\bar{m}$ has a motion of translation $\bar{v}$ and of rotation ω_1 in its own plane. If any point of the disc is suddenly fixed, find the angular velocity ω.*



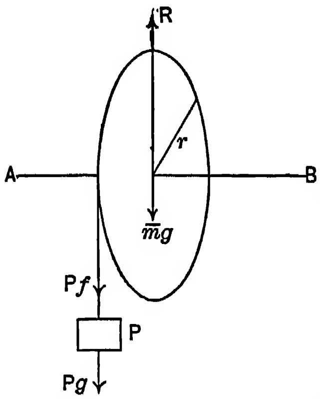
ANS. Let p be the perpendicular distance between the fixed point P and the direction of motion of translation $\bar{v}$, and d the distance between P and O. Let $I = \bar{m}\kappa^2$ be the moment of inertia for the axis at O, and $I' = \bar{m}(\kappa^2 + d^2)$ for the axis at P.

The moment of momentum about axis at P before stoppage is

$$\bar{m}\bar{v}p + I'\omega_1,$$

and after stoppage $I'\omega$. Hence

$$I'\omega = \bar{m}\bar{v}p + I'\omega_1, \quad \text{or} \quad \omega = \omega_1 + \frac{\bar{v}p}{\kappa^2 + d^2}.$$



(6) *A disc of mass $\bar{m} = 8$ lbs. and radius $r = 2$ feet rotates about a fixed horizontal axis AB. A perfectly flexible string wound on the disc has a mass $P = 16$ lbs. attached to its lower end. Find the distance described by P in $t = 2$ seconds, neglecting friction and mass of the string. (Take $g = 32$).*

ANS. The moment of inertia for axis AB is $I = \frac{\bar{m}}{2}r^2$. The impressed forces are the upward reaction R at O and the weight $\bar{m}g$ of the disc and the weight Pg of P. Let f be the acceleration of P, then Pf downward is the effective force on P.

The moment of the effective forces of the particles of the disc is $I\alpha$ (page 317), and the sum of the components of the effective forces of the particles of the disc in any direction is zero (page 316).

Reversing the effective forces, we have, by D'Alembert's principle,

$$R - \bar{m}g - Pg + Pf = 0,$$

$$Pgr - Pfr - I\alpha = 0.$$

But $I = \frac{\bar{m}}{2}r^2$ and $r\alpha = f$. Hence we obtain from the second of these equations

$$f = \frac{Pg}{P + \frac{\bar{m}}{2}} = \frac{4}{5}g = 25.6 \text{ ft.-per-sec. per sec.},$$

and from the first equation

$$R = \bar{m}g + Pg - \frac{4}{5}Pg = (\bar{m} + \frac{1}{5}P)g \text{ poundals},$$

or $R = \bar{m} + \frac{1}{5}P = 11.2$ pounds.

We see that the reaction R is less than the weight $\bar{m} + P = 24$ pounds of the apparatus.

Since f is uniform, we have for the distance described in t seconds starting from rest

$$s = \frac{1}{2}ft^2 = \frac{Pgt^2}{2P + \bar{m}} = 51.2 \text{ feet}.$$

NOTE.—The mass reduced to the circumference (page 322) is $P + \frac{I}{r^2} = P + \frac{\bar{m}}{2}$. We have then directly

$$(\text{reduced mass}) \times \text{acceleration} = \text{moving force}, \quad \text{or} \quad \left(P + \frac{\bar{m}}{2}\right)f = Pg.$$

(7) *A horizontal uniform disc is free to revolve about a vertical axis through its centre. A man walks around on the outer edge. Find the angular distance described by the man and the disc when he has walked once round.*

ANS. Let M be the mass of the man, and D the mass of the disc, and r its radius. Then $I = \frac{D}{2}r^2$.

Let α be the angular acceleration of the disc, and F the force exerted on its circumference. Then (page 317)

$$Fr = I\alpha, \quad \text{or} \quad \alpha = \frac{Fr}{I} = \frac{2F}{Dr}.$$

If α_1 is the angular acceleration of the man, we have

$$Fr = Mr^2\alpha_1, \quad \text{or} \quad F = Mr\alpha_1.$$

Hence

$$\alpha = \frac{2M}{D}\alpha_1.$$

The angular distance described by the disc is then $\theta = \frac{1}{2}\alpha t^2$, and by the man $\theta_1 = \frac{1}{2}\alpha_1 t^2$, and when the man has walked once round we have

$$\frac{1}{2}\alpha t^2 + \frac{1}{2}\alpha_1 t^2 = 2\pi.$$

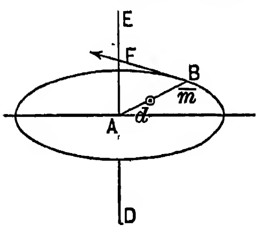
Inserting the value of α_1, we have for the angular distance of the man

$$\frac{1}{2}\alpha_1 t^2 = \frac{2\pi D}{D+2M},$$

and for the angular distance of the disc

$$\frac{1}{2}\alpha t^2 = \frac{4\pi M}{D+2M}.$$

(8) *Let a body of mass* $\overline{\text{m}}$ *on the horizontal arm AB be free to rotate about the vertical axis ED. Let the body be acted upon by a horizontal force F of constant magnitude always at right angles to AB at the distance AB = r. Let the distance AO of the centre of mass O from the axis be d. Find the number of turns which the body will make about the axis in the time t, neglecting the mass of the arm.*



ANS. Let κ be the radius of gyration of the body for an axis through O parallel to ED. Then the moment of inertia of the body for axis ED is

$$I' = \overline{\text{m}}(\kappa^2 + d^2),$$

and we have (page 317)

$$\alpha = \frac{Fr}{\overline{\text{m}}(\kappa^2 + d^2)}.$$

If θ is the angular distance, we have $\theta = \frac{1}{2}\alpha t^2$ or

$$\theta = \frac{Frt^2}{2\overline{\text{m}}(\kappa^2 + d^2)}.$$

The number of complete revolutions will then be

$$n = \frac{\theta}{2\pi} = \frac{Frt^2}{4\pi\overline{\text{m}}(\kappa^2 + d^2)}.$$

If the body is a sphere 2 feet in diameter, weighing 100 lbs., the centre 5 feet from the axis, and F is a force of 25 pounds at the end of a lever 8 feet long, find the number of turns in 5 minutes. ($g = 32$.)

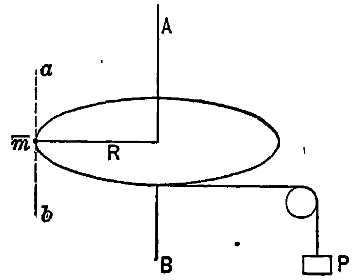
ANS.
$$n = \frac{25g \times 8 \times 300^2}{4\pi \times 100\left(\frac{2}{5} + 25\right)} = \frac{7200000}{127\pi} = 1845\tfrac{45}{899} \text{ turns.}$$

The time necessary to make one turn is

$$t=\sqrt{\frac{4\pi\times 100\left(\frac{2}{5}+25\right)}{25g\times 8}}=2.23 \text{ sec.}$$

(9) *A sphere whose mass is $\overline{\text{m}}$ rests upon the rim of a horizontal disc of mass D. A perfectly flexible string passes round the disc and over a pulley and has a mass P at its lower end. Disregarding friction and mass of pulley and string, find the distance described by P in the time t.*

ANS. Let R be the radius of the disc and r the radius of the sphere. If the sphere *moves with the disc as if it were part of it*, i.e., rotates about the axis ab in the same time that it rotates about the parallel axis AB, we have the moment of inertia of the sphere relative to the axis AB



$$\frac{2}{5}\overline{\text{m}}r^2+\overline{\text{m}}R^2.$$

In this case we have for the angular acceleration about AB

$$\frac{f}{R}=\alpha=\frac{P(g-f)R}{\frac{2}{5}\cdot\overline{\text{m}}r^2+\overline{\text{m}}R^2+\frac{D}{2}R^2}.$$

Hence the acceleration f of P is

$$f=R\alpha=\frac{Pg}{P+\overline{\text{m}}\left(1+\frac{2}{5}\frac{r^2}{R^2}\right)+\frac{D}{2}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The distance described by P is then

$$s=\frac{1}{2}ft^2=\frac{Pgt^2}{2P+2\overline{\text{m}}\left(1+\frac{2r^2}{5R^2}\right)+D}.$$

If the sphere *does not rotate about the axis ab*, as when, for instance, it is hung from the rim, we may consider it as a particle, and its moment of inertia is then $\overline{\text{m}}R^2$. We have then

$$f=\frac{Pg}{P+\overline{\text{m}}+\frac{D}{2}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Hence we have

$$s=\frac{Pgt^2}{2P+2\overline{\text{m}}+D}.$$

If the sphere has an angular acceleration α_1, not equal to α, about ab in same direction as the disc, we have for the moment of the force causing this acceleration $\frac{2}{5}\overline{\text{m}}r^2\alpha_1$. Hence by D'Alembert's principle

$$P(g-f)R-\frac{2}{5}\overline{\text{m}}R^2\alpha_1-\overline{\text{m}}R^2\alpha-\frac{D}{2}R^2\alpha=0, \quad \text{or} \quad f=\frac{Pg-\frac{2}{5}\frac{\overline{\text{m}}r^2}{R}\alpha_1}{P+\overline{\text{m}}+\frac{D}{2}}.$$

(10) *A hollow circular disc whose outer radius is a_1 and inner radius b_1 and thickness t_1 revolves about an axis perpendicular to its plane. Find the thickness t_2 of an equivalent disc whose outer radius is a_2 and inner radius b_2.*

ANS. If the discs are equivalent, the same force moment should give the same angular acceleration α to each, or

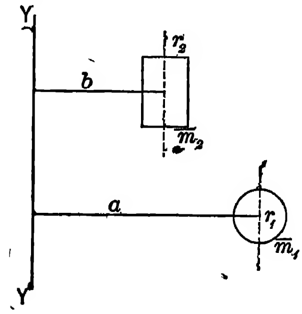
$$I_1\alpha=I_2\alpha, \qquad \text{or} \qquad I_1=I_2.$$

But $I_1=\overline{\text{m}}_1(a_1^2+b_1^2)=\delta\pi t_1(a_1^2-b_1^2)(a_1^2+b_1^2)=\delta\pi t_1(a_1^4-b_1^4)$.

In the same way $I_2=\delta\pi t_2(a_2^4-b_2^4)$. Hence

$$t_2=\frac{a_1^4-b_1^4}{a_2^4-b_2^4}\cdot t_1.$$

(11) *A sphere of radius r_1 rotates about the axis YY at a distance a. Find the height h of an equivalent cylinder of base radius r_2, whose axis is parallel to YY at a distance b.*



ANS. The moment of inertia of the sphere of mass m_1 relative to YY is

$$I_1' = \frac{2\overline{m}_1 r_1^2}{5} + \overline{m}_1 a^2.$$

The moment of inertia for a cylinder of mass $\overline{m}_2$ is

$$I_2' = \frac{\overline{m}_2 r_2^2}{2} + \overline{m}_2 b^2.$$

We have for equivalence $I_1' = I_2'$. Hence if the cylinder and sphere rotate about their own axes parallel to YY in the same time they rotate about YY,

$$\frac{2\overline{m}_1 r_1^2}{5} + \overline{m}_1 a^2 = \frac{\overline{m}_2 r_2^2}{2} + \overline{m}_2 b^2.$$

But if δ_1 is the density of the sphere and δ_2 of the cylinder,

$$\overline{m}_1 = \frac{4\delta_1 \pi r_1^3}{3} \quad \text{and} \quad \overline{m}_2 = \delta_2 \pi r_2^2 h.$$

Hence

$$h = \frac{8\delta_1 r_1^3}{15\delta_2 r_2^2} \cdot \frac{2r_1^2 + 5a^2}{r_2^2 + 2b^2}.$$

If the cylinder and sphere rotate about the axis YY without turning on their own axes, they can be treated as particles and we have

$$\overline{m}_1 a^2 = \overline{m}_2 b^2, \quad \text{and} \quad h = \frac{4\delta_1 r_1^3 a^2}{3\delta_2 r_2^2 b^2}.$$

If the cylinder and sphere have the angular velocity or acceleration ω_1 or α_1 about their own axes and ω or α about YY in the same or opposite directions, we have

$$\frac{2}{5}\overline{m}_1 r_1^2 \omega_1 \pm \overline{m}_1 a^2 \omega = \overline{m}_2 \frac{r_2^2}{2}\omega_1 \pm \overline{m}_2 b^2 \omega,$$

or

$$\frac{2}{5}\overline{m}_1 r_1^2 \alpha_1 \pm \overline{m}_1 a^2 \alpha = \overline{m}_2 \frac{r_2^2}{2}\alpha_1 \pm \overline{m}_2 b^2 \alpha,$$

Hence

$$h = \frac{8\delta_1 r_1^3}{15\delta_2 r_2^2} \cdot \frac{2r_1^2\omega_1 \pm 5a^2\omega}{r_2^2\omega_1 \pm 2b^2\omega} \quad \text{or} \quad h = \frac{8\delta_1 r_1^3}{15\delta_2 r_2^2} \cdot \frac{2r_1^2\alpha_1 \pm 5a^2\alpha}{r_2^2\alpha_1 \pm 2b^2\alpha}.$$

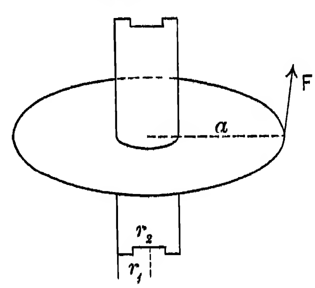
(12) *Upon a vertical hollow axle whose outer radius is r_1 and inner radius r_2, and whose length is l, there is fixed a circular disc of radius a at right angles to the axle. Under the action of a force F of constant magnitude acting with the lever-arm a the angular velocity ω_1 is attained. Find the time t_1 of attaining this velocity. If now the force F ceases to act, find the time t of coming to rest and the number of revolutions in that time.*

ANS. Let the mass of the axle be A, and of the disc D. Then the moment of inertia of the axle is (page 46)

$$\frac{A}{2}(r_1^2 + r_2^2),$$

and the moment of inertia of the disc is

$$\frac{D}{2}(a^2 + r_1^2).$$



The total moment of inertia for the centre line of the axle is then

$$I = \frac{A}{2}(r_1^2 + r_2^2) + \frac{D}{2}(a^2 + r_1^2),$$

and this axis is a principal axis.

The pressure on the foot of the axle is $(D + A)g$ poundals. The moment of the friction for hollow flat pivot (page 225) is

$$M = \frac{2}{3}\mu(D + A)g\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right),$$

where μ is the coefficient of kinetic friction.

If α_1 is the angular acceleration, we have the moment of the effective forces $I\alpha_1$ (page 317). Hence, by D'Alembert's principle,

$$Fa - M - I\alpha_1 = 0, \quad \text{or} \quad \alpha_1 = \frac{Fa - M}{I}.$$

Hence the angular velocity at the end of the time t_1 is

$$\omega_1 = \alpha_1 t_1, \quad \text{or} \quad t_1 = \frac{\omega_1}{\alpha_1} = \frac{I\omega_1}{Fa - M} = \frac{\left[\frac{A}{2}(r_1^2 + r_2^2) + \frac{D}{2}(a^2 + r_1^2)\right]\omega_1}{Fa - \frac{2}{3}\mu(D + A)g\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)}.$$

If the force F now ceases, we have the angular retardation

$$\alpha = -\frac{M}{I},$$

and the angular velocity at the end of any time t from the instant the force F ceases is

$$\omega = \omega_1 + \alpha t = \omega_1 - \frac{Mt}{I}.$$

When the apparatus comes to rest $\omega = 0$, and the time t of coming to rest is

$$t = \frac{I\omega_1}{M} = \frac{\left[\frac{A}{2}(r_1^2 + r_2^2) + \frac{D}{2}(a^2 + r_1^2)\right]\omega_1}{\frac{2}{3}\mu(D + A)g\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)}.$$

The number of radians described in this time t is

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2 = \frac{I\omega_1^2}{2M}.$$

The number of revolutions is then

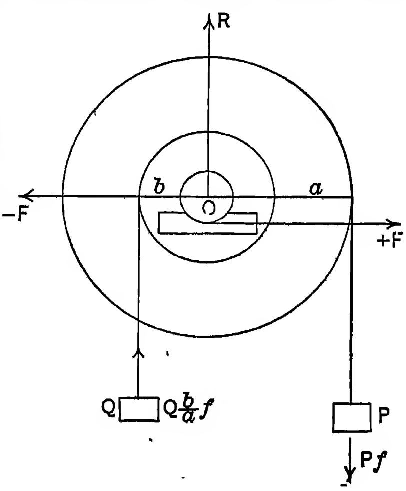
$$n = \frac{\theta}{2\pi} = \frac{I\omega_1^2}{4\pi M} = \frac{\left[\frac{A}{2}(r_1^2 + r_2^2) + \frac{D}{2}(a^2 + r_1^2)\right]\omega_1^2}{\frac{8}{3}\pi\mu(D + A)g\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)}.$$

Again, the kinetic energy at the instant the force F ceases is (page 321) $\frac{I}{2}\omega_1^2$. The work of the friction for one revolution is $2\pi M$. In n revolutions the work is $2\pi n M$. We have then, in coming to rest,

$$2\pi n M = \frac{I}{2}\omega_1^2, \quad \text{or} \quad n = \frac{I\omega_1^2}{4\pi M},$$

as before.

(13) *Suppose a wheel and axle composed of hollow discs for the wheel and axle and a solid cylinder for the journal. The radius of the wheel is $a = 3$ ft., of the axle $b = 2$ ft., of the journal $r = 1$ inch. Let the mass of the wheel be $W = 5$ lbs., of the axle $A = 3$ lbs., of the journal $J = 2$ lbs. Let the moving mass be $P = 10$ lbs. and the mass lifted $Q = 5$ lbs. Let the string be perfectly flexible, and disregard its mass. Let P start from rest and fall for $t = 5$ sec. Discuss the motion of the apparatus, taking into account the mass of the wheel, axle and journal, and the friction; the coefficient of friction being $\mu = 0.07$. Take $g = 32\frac{1}{6}$ ft.-per-sec. per sec.* (Compare example (6), page 307.)



ANS. We have for principal axis through the centre of mass O

Moment of inertia of wheel $= \frac{W}{2}(a^2 + b^2) = 32.5$ lb.-ft.²

" " " " axle $= \frac{A}{2}(b^2 + r^2) = 6\frac{1}{96}$ "

" " " " journal $= \frac{J}{2}r^2 = \frac{1}{144}$ "

Hence moment of inertia of wheel, axle and journal is

$$I = \frac{W}{2}(a^2 + b^2) + \frac{A}{2}(b^2 + r^2) + \frac{J}{2}r^2 = 38\tfrac{149}{288}\text{ lb.-ft.}^2$$

The impressed forces are the upward reaction R at the centre O, the downward weights Pg, Qg, Wg, Ag, Jg, the friction $+F$ and the equal and opposite reaction $-F$ of the bearing. The moment of the friction is then the moment of a couple, and is Fr at any point in its plane (page 185).

The effective forces are Pf down and $Q\frac{b}{a}f$ up, and the effective forces of the particles of the wheel, journal and axle. The sum of these in any direction is zero (page 316), and their moment about O is $I\alpha$ (page 317).

If we reverse the effective forces and then apply D'Alembert's principle, we have

$$+F - F = 0, \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$+R - Pg - Qg - Wg - Ag - Jg + Pf - Q\frac{b}{a}f = 0, \quad \ldots \ldots \ldots \quad (2)$$

$$-Pga + Qgb + Fr - I\alpha + Pfa + Q\frac{b^2}{a}f = 0. \quad \ldots \ldots \ldots \quad (3)$$

From (2) we have the pressure on the bearing,

$$R = (W + A + J)g + Q\left(g + \frac{b}{a}f\right) + P(g - f),$$

or

$$R = (P + Q + W + A + J)g - \left(P - Q\frac{b}{a}\right)f \ldots \text{ poundals.}$$

We see that the pressure on the bearing is less than the weight of the apparatus.

The friction for new bearing is then (page 229)

$$F = \frac{\mu\beta}{\sin\beta}R = \frac{\mu\beta}{\sin\beta}\left[(P + Q + W + A + J)g - \left(P - Q\frac{b}{a}\right)f\right] \text{ poundals,}$$

where μ is the coefficient of kinetic friction and β is the bearing angle. We have also $a\alpha = f$, or $\alpha = \frac{f}{a}$. Inserting this value for α and F in (3), we have for the acceleration f at the circumference of the wheel

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{\mu r\beta}{a\sin\beta}\left(P + Q + W + A + J\right)g}{\left[P + Q\frac{b^2}{a^2} + \frac{W}{2a^2}(a^2 + b^2) + \frac{A}{2a^2}(b^2 + r^2) + \frac{J}{2a^2}r^2\right] - \frac{\mu r\beta}{a\sin\beta}\left(P - Q\frac{b}{a}\right)}. \quad \ldots \quad (4)$$

If we disregard mass of wheel, axle and journal, we have the same value for f as already found in example (6), page 307.

If β is small, $\sin\beta = \beta$, and we have for the given numerical values

$$f = 0.40136g = 12.912 \text{ ft.-per-sec. per sec., and } \alpha = \frac{f}{a} = 4.304\,\frac{\text{radians}}{\text{sec.}^2}.$$

The acceleration of Q is then

$$\frac{b}{a}f = 8.608 \text{ ft.-per-sec. per sec.}$$

The velocity of P at the end of the time $t = 5$ sec. is

$$v = ft = 64.56 \text{ ft. per sec.},$$

and the angular velocity of the wheel is

$$\omega = \frac{v}{a} = 21.52 \text{ radians per sec.}$$

The velocity of Q at the end of the time $t = 5$ sec. is

$$\frac{b}{a}v = 43.04 \text{ ft. per sec.}$$

The pressure R on the bearing is

$$R = 22.32426g \text{ poundals} = 22.32426 \text{ pounds},$$

whereas the weight of the apparatus is 25 pounds.

The friction is

$$F = \mu R = 1.5627g \text{ poundals} = 1.5627 \text{ pounds}.$$

The moment of the friction is

$$Fr = 0.13022g \text{ poundal-ft.} = 0.13022 \text{ pound-ft.}$$

The moment of the effective forces of the particles of the wheel, axle and journal is

$$I\alpha = 5.1531g \text{ poundal-ft.} = 5.1531 \text{ pound-ft.}$$

The distance s described by P is

$$s = \frac{1}{2}ft^2 = 161.4 \text{ ft.}$$

The distance described by Q is

$$\frac{b}{a}s = 107.6 \text{ ft.}$$

The tension T_r on right-hand string is

$$T_r = P(g - f) = 5.9864g \text{ poundals} = 5.9864 \text{ pounds}.$$

The tension T_l on left-hand string is

$$T_l = Q(g + \frac{b}{a}f) = 6.3378g \text{ poundals} = 6.3378 \text{ pounds}.$$

Moment of tension on right	= 17.9592	pound-ft.,
" " " " left	= 12.6756	"
Difference	= 5.2836	"

and this difference we see is equal to $Fr + I\alpha$, the moment of friction and effective forces of wheel, axle and journal, as should be.

$$\text{Work of } P = 5.9864 \times 161.4 = 966.205 \text{ ft.-lbs.} = Ps - \frac{Pv^2}{2g},$$

$$\text{" on } Q = 6.3378 \times 107.6 = 681.947 \text{ "} = Q\frac{b}{a}s + \frac{Qb^2v^2}{2a^2g}.$$

$$\text{Difference} = 284.258 \text{ "}$$

This difference must be work on wheel and work of friction.

Now work of friction is $F\frac{r}{a}s = 7.005$ ft.-lbs.

Hence work on wheel $= 277.253$ ft.-lbs., and is equal to $\frac{I}{2g}\omega^2$ (page 321), as should be.

The power of P (page 262) $= \frac{966.205}{5} = 193.241$ ft.-lbs. per sec., or

$$\frac{193.241}{550} = 0.35 \text{ horse-power.}$$

The efficiency of the apparatus (page 267) is

$$\epsilon = \frac{681.947}{966.205} = 0.70.$$

Note.—The total mass reduced to the circumference (page 322) is

$$M_a = P + Q\frac{b^2}{a^2} + \frac{W}{2a^2}(a^2+b^2) + \frac{A}{2a^2}(b^2+r^2) + \frac{J}{2a^2}r^2.$$

The weight Qg reduced to the circumference is $Q\frac{b}{a}g$, and the force of friction F reduced to the circumference is $F\frac{r}{a}$.

The moving force at the circumference is then

$$\text{moving force} = \left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F.$$

We have then directly, reduced mass × acceleration = moving force, or

$$M_a f = \left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F,$$

or

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F}{M_a}. \qquad (5)$$

If we substitute the values of M_a and F, we obtain (4).

(14) *Suppose a wheel and axle composed of hollow discs for the rim or outer circumference C, the hub H, and the axle A; of a solid cylinder for the journal J, and of four spokes, each spoke S being a bar of uniform cross-section. Let the outer radius of C be $a = 20$ inches, and the inner radius $r_4 = 19$ inches; the outer radius of H be $r_1 = 8$ inches and the inner radius $b = 6$ inches; the radius of the journal J is $r = 1$ inch. Let the mass of the rim or outer circumference be $C = 40$ lbs., of the hub $H = 12$ lbs., of the axle $A = 10$ lbs., of the journal $J = 2$ lbs, and of each spoke $S = 3\frac{3}{4}$ lbs. Let the moving mass $P = 60$ lbs. and the mass lifted $Q = 160$ lbs. Let the string be perfectly flexible, and disregard its mass. Let P start from rest and fall for a time $t = 3$ sec. Discuss the motion of the apparatus, taking into account the mass of the wheel, axle, journal and spokes, and the friction; the coefficient of kinetic friction being $\mu = 0.07$. Take $g = 32\frac{1}{6}$ ft.-per-sec. per sec.*

ANS. The moment of inertia of a spoke for axis through its centre of mass at right angles to plane of wheel is (page 38)

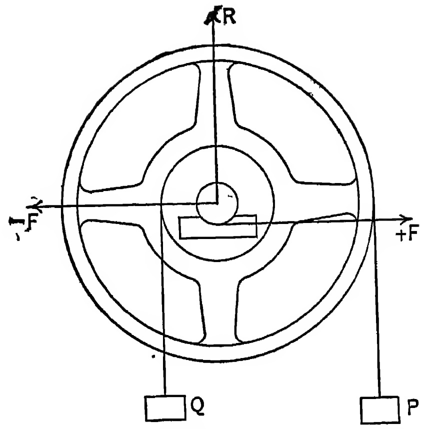
$$\frac{S}{3}\left(\frac{r_4 - r_1}{2}\right)^2.$$

For parallel axis through centre of wheel it is then (page 33)

$$\frac{S}{3}\left(\frac{r_4 - r_1}{2}\right)^2 + S\left(\frac{r_4 + r_1}{2}\right)^2.$$

For four spokes we have then

$$\frac{4S}{3}\left(\frac{r_4 - r_1}{2}\right)^2 + 4S\left(\frac{r_4 + r_1}{2}\right)^2 = \frac{605}{576} \text{ lb.-ft.}^2$$



The moment of inertia of the rim is (page 46)

$$\frac{C}{2}(a^2 + r_4^2) = \frac{3805}{36} \text{ lb.-ft.}^2;$$

of the hub,

$$\frac{H}{2}(r_1^2 + b^2) = \frac{25}{6} \text{ lb.-ft.}^2;$$

of the axle,

$$\frac{A}{2}(b^2 + r^2) = \frac{185}{144} \text{ lb.-ft.}^2;$$

of the journal,

$$\frac{Jr^2}{2} = \frac{1}{144} \text{ lb.-ft.}^2$$

The total moment of inertia for the axis through the centre of the wheel at right angles to its plane is then

$$I = \frac{7181}{64} \text{ lb.-ft.}^2,$$

and this axis is a principal axis.

The impressed forces are the upward reaction R, the weights Pg, Qg, Cg, Hg, Ag, Jg, $4Sg$, the friction $+F$ and the equal and opposite reaction $-F$ of the bearing.

The effective forces are Pf down and $Q\frac{b}{a}f$ up and the effective forces of the particles of the wheel. If we reverse these forces and apply D'Alembert's principle, we have

$$+R - Pg - Qg - Cg - Hg - Ag - Jg - 4Sg + Pf - Q\frac{b}{a}f = 0, \quad \ldots \quad (1)$$

$$-Pga + Qgb + Fr - I\alpha + Pfa + Q\frac{b^2}{a}f = 0. \quad \ldots \quad (2)$$

From (1) we have the pressure on the bearing,

$$R = (C + H + A + J + 4S + Q + P)g - \left(P - Q\frac{b}{a}\right)f.$$

We see that the pressure on the bearing is less than the weight of the apparatus. Let the mass of the apparatus be M, so that

$$M = C + H + A + J + 4S + Q + P.$$

Then we can write

$$R = Mg - \left(P - Q\frac{b}{a}\right)f.$$

The friction for new bearing is then (page 229)

$$F = \frac{\mu\beta}{\sin\beta}R = \frac{\mu\beta}{\sin\beta}\left[Mg - \left(P - Q\frac{b}{a}\right)f\right],$$

where μ is the coefficient of kinetic friction and β is the bearing angle. We have also $a\alpha = f$, or $\alpha = \frac{f}{a}$. Inserting these values for α and F in (2), we have for the acceleration f at the circumference

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{\mu r\beta}{a\sin\beta}Mg}{P + Q\frac{b^2}{a^2} + \frac{I}{a^2} - \frac{\mu r\beta}{a\sin\beta}\left(P - Q\frac{b}{a}\right)}.$$

If β is small, $\sin \beta = \beta$, and we have for the given numerical values

$$f = 0.09g = 2.895 \text{ ft.-per-sec. per sec., and } \alpha = \frac{f}{a} = 1.737 \frac{\text{radians}}{\text{sec.}^2}.$$

The acceleration of Q is then

$$\frac{b}{a}f = 0.8685 \text{ ft.-per-sec. per sec.}$$

The velocity of P at the end of the time $t = 3$ sec. is

$$v = ft = 8.685 \text{ ft. per sec.},$$

and the angular velocity of the wheel is

$$\omega = \frac{v}{a} = 5.211 \text{ radians per sec.}$$

The velocity of Q at the end of $t = 3$ sec. is

$$\frac{b}{a}v = 2.6055 \text{ ft. per sec.}$$

The pressure R on the bearing is

$$R = 297.92g \text{ poundals} = 297.92 \text{ pounds},$$

whereas the weight of the apparatus is 299 pounds.

The friction is

$$F = \mu R = 20.8544g \text{ poundals} = 20.8544 \text{ pounds.}$$

The moment of the friction is

$$Fr = 1.737875g \text{ poundal-ft.} = 1.737875 \text{ pound-ft.}$$

The moment of the effective forces for all rotating particles is

$$I\alpha = 227.8666 \text{ poundal-ft.} = 227.866 \text{ pound-ft.}$$

The distance s described by P is

$$s = \frac{1}{2}ft^2 = 13.0275 \text{ ft.}$$

The distance described by Q is

$$\frac{b}{a}s = 3.90825 \text{ ft.}$$

The tension T_r on the right-hand string is

$$T_r = P(g - f) = 54.6g \text{ poundals} = 54.6 \text{ pounds.}$$

The tension T_l on the left-hand string is

$$T_l = Q\left(g + \frac{b}{a}f\right) = 164.32g \text{ poundals} = 164.32 \text{ pounds.}$$

Moment of tension on right = 91.00 pound-ft.,
" " " left = 82.16 "
Difference = 8.84 "

and this difference we see is equal to $Fr + I\alpha$, the moment of friction and effective forces of rotating particles, as should be.

$$\begin{aligned} \text{Work of } P &= 54.6 \times 13.0275 = 711.3015 \text{ ft.-lbs.} = Ps - \frac{Pv^2}{2g}, \\ \text{" on } Q &= 164.32 \times 3.90825 = 642.2036 \text{ "} \\ \text{Difference} &= 69.0979 \text{ "} = Q\frac{b}{a}s + \frac{Qb^2v^2}{2a^2g}. \end{aligned}$$

This difference must be work on wheel and work of friction. Now, work of friction is

$$F\frac{r}{a}s = 13.5841 \text{ ft.-lbs.}$$

Hence work on wheel = 55.5138 ft.-lbs., and this is equal to $\frac{I}{2g}\omega^2$ (page 321), as should be.

$$\text{The power of } P = \frac{711.3015}{3} = 237.1005 \text{ ft.-lbs. per sec., or}$$

$$\frac{237.1005}{550} = 0.43 \text{ horse-power.}$$

The efficiency of the apparatus is

$$\frac{642.2036}{711.3015} = 0.93.$$

Note.—Just as in example (13), the moving force reduced to the circumference is $\left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F$. If then the reduced mass (page 322) is M_a, we have, just as before,

$$f = \frac{\left(P - Q\frac{b}{a}\right)g - \frac{r}{a}F}{M_a}.$$

In the present case the reduced mass is

$$M_a = P + Q\frac{b^2}{a^2} + \frac{I}{a^2}.$$

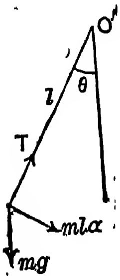
If we substitute this and the value for F, we have f directly.

CHAPTER IV.

COMPOUND PENDULUM. CENTRE OF OSCILLATION. CENTRE OF PERCUSSION. EXPERIMENTAL DETERMINATION OF MOMENT OF INERTIA. EXPERIMENTAL DETERMINATION OF g.

Simple Pendulum.—A particle suspended from a fixed point O' by a perfectly flexible inextensible string without mass, and swinging under the action of gravity, is called a *simple* pendulum. It is then a purely ideal conception.

Let the mass of the particle be m, the length of string l, the angle with the vertical at any instant θ, the angular acceleration α, and the angular velocity ω.

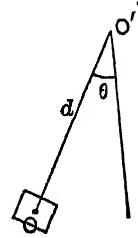


The impressed forces are the tension T in the string and the weight mg. The effective forces are $ml\alpha$ at right angles to l, and $ml\omega^2$ along l towards O'.

By D'Alembert's principle, reversing the effective forces and taking moments about O',

$$mg \times l \sin\theta - ml^2\alpha = 0, \quad \text{or} \quad \alpha = \frac{g \sin\theta}{l}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Instead of a particle of mass m suppose we have a body of mass $\overline{\mathrm{m}}$. Let this body have the angular acceleration α' about a principal axis through its centre of mass O, and the centre of mass the acceleration α about a parallel axis at O'. Let the distance OO' be d. Then, by the principle of page 317, the moment of the effective forces of rotation about axis at O' is



$$\overline{\mathrm{m}}d^2\alpha + I\alpha',$$

where I is the moment of inertia of the body for axis at O, and, by D'Alembert's principle,

$$\overline{\mathrm{m}}g \times d \sin\theta - \overline{\mathrm{m}}d^2\alpha - I\alpha' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

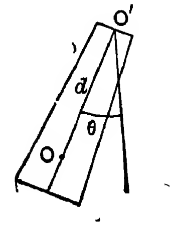
Now if $\alpha' = 0$, that is, if the body has no angular acceleration about the axis through the centre of mass O, we have, as before,

$$\alpha = \frac{g \sin\theta}{d},$$

and we still have a simple pendulum of length $O'O = d$, and can treat the body as if it were a particle of equal mass at O.

But if the body has an angular acceleration about the axis through O, we can no longer treat the body as a particle and we have no longer a simple pendulum.

Compound Pendulum.—If $\alpha' = \alpha$, then the body has the same angular acceleration about the axis through O that it has about the parallel axis through O'. This is the case of a rigid body swinging under the action of gravity about a fixed axis at O'. Such a body is called a *compound* or physical pendulum. It is an actual pendulum and not a purely ideal conception.



We have in such case, from (2),

$$\alpha = \frac{\overline{\mathrm{m}} gd \sin\theta}{I + \overline{\mathrm{m}} d^2},$$

or, since $I = \overline{\mathrm{m}}\kappa^2$, where κ is the radius of gyration for the axis through the centre of mass O,

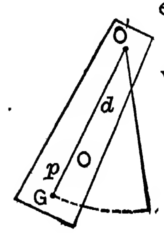
$$\alpha = \frac{gd\sin\theta}{\kappa^2 + d^2}. \quad \quad (3)$$

EQUIVALENT SIMPLE PENDULUM.—If now we equate (1) and (3), we shall obtain the length l of the *equivalent* simple pendulum, that is, the length of a simple pendulum which has at any instant the same angular acceleration as the actual pendulum, and which therefore swings in the same time. We have then, since $\overline{\mathrm{m}}(\kappa^2 + d^2) = I' =$ the moment of inertia relative to the axis at O',

$$l = \frac{\kappa^2 + d^2}{d} = \frac{I'}{\overline{\mathrm{m}} d}. \quad \quad (4)$$

That is, *the length of the equivalent simple pendulum is equal to the moment of inertia relative to the axis at O' divided by the moment of the mass relative to this axis.*

CENTRE OF OSCILLATION.—If we lay off this distance l given by (4) along $O'O$, we obtain a point G. This point is called the *centre of oscillation* or *centre of gyration*, because it is the point at which, if the whole mass were concentrated, we should have an equivalent simple pendulum, which would vibrate in the same time.



Let $OG = p$ be the distance of this point from the centre of mass O. Then we have $p = l - d$, or, from (4),

$$p = \frac{\kappa^2}{d}, \quad \text{or} \quad p : \kappa :: \kappa : d,$$

or

$$p = \frac{\overline{\mathrm{m}}\kappa^2}{\overline{\mathrm{m}} d} = \frac{I}{\overline{\mathrm{m}} d}. \quad \quad (5)$$

That is, *the radius of gyration κ is a mean proportional between the distances p and d of the centres of oscillation and suspension from the centre of mass. Also, the distance p is equal to the moment of inertia I relative to a parallel axis at the centre of mass O divided by the moment of the mass relative to the axis of suspension at O'.*

Suppose now the body turned end for end and suspended from the point G instead of O'. Then, from (4), the length of the equivalent simple pendulum will be

$$l = \frac{\kappa^2 + p^2}{p},$$

or, inserting the value of p from (5),

$$l = \frac{\kappa^2 + d^2}{d},$$

or just the same as before.

Hence *the centre of suspension and oscillation can be interchanged without changing the time of vibration.*

TIME OF VIBRATION.—The time of vibration of the simple pendulum (page 138) is

$$t = \pi\sqrt{\frac{l}{g}}. \qquad (6)$$

If we put for l its value from (4), we have for the time of vibration of the compound pendulum

$$t = \pi\sqrt{\frac{\kappa^2 + d^2}{gd}} = \pi\sqrt{\frac{I'}{\overline{\mathrm{m}}gd}}. \qquad (7)$$

Example.—*The bob of a heavy pendulum contains a spherical cavity filled with water. Determine the motion.*

ANS. Let $\overline{\mathrm{m}}$ be the mass of the pendulum, $\overline{\kappa}$ its radius of gyration for axis through centre of mass O *of the pendulum*, d the distance $O'O$. Let m be the mass of the water, and s the distance of its centre of mass from O'.

The force between the water and the cavity is always normal to the spherical boundary. There is then no force tending to cause angular acceleration of the water about its centre of mass. We can therefore treat the water mass as a particle.

The moment of inertia of the pendulum and water relative to the axis at O' is then

$$I' = \overline{\mathrm{m}}(\overline{\kappa}^2 + d^2) + ms^2.$$

The mass moment is $\overline{\mathrm{m}}d + ms$. We have then for the length of equivalent simple pendulum, from (4),

$$l = \frac{\overline{\mathrm{m}}(\overline{\kappa}^2 + d^2) + ms^2}{\overline{\mathrm{m}}d + ms}.$$

The time of vibration is then, from (6),

$$t = \pi\sqrt{\frac{\overline{\mathrm{m}}(\overline{\kappa}^2 + d^2) + ms^2}{(\overline{\mathrm{m}}d + ms)g}}.$$

If the water were to become solid and rigidly connected with the cavity, let κ be its radius of gyration for a diameter. Then

$$I' = \overline{\mathrm{m}}(\overline{\kappa}^2 + d^2) + m(\kappa^2 + s^2),$$

and

$$t = \pi\sqrt{\frac{\overline{\mathrm{m}}(\overline{\kappa}^2 + d^2) + m(\kappa^2 + s^2)}{(\overline{\mathrm{m}}d + ms)g}}.$$

This latter time of vibration is evidently greater than the first.

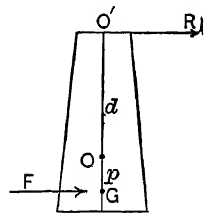
Significance of the Term 'Radius of Gyration.'—We see from (5) that if we make the distance $O'O = d$ equal to κ, we have the distance $OG = p$ *also equal to* κ.

That is, if the distance from the centre of mass O to the axis of suspension O' is made equal to κ, the distance from the centre of mass O to the centre of oscillation or gyration G is also κ.

If then we describe a sphere about O with the radius κ, the body, if suspended from any point on the surface of this sphere, will vibrate in the same time.

Hence the distance κ is called the "*the radius of gyration*" (page 32).

Centre of Percussion.—Suppose a force F to act at some point G in the line $O'O$ at right angles to the plane of $O'O$ and the axis of suspension at O'. Then, by D'Alembert's principle, we have



$$F(d+p) = I'\alpha = \overline{m}(\kappa^2 + d^2)\alpha.$$

Hence

$$\alpha = \frac{F(d+p)}{\overline{m}(\kappa^2+d^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α is the angular acceleration about the axis at O'.

Now the pressure on the fixed axis at O' is R parallel to F. Since the centre of mass moves as if all the mass and all the impressed forces were collected at the centre of mass, if $\bar{f}$ is the acceleration of the centre of mass we have

$$\overline{m}\bar{f} = F + R.$$

But $\bar{f} = d\alpha$. Hence

$$\overline{m}d\alpha = F + R, \quad \text{or} \quad \alpha = \frac{F+R}{\overline{m}d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating (1) and (2), we have for the reaction R of the axis at O'

$$R = \frac{F(pd-\kappa^2)}{\kappa^2+d^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $pd > \kappa^2$, we see, from (3), that R is positive, or in the same direction as F.

If $pd < \kappa^2$, we have R negative, or opposite in direction to F.

If $pd = \kappa^2$, or

$$OG = p = \frac{\kappa^2}{d} = \frac{I}{\overline{m}d}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the reaction R of the axis at O is zero. In this latter case we have then

$$O'G = p + d = \frac{\kappa^2+d^2}{d} = \frac{I'}{\overline{m}d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The point G given by (5) is called the CENTRE OF PERCUSSION, because if a body is struck at this point so that the force F is at right angles to the plane of $O'O$ and the axis of suspension, there will be *no reaction of the axis.*

We see, from page 337, that *the centre of percussion is the same as the centre of oscillation.*

We can find equation (4) directly from our equations (15), page 161, for the position of the instantaneous axis of acceleration.

Thus taking origin at O, and the plane of rotation as the plane of XY, we have

$$\bar{f} = \bar{f}_{nx}, \qquad \bar{f}_{ny} = 0, \qquad \bar{f}_{nz} = 0,$$
$$\alpha_x = 0, \qquad \alpha_y = 0, \qquad \alpha_z = \alpha.$$

Therefore, from equations (15), page 161, we have for the position of the instantaneous axis of acceleration

$$p_x = 0, \qquad p_y = \frac{\bar{f}}{\alpha}, \qquad p_z = 0.$$

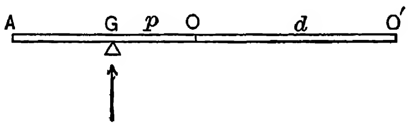
But $\overline{\mathrm{m}}\bar{f} = F$, if there is no reaction R. Hence $\bar{f} = \frac{F}{\overline{\mathrm{m}}}$. Also, by D'Alembert's principle, $Fp = I\alpha$, or $\alpha = \frac{Fp}{I}$. Substituting, we have

$$p_y = \frac{I}{\overline{\mathrm{m}}d},$$

which is the same as equation (4).

To prevent a hammer from "jarring" the hand, or reacting upon a fixed axis about which it turns, the direction of the force at the moment of striking should pass through the centre of percussion G at right angles to the line $O'O$.

Suppose we grasp a prismatic rod $O'A$ of length l at the end O' and strike an obstacle with a force F at right angles to the line $O'O$, where O is the centre of mass.



Then, from (3), the force R at O is

$$R = \frac{pd - \kappa^2}{\kappa^2 + d^2}F.$$

In the case of the rod $I = \frac{\overline{\mathrm{m}}l^2}{12}$ (page 38), and hence $\kappa^2 = \frac{l^2}{12}$, and $d = \frac{l}{2}$. Hence

$$R = \frac{6p - l}{4l}F.$$

If then the obstacle is struck by O, $p = 0$ and $R = -\frac{F}{4}$, that is, the force on the hand is one fourth of that on the obstacle and opposite in direction.

If the obstacle is struck by A, $p = \frac{l}{2}$ and $R = +\frac{F}{2}$, or the force on the hand is one half of that on the obstacle and in the same direction.

If the obstacle is struck by G, so that $p = \frac{l}{6}$, $R = 0$, and there will be no force on the hand. G is the centre of percussion.

Experimental Determination of Moment of Inertia.—From the principles of page 337 we can determine experimentally the moment of inertia of a body relative to any axis.

1st METHOD.—Thus we have, from equation (7), page 338,

$$I' = \overline{\mathrm{m}}d\frac{gt^2}{\pi^2}. \qquad (1)$$

We must first, then, determine the mass $\overline{\mathrm{m}}$ of the body and locate its centre of mass O. Then suspend the body from an axis at O' and measure the distance $O'O = d$. Then swing the body about this axis and note the time t of vibration. The moment of inertia I' relative to the axis at O is then given by (1). The moment of inertia I relative to a parallel axis through the centre of mass is then given by

$$I = I' - \overline{\mathrm{m}}d^2.$$

2d METHOD.—From equation (4), page 337, we have for the distance $O'G$ from the axis of suspension to the centre of oscillation

$$O'G = \frac{I'}{\overline{\mathrm{m}}d}, \quad \text{or} \quad I' = \overline{\mathrm{m}}d \times O'G. \qquad (2)$$

We must first, then, determine the mass $\overline{\mathrm{m}}$ of the body and locate its centre of mass O as before. Then suspend the body from an axis at O' and measure the distance $O'O = d$.

Then swing the body about this axis and note the time of vibration.. Then turn the body over and suspend it from another point G in the line $O'O$ at such a distance from O, found by trial, that the time of vibration about a parallel axis at G is unchanged, and measure $O'G$. The point G is the centre of oscillation, and the moment of inertia I' relative to the axis at O is then given by (2). The moment of inertia for a parallel axis through the centre of mass is then, as before,

$$I = I' - \overline{\mathrm{m}}d^2.$$

This method does not involve knowing the value of g.

Experimental Determination of g.—Having thus found $\overline{\mathrm{m}}$, s and I' and the time of vibration t, we have, from equation (7), page 338,

$$g = \frac{\pi^2 I'}{\overline{\mathrm{m}}dt^2}.$$

We can thus determine g by pendulum observations.

The quantity $\dfrac{\pi^2 I'}{\overline{\mathrm{m}}d}$ is the pendulum constant which must be accurately determined. Once known, the observation of t at any locality gives at once the value of g.

CHAPTER V.

IMPACT.

Impact.—When two moving bodies come into collision the straight line normal to the compressed surfaces at the point of contact is the line of impact. If the centre of mass of the two bodies is upon this line, the impact is *central;* if not, we have *eccentric* impact.

When we consider the direction of motion, we can distinguish *direct* impact when the line of impact coincides with the direction of motion, and *oblique* impact when the line of impact does not coincide with the direction of motion.

FIG. 1.

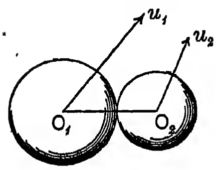
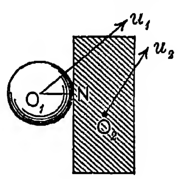


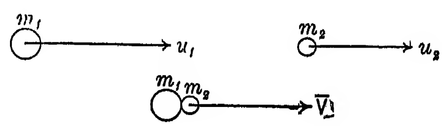
FIG. 2.



Thus in Fig. 1 if the two bodies move in the directions u_1 and u_2, we have *oblique central* impact, and in Fig. 2 we have *oblique eccentric* impact. If in Fig. 1 the directions of motion u_1 and u_2 coincided with O_1O_2, we should have *direct central* impact. If in Fig. 2 the direction of motion u_1 coincided with O_1N, and u_2 were parallel, we should have *direct eccentric* impact.

Direct Central Impact—Non-Elastic.—We can evidently consider the bodies in direct central impact as particles. Let m_1 and u_1 be the mass and initial velocity of one particle before impact, and m_2, u_2 of the other before impact. Let u_1 be greater than u_2 and in the same straight line. Let the direction of u_1 be positive, the opposite direction negative.

When the bodies meet there is a short interval of compression, at the end of which both masses have a common velocity $\bar{v}$. If the bodies are non-elastic, they remain in contact with the common velocity $\bar{v}$.



By the principle of conservation of the centre of mass (page 300), if there are no external forces the motion of the centre of mass is unaffected by the impact.

We have then

$$\bar{v} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}.$$

Or, by the principle of conservation of momentum (page 300), we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\bar{v}, \quad \ldots \ldots \ldots \quad (1)$$

or *the momentum before equals the momentum after impact.*

In equation (1) a velocity opposite to u_1 is to be taken as negative.

The energy before impact is

$$\mathcal{E}_1 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2,$$

and the energy after impact is

$$\mathcal{E} = \frac{1}{2}(m_1 + m_2)\bar{v}^2.$$

The loss of energy is then

$$\mathcal{E}_1 - \mathcal{E} = \frac{m_1 m_2 (u_1 - u_2)^2}{2(m_1 + m_2)}.$$

This work causes deformation and rise of temperature.

If we take mass in lbs. and velocity in ft. per sec., equation (2) gives loss of energy in ft.-poundals. If we wish foot-pounds, we must divide by g, or in foot-pounds

$$\mathcal{E}_1 - \mathcal{E} = \frac{m_1 m_2 (u_1 - u_2)^2}{2g(m_1 + m_2)}. \qquad \qquad (2)$$

We call $\dfrac{m_1 m_2}{m_1 + m_2}$ the *harmonic mean* between m_1 and m_2, and $\dfrac{(u_1 - u_2)^2}{2g}$ is the height due to the difference of the velocities.

Hence *the loss of energy in non-elastic impact is equal to the product of the harmonic mean of the masses and the height due to the difference of their velocities.*

If the mass m_2 is at rest, we have $u_2 = 0$, and

$$\bar{v} = \frac{m_1 u_1}{m_1 + m_2}, \qquad \mathcal{E}_1 - \mathcal{E} = \frac{m_1 m_2 u_1^2}{2g(m_1 + m_2)}.$$

If the bodies move toward each other, u_2 is negative, and

$$\bar{v} = \frac{m_1 u_1 - m_2 u_2}{m_1 + m_2}, \qquad \mathcal{E}_1 - \mathcal{E} = \frac{m_1 m_2 (u_1 + u_2)^2}{2g(m_1 + m_2)}.$$

In this case if the momentums of the bodies are equal, or $m_1 u_1 = m_2 u_2$, $\bar{v} = 0$, or the bodies come to rest. If, on the contrary, the masses are equal, we have

$$\bar{v} = \frac{u_1 - u_2}{2}, \qquad \mathcal{E}_1 - \mathcal{E} = \frac{m}{2} \cdot \frac{(u_1 + u_2)^2}{2g}.$$

If the bodies move in the same direction and the mass of the one in advance, m_2, is infinite, we have

$$\bar{v} = u_2, \qquad \mathcal{E}_1 - \mathcal{E} = \frac{m_1 (u_1 - u_2)^2}{2g},$$

or the velocity of the infinite mass is not changed by the impact. If the infinite mass is at rest, or $u_2 = 0$, we have

$$\bar{v} = 0, \qquad \mathcal{E}_1 - \mathcal{E} = \frac{m_1 u_1^2}{2g}.$$

Examples.—(1) *A non-elastic body of mass $m_1 = 50$ lbs. moving with a velocity $u_1 = 7$ ft. per sec. impinges centrally upon another of mass $m_2 = 30$ lbs. moving in the same direction with a velocity of $u_2 = 3$ feet per sec. Find the velocity with which the two move on together after the collision, and the work expended in deformation and heating.*

ANS. $\bar{v} = 5\frac{1}{2}$ ft. per sec. in the direction of u_1; $\frac{150}{g}$ ft.-pounds.

(2) *In order to cause a non-elastic body weighing 120 lbs. to change its velocity from* $1\frac{1}{2}$ *to 2 feet per sec., we let a non-elastic body weighing 50 lbs. strike it. Find the velocity of the latter body, and the work expended in deformation and heating.*

ANS. 3.2 ft.-per-sec.; $\dfrac{867}{17g}$ ft.-pounds.

(3) *Two non-elastic masses of 3 and 5 tons impinge centrally with velocities of 4 and 5.5 ft. per sec. respectively. Find their final velocity when moving in the same and opposite directions.*

ANS. $4\frac{15}{16}$ ft. per sec.; $1\frac{15}{16}$ ft. per sec. in the direction of the larger velocity.

(4) *Two non-elastic bodies of 3 lbs. and 1 oz. are moving in opposite directions and impinge centrally. The first has a velocity of* $3\frac{1}{2}$ *and the latter of 9 ft. per sec. In what direction do they move after impact?*

ANS. In the direction of the first with a velocity $3\frac{12}{49}$ ft. per sec.

(5) *A non-elastic body whose mass is 16 lbs. moving with a velocity of 25 miles an hour impinges centrally on another moving in the opposite direction. The two come to rest. If the mass of the latter were 28 lbs., find its velocity. If the velocity of the latter were 66 ft. per sec., find its mass.*

ANS. $14\frac{2}{7}$ miles per hour; $8\frac{8}{9}$ lbs.

(6) *A number of non-elastic balls of masses* $m_1, m_2, m_3, \ldots m_n$ *lie on a straight line at rest. If the first have a velocity of* u_1 *toward the others, what will be the ultimate velocity of all?*

ANS. $v_n = \dfrac{m_1 u_1}{m_1 + m_2 + \ldots m_n}$.

(7) *If in the preceding example the initial velocities are* $u_1, u_2, u_3 \ldots u_n$, *find the ultimate velocity.*

ANS. $v_n = \dfrac{m_1 u_1 + m_2 u_2 + \ldots m_n u_n}{m_1 + m_2 + \ldots m_n}$.

(8) *If in a machine 16 impacts per minute occur between two non-elastic masses* $m_1 = 100$ *lbs. and* $m_2 = 1200$ *lbs. moving in the velocities* $u_1 = 5$ *and* $u_2 = 2$ *ft. per sec., find the loss of energy.*

ANS. $\dfrac{16}{60} \cdot \dfrac{(5-2)^2}{2g} \cdot \dfrac{1000 \times 1200}{2200} = 20.3$ ft.-pounds per sec.

(9) *If two trains* $m_1 = 120000$ *lbs. and* $m_2 = 160000$ *lbs. come into collision with the opposite velocities* $u_1 = 20$ *and* $u_2 = 15$ *ft. per sec., find the loss of energy which is expended in the destruction of the cars, considering them as non-elastic.*

ANS. $\dfrac{(20+15)^2}{2g} \cdot \dfrac{120000 \times 160000}{280000} = 1\,302\,000$ ft.-pounds.

Direct Central Impact—Perfect Elasticity.—When two bodies collide, and the impact is direct central, there is a short interval of compression at the end of which both masses have a common velocity $\bar{v}$. If the bodies are non-elastic, they remain in contact. If, however, they are elastic, there is another short interval of expansion at the end of which m_1 has the final velocity v_1, and m_2 the final velocity v_2. If the bodies are *perfectly* elastic, the works of compression and of expansion are equal, and hence *no energy is lost.*

By the principle of conservation of the centre of mass (page 300) we have the velocity $\bar{v}$ of the centre of mass,

$$\bar{v} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2},$$

or

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2. \quad \ldots \ldots \ldots \quad (1)$$

That is, by the principle of conservation of momentum *the momentum before equals the momentum after impact.*

Also, since for perfectly elastic impact no energy is lost, the energy before must equal the energy after impact, or

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2. \quad \ldots \ldots \ldots \quad (2)$$

From (1) and (2) we have at once

$$\left.\begin{aligned} v_1 &= u_1 - \frac{2m_2(u_1 - u_2)}{m_1 + m_2}, \\ v_2 &= u_2 + \frac{2m_1(u_1 - u_2)}{m_1 + m_2}. \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (3)$$

If we subtract the first of these from the second, we have

$$v_2 - v_1 = u_1 - u_2.$$

That is, *the velocity of approach* $(u_1 - u_2)$ *equals the velocity of separation* $(v_2 - v_1)$.

The loss and gain of velocity are

$$u_1 - v_1 = \frac{2m_2(u_1 - u_2)}{m_1 + m_2}, \qquad v_2 - u_2 = \frac{2m_1(u_1 - u_2)}{m_1 + m_2},$$

or *twice as much* as for non-elastic impact.

We take velocities in the direction of u_1 positive, in the opposite direction negative.

If the mass m_2 is at rest, we have $u_2 = 0$, and

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} \cdot u_1, \qquad v_2 = \frac{2m_1}{m_1 + C_2} \cdot u_1.$$

If the bodies move toward each other, u_2 is negative, and

$$v_1 = u_1 - \frac{2m_2(u_1 + u_2)}{m_1 + m_2}, \qquad v_2 = -u_2 + \frac{2m_1(u_1 + u_2)}{m_1 + m_2}.$$

In this case, if the momentums of the bodies are equal, or $m_1u_1 = m_2u_2$, we have

$$v_1 = -u_1, \qquad v_2 = +u_2.$$

That is, the bodies after impact move in opposite directions with the same speeds they originally had. If, on the other hand, the masses are equal, we have

$$v_1 = -u_2, \qquad v_2 = u_1.$$

That is, each body returns with the velocity the other body had before impact.

If the bodies move in the same direction and the mass m_2 of the one in advance is infinite, we have

$$v_2 = u_2,$$

or the velocity of the infinite mass is not changed by the impact. If the infinite mass is at rest, or $u_2 = 0$, we have

$$v_1 = -u_1, \qquad v_2 = 0.$$

That is, the velocity of the impinging body is reversed.

Examples.—(1) *Two perfectly elastic balls weighing 10 lbs. and 16 lbs. collide centrally with the velocities 12 and 6 feet per sec. Find the loss and gain of velocity and the final velocities after collision, if the initial velocities are in the same and in opposite direction.*

ANS. In the first case the final velocities are $v_1 = +4\frac{8}{13}$ and $v_2 = +10\frac{8}{13}$ ft. per sec. The first body loses, then, $7\frac{5}{13}$ and the other gains $4\frac{8}{13}$ ft. per sec.

In the second case the final velocities are $v_1 = -10\frac{2}{13}$ and $v_2 = +7\frac{11}{13}$ ft. per sec. Each body then rebounds with these velocities. The first body loses $22\frac{2}{13}$ and the other gains $13\frac{11}{13}$ ft. per sec.

(2) *A number of perfectly elastic balls of masses $m_1, m_2, m_3 \ldots m_n$ lie in a straight line at rest. If the first have a velocity of u_1 toward the others, find their velocities after impact.*

ANS. The velocity of the first is

$$v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2}.$$

The velocity of any intermediate ball is

$$v_n = \frac{2^{n-1} m_1 \,.\, m_2 \ldots m_{n-1} \,.\, (m_n - m_{n+1})u_1}{(m_1 + m_2)(m_2 + m_3) \ldots (m_n + m_{n+1})}.$$

The velocity of the last ball is

$$v_n = \frac{2^{n-1} m_1 \,.\, m_2 \ldots m_{n-1} u_1}{(m_1 + m_2)(m_2 + m_3) \ldots (m_{n-1} + m_n)}.$$

(3) *In the preceding example let there be four balls, the mass of the first* m_1, *of the second* $m_2 = am_1$, *of the third* $m_3 = am_2 = a^2m_1$, *of the fourth* $m_4 = am_3 = a^3m_1$.

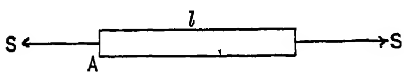
ANS. $v_1 = \frac{1 - a}{1 + a}u_1, \qquad v_2 = \frac{2}{1 + a}u_1, \qquad v_3 = \frac{4}{(1 + a)^2}, \qquad v_4 = \frac{8}{(1 + a)^3}u_1.$

If, for example, the mass of each ball is one half that of the preceding,

$$v_1 = \frac{1}{3}u_1, \qquad v_2 = \frac{4}{3}u_1, \qquad v_3 = \frac{16}{9}u_1, \qquad v_4 = \frac{64}{27}u_1.$$

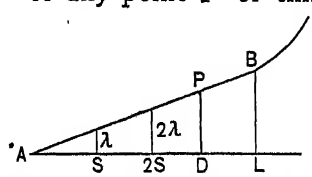
Coefficient of Elasticity.—No body is perfectly elastic or absolutely non-elastic. We deal with bodies more or less elastic, that is, imperfectly elastic.

Let a prismatic body of length l and area of cross-section A be acted upon by a stress S in its axis, and let the elongation or compression or, in general, the *strain* be denoted by λ.



We know by experiment that within a certain limit, twice, three times, or four times, etc., the stress S will cause a strain of 2λ, 3λ, 4λ, etc. The limit up to which this law of proportionality of stress to strain holds true, for any material, is called the *elastic limit* for that material for the kind of stress under consideration.

Thus if we lay off the successive stresses to any convenient scale horizontally, and lay off the corresponding observed strains to any convenient scale vertically, we obtain within the elastic limit a straight line AB. The co-ordinates AD and DP of any point P of this line give the stress and corresponding strain. The point B at which the straight line begins to curve gives the stress AL, and this is the elastic limit stress.



We see, then, that within the elastic limit the ratio $\frac{S}{\lambda}$ of stress to strain is constant. Now if A is the area of cross-section of the prism, then $\frac{S}{A}$ is the unit stress, or stress per square inch. Also, if l is the original length, then $\frac{\lambda}{l}$ is the unit strain, or strain per unit of length. If then the experiment were made upon a prism of one unit area and one unit length, we should have within the elastic limit the ratio $\frac{S}{A} \div \frac{\lambda}{l}$, or $\frac{Sl}{A\lambda}$ constant. This constant for any material is called the *coefficient of elasticity* for that material, for the kind of stress under consideration. We denote it by E. We have then within the elastic limit

$$E = \frac{Sl}{A\lambda},$$

and we can define the coefficient of elasticity in any case as the *unit stress divided by the unit strain.*

Values of E for various materials, as determined by experiment, are given on page 478.

Examples.—(1) *A wrought-iron rod 30 feet long and 4 square inches in cross-section is subjected to a tensile stress of 4000 pounds. The elongation is found to be 0.01 ft. Find E.*

ANS. The unit stress is $\frac{S}{A} = \frac{4000}{4} = 1000$ pounds per square inch. The unit strain is $\frac{\lambda}{l} = \frac{1}{3000}$ ft. per foot of length. We have then

$$E = \frac{Sl}{A\lambda} = 30\,000\,000 \text{ pounds per square inch.}$$

(2) *Taking E for wrought iron as thus determined, find the compression of a wrought-iron prism 2 ft. long and 12 square inches cross-section under a stress of 150000 pounds.*

ANS. $\lambda = \frac{Sl}{AE} = \frac{150000 \times 24}{12 \times 30\,000\,000} = \frac{1}{100}$ inch.

Modulus of Elasticity.—Let F_c be the force of compression when two bodies collide at the end of the period of compression, and F_r the force of expansion or force of restitution at the beginning of the period of expansion. If a body is non-elastic, F_r is zero. If a body is perfectly elastic, $F_r = F_c$. Strictly speaking, no bodies are absolutely non-elastic or perfectly elastic. We deal with bodies imperfectly elastic where F_r is less than F_c. The ratio $\frac{F_r}{F_c}$ of the force of restitution to the force of compression is found by experiment to be a constant for any body so long as the limit of elasticity is not exceeded. This ratio we denote by e and call it the *modulus of elasticity*. We have then

$$e = \frac{F_r}{F_c}.$$

If we let a sphere of mass m fall from a height h upon a relatively very large rigidly supported mass of the same material, it will rebound to a height h', less than the original height. The velocity with which it strikes is $u_1 = \sqrt{2gh}$, and the velocity of rebound is $v_1 = \sqrt{2gh'}$. If the interval of compression t is very short, we have the force of compression

$$F_c = \frac{mu_1}{t} = \frac{m\sqrt{2gh}}{t},$$

and the force of restitution

$$F_r = \frac{mv_1}{t} = \frac{m\sqrt{2gh'}}{t}.$$

We have then

$$e = \frac{\sqrt{h'}}{\sqrt{h}}.$$

We can thus experimentally determine the modulus of elasticity e for various materials. For perfect elasticity $e = 1$, and for non-elastic bodies $e = 0$.

The following average values have been thus determined by experiment:

Cast iron, $e = 1$ nearly. Glass, $e = \frac{15}{16}$. Steel, $e = \frac{5}{9}$. Clay, wood, $e = 0$ nearly.

Direct Central Impact—Imperfect Elasticity.—Let F_c be the force at the end of the period of compression when both bodies are moving with the common velocity

$$\bar{v} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}.$$

Let λ_1 be the compression of m_1, and λ_2 of m_2. Then the work of compression is, since $\frac{1}{2}F_c$ is the average force,

$$\frac{1}{2}F_c(\lambda_1 + \lambda_2).$$

This work should equal the energy lost during compression, or

$$\frac{1}{2}F_c(\lambda_1 + \lambda_2) = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)\bar{v}^2,$$

or putting for $\bar{v}$ its value, we obtain for F_c in pounds, if we take mass in lbs. and distance in feet and velocity in ft. per sec.,

$$F_c = \frac{m_1m_2}{(m_1 + m_2)g} \cdot \frac{(u_1 - u_2)^2}{\lambda_1 + \lambda_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let F_r' be the force of restitution at the beginning of the period of expansion of m_1, and λ_1' its expansion. Let F_r'' be the force of restitution at the beginning of the period of expansion of m_2, and λ_2' its expansion. Then the work of expansion is

$$\frac{1}{2}F_r'\lambda_1' + \frac{1}{2}F_r''\lambda_2'.$$

The total energy lost then is, in foot-pounds,

$$\frac{1}{2}F_c(\lambda_1 + \lambda_2) - \frac{1}{2}F_r'\lambda_1' - \frac{1}{2}F_r''\lambda_2' = \frac{1}{2g}m_1u_1^2 + \frac{1}{2g}m_2u_2^2 - \frac{1}{2g}m_1v_1^2 - \frac{1}{2g}m_2v_2^2. \quad . \quad (2)$$

Now, from page 346, we have

$$\lambda_1 = \frac{F_cl_1}{A_1E_1} \quad \text{and} \quad \lambda_2 = \frac{F_cl_2}{A_2E_2},$$

where l_1 and l_2 are the lengths of the masses m_1 and m_2, A_1 and A_2 their areas of cross-section, and E_1 and E_2 their coefficients of elasticity.

For the sake of simplicity we put the quantities

$$\frac{F_c}{\lambda_1} = \frac{A_1E_1}{l_1} = H_1 \quad \text{and} \quad \frac{F_c}{\lambda_2} = \frac{A_2E_2}{l_2} = H_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The quantity $H = \frac{AE}{l}$ in general we call the *hardness* of a body.

The hardness of a body is then the ratio of the stress to strain.

We can write, then,

$$\lambda_1 = \frac{F_c}{H_1} \quad \text{and} \quad \lambda_2 = \frac{F_c}{H_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where H_1 and H_2 are given by (3).

We also have from page 347, since stress and strain are proportional,

$$\frac{F_r'}{F_c} = \frac{\lambda_1'}{\lambda_1} = e_1 \quad \text{and} \quad \frac{F_r''}{F_c} = \frac{\lambda_2'}{\lambda_2} = e_2,$$

where e_1 and e_2 are the moduli of elasticity for the masses m_1 and m_2. We have then

$$F_r' = e_1F_c, \qquad F_r'' = e_2F_c, \qquad \lambda_1' = e_1\lambda_1, \qquad \lambda_2' = e_2\lambda_2. \quad . \quad . \quad . \quad . \quad (5)$$

If we substitute the values for λ_1, λ_2, F_r', F_r'', λ_1' and λ_2', given by equations (4) and (5), in equations (1) and (2), we obtain

$$F_c = (u_1 - u_2)\sqrt{\frac{m_1 m_2}{(m_1 + m_2)g} \cdot \frac{H_1 H_2}{H_1 + H_2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$$\frac{1}{2}F_c^2\left[\frac{1 - e_1^2}{H_1} + \frac{1 - e_2^2}{H_2}\right] = \frac{1}{2g}m_1 u_1^2 + \frac{1}{2g}m_2 u_2^2 - \frac{1}{2g}m_1 v_1^2 - \frac{1}{2g}m_2 v_2^2.$$

We also have, from the principle of conservation of momentum (page 300),

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2.$$

From these three equations we obtain

$$\left.\begin{aligned} u_1 - v_1 &= (u_1 - u_2)\frac{m_2}{m_1 + m_2}\left[1 + \sqrt{\frac{e_2^2 H_1 + e_1^2 H_2}{H_1 + H_2}}\right], \\ v_2 - u_2 &= (u_1 - u_2)\frac{m_1}{m_1 + m_2}\left[1 + \sqrt{\frac{e_2^2 H_1 + e_1^2 H_2}{H_1 + H_2}}\right]. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Equations (7) are general and give the final velocities v_1 and v_2 of m_1 and m_2. If both masses are of the same material, $H_1 = H_2$ and $e_1 = e_2 = e$, and we have

$$\left.\begin{aligned} v_1 &= u_1 - (u_1 - u_2)\frac{(1 + e)m_2}{m_1 + m_2}, \\ v_2 &= u_2 + (u_1 - u_2)\frac{(1 + e)m_1}{m_1 + m_2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

If the bodies are perfectly elastic, we have $e_1 = e_2 = 1$ and equations (7) reduce to equations (3), page 345. If the bodies are non-elastic, $e_1 = 0$, $e_2 = 0$, $v_1 = v_2 = \bar{v}$, and equations (7) reduce to equation (1), page 342.

Velocities in the direction of u_1 are positive, in the opposite direction negative.

Examples.—(1) *If an iron sledge of mass $m_1 = 50$ lbs., $l_1 = 6$ inches long and $A_1 = 4$ square inches area of face, strikes an immovable lead plate $l_2 = 1$ inch thick and $A_2 = 2$ square inches area, with a velocity $u_1 = 50$ ft. per sec., find the force of impact and the compression of the sledge and plate, taking $E_1 = 29\,000\,000$ and $E_2 = 700\,000$ pounds per square inch.*

ANS. We have $H_1 = \dfrac{4 \times 29\,000\,000}{\frac{1}{2}} = 232\,000\,000$ pounds per ft.

$H_2 = \dfrac{2 \times 700\,000}{\frac{1}{12}} = 16\,800\,000$ pounds per ft. We also have $u_2 = 0$ and $m_2 = \infty$; hence, from (6), the force of impact is

$$F_c = u_1\sqrt{\frac{m_1}{g} \cdot \frac{H_1 H_2}{H_1 + H_2}} = 50\sqrt{\frac{50}{g} \cdot \frac{232\,000\,000 \times 16\,800\,000}{248\,800\,000}},$$

or, taking $g = 32$ ft.-per-sec. per sec.,

$$F_c = 247\,350 \text{ pounds.}$$

From (4) we have for the compression of the sledge and plate

$$\lambda_1 = \frac{F_c}{H_1} = 0.0016 \text{ ft.} = 0.0127 \text{ in.}, \qquad \lambda_2 = \frac{F_c}{H_2} = 0.0147 \text{ ft.} = 0.177 \text{ in.}$$

(2) *In the preceding example consider the sledge as perfectly elastic and the plate as inelastic.*

ANS. We have $u_2 = 0$, $m_2 = \infty$, $e_1 = 1$ and $e_2 = 0$.

Hence, from (7),

$$v_1 = u_1 - u_1\left[1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right] = 50 - 50\left[1 + \sqrt{\frac{16\,800\,000}{248\,800\,000}}\right] = -13 \text{ ft. per sec.}$$

That is, the sledge rebounds with this velocity.

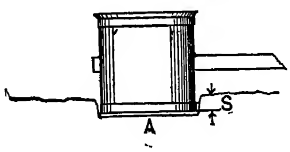
(3) *Find the velocities after impact of two steel masses, if the velocities before impact are $u_1 = 10$ and $u_2 = -6$ ft. per sec., and the masses $m_1 = 30$ lbs., $m_2 = 40$ lbs., taking $e = \frac{5}{9}$.*

$$\text{ANS. } v_1 = 10 - 16\,.\frac{40}{70}\left(1 + \frac{5}{9}\right) = -4.22 \text{ ft. per sec.,}$$

$$v_2 = -6 + 16\,\frac{30}{70}\left(1 + \frac{5}{9}\right) = +4.665 \text{ ft. per sec.}$$

That is, m_1 rebounds and m_2 rebounds.

Earth Consolidation.—When a maul strikes a mass of soft earth it compresses the earth with a certain force F. Let s be the depth of penetration, m the mass of the maul, and h the height from which it is let fall. Then the energy of the maul before it is dropped is mh. Since this energy is expended in compressing the soil, we have



$$Fs = mh, \quad \text{or} \quad F = \frac{mh}{s}.$$

If we divide this force F by the cross-section A of the maul, we have for the unit force of compression

$$\frac{F}{A} = \frac{mh}{As}.$$

The resistance F of soils to the penetration of the maul is generally variable and increases with the depth s of penetration. In many cases we may assume it to increase directly with the penetration. In such case we should have

$$\frac{1}{2}Fs = mh, \quad \text{or} \quad F = \frac{2mh}{s}, \quad \text{and} \quad \frac{F}{A} = \frac{2mh}{As},$$

or twice as much as before.

If A is taken in square inches and s and h in feet or inches and m in lbs., $\frac{F}{A}$ is the number of pounds per square inch resistance of the soil. Allowing a factor of safety of 10, we could then safely load the compacted soil up to $\frac{1}{10}\frac{F}{A}$.

Example.—*A maul whose mass is 120 lbs. falls from a height of 18 inches, and the earth is compressed one eighth of an inch by the last blow. The cross-section is 16 square inches. Taking a factor of safety of 10, find the safe load.*

ANS. $\frac{F}{A} = \frac{mh}{AS} = \frac{120 \times 18}{16 \times \frac{1}{8}} = 1080$ pounds per sq. inch. Taking a factor of safety of 10, we have 108 pounds per sq. inch safe load.

Pile-driving.—Let m_1 be the mass of the ram and h the height of fall, so that the velocity of impact is $u_1 = \sqrt{2gh}$.

The energy of the ram is $m_1 h$. Let s be the depth of penetration for a blow, and λ_1 the compression of ram, and λ_2 of the pile. Then we have, if F is the force of compression,

$$Fs + \frac{1}{2}F(\lambda_1 + \lambda_2) = m_1 h. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From equations (4) and (6), page 349, we have, making $u_2 = 0$ and $u_1 = \sqrt{2gh}$,

$$\lambda_1 + \lambda_2 = \sqrt{\frac{2m_1 m_2 h}{m_1 + m_2} \cdot \frac{H_1 + H_2}{H_1 H_2}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where m_2 is the mass of the pile, and H_1, H_2 the hardness of ram and pile, so that

$$H_1 = \frac{A_1 E_1}{l_1}, \qquad H_2 = \frac{A_2 E_2}{l_2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where A_1 and A_2 are the cross-sections of ram and pile, l_1 and l_2 their lengths, and E_1, E_2 their coefficients of elasticity.

Since E_1 and E_2 are given in pounds per sq. in. (page 478). if we take A_1 and A_2 in square inches and l_1, l_2 and h in feet, and m_1 and m_2 in lbs., H_1 and H_2 will be pounds per ft. and $\lambda_1 + \lambda_2$ will be given in feet.

Inserting this value of $\lambda_1 + \lambda_2$, we have

$$F = \frac{m_1 h}{s + \sqrt{\frac{m_1 m_2 h}{2(m_1 + m_2)} \cdot \frac{H_1 + H_2}{H_1 H_2}}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (4) we can find the resistance of the pile by measuring the penetration s.

Since the pile is wood and very long compared to the iron ram, l_2 is much greater than l_1, and E_2 much less than E_1, and hence H_1 is very large compared to H_2, and we have, approximately,

$$\frac{H_1 + H_2}{H_1 H_2} = \frac{1 + \frac{H_2}{H_1}}{H_2} = \frac{1}{H_2}.$$

Hence we have, approximately,

$$F = \frac{m_1 h}{s + \sqrt{\frac{m_1 m_2 h}{2(m_1 + m_2) H_2}}}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where m_1 and m_2 are to be taken in lbs., h and s in feet or inches, and H_2 in pounds per ft. or inch. For wood we can take $E_2 = 1\ 560\ 000$ pounds per sq. inch (page 478),

If we take a factor of safety of 6 or 10, we can safely load the pile up to $\frac{1}{6}$ or $\frac{1}{10}F$.

Example.—*A pile whose cross-section is $A_2 = 1$ sq. ft. and length $l_2 = 25$ ft. and mass $m_2 = 1200$ lbs. is driven by the last tally of ten blows of a ram of mass $m_1 = 2000$ lbs. falling $h = 6$ ft, a distance of 2 inches. Taking $E_2 = 1\ 560\ 000$, find the load the pile can safely sustain for a factor of safety of 6.*

ANS. $H_2 = \frac{A_2 E_2}{l_2} = \frac{144 \times 1\ 560\ 000}{25} = 8\ 985\ 600$ pounds per ft.

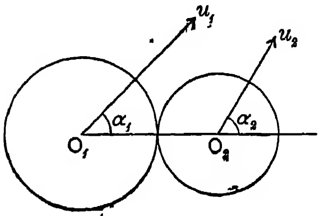
The mean depth of penetration for one blow is $s = \frac{2}{10}$ inch $= \frac{1}{60}$ ft. Hence

$$F = \frac{2000 \times 6}{\frac{1}{60} + \sqrt{\frac{2000 \times 1200 \times 6}{2(2000 + 1200) \times 8\,985\,600}}} = 378\,000 \text{ pounds.}$$

For a factor of safety of 6 we have

$$F = \frac{378\,000}{6} = 63\,000 \text{ pounds.}$$

Oblique Central Impact.—If the directions of motion u_1, u_2 make the angles α_1, α_2 with the line of impact O_1O_2, we can resolve each velocity into components $u_1 \cos \alpha_1$ and $u_2 \cos \alpha_2$ along the line of impact, and $u_1 \sin \alpha_1$, $u_2 \sin \alpha_2$ at right angles to this line. These latter components are unchanged by the impact. As to the former, we have from equations (8), page 349, if the two bodies are of the same material and e is the modulus of elasticity,



$$\left.\begin{aligned} v_1 &= u_1 \cos \alpha_1 - (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) \frac{(1+e)m_2}{m_1 + m_2}, \\ v_2 &= u_2 \cos \alpha_2 + (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) \frac{(1+e)m_1}{m_1 + m_2}, \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where m_1 and m_2 are the masses of the two bodies, and v_1; v_2 the final velocities *along the line of impact*.

The actual velocity w_1 of the first body after impact is then

$$w_1 = \sqrt{v_1^2 + u_1^2 \sin^2 \alpha_1}, \quad . \quad . \quad . \quad (2)$$

making an angle β_1 with O_1O_2 given by

$$\tan \beta_1 = \frac{u_1 \sin \alpha_1}{v_1}. \quad . \quad . \quad . \quad (3)$$

The actual velocity w_2 of the second body after impact is

$$w_2 = \sqrt{v_2^2 + u_2^2 \sin^2 \alpha_2}, \quad . \quad . \quad . \quad (4)$$

making an angle β_2 with O_1O_2 given by

$$\tan \beta_2 = \frac{u_2 \sin \alpha_2}{v_2}. \quad . \quad . \quad . \quad (5)$$

If the mass m_2 is at rest and indefinitely great, we have $m_2 = \infty$, $u_2 = 0$, and, from (1),

$$\left.\begin{aligned} v_1 &= -eu_1 \cos \alpha_1, \\ v_2 &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

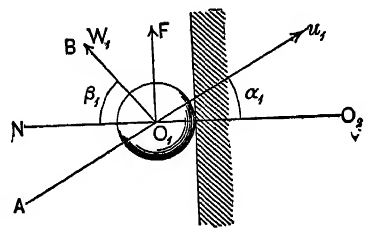
and, from (2) and (4),

$$\left.\begin{aligned} w_1 &= \sqrt{u_1^2(\sin^2 \alpha_1 + e^2 \cos^2 \alpha_1)}, \\ w_2 &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

making the angle with O_1O_2 given by

$$\tan \beta_1 = -\frac{\sin \alpha_1}{e \cos \alpha_1} = -\frac{1}{e} \tan \alpha_1. \quad \ldots \ldots \ldots \quad (8)$$

For non-elastic bodies $e = 0$, and for perfectly elastic bodies $e = 1$. For non-elastic bodies, then, from (6), (7), (8), we have $v_1 = 0$, $w_1 = u_1 \sin\alpha_1$, $\tan \beta_1 = \infty$ or $\beta = 90°$. That is, the velocity v_1 along the line of impact is annihilated, that at right angles is unchanged, and the body moves after impact in the direction O_1F at right angles to O_1O_2 with the speed $u_1 \sin \alpha_1$.



For perfectly elastic bodies $v_1 = -u_1 \cos \alpha_1$, $w_1 = u_1$, $\tan \beta_1 = -\tan \alpha_1$. That is, the velocity along the line of impact is reversed, the angle of reflection NO_1B is equal to the angle of incidence NO_1A, and the body moves after impact in the direction O_1B with the original speed u_1.

For imperfect elasticity we have, from (8),

$$e = -\frac{\tan \alpha_1}{\tan \beta_1},$$

or the modulus of elasticity is equal to the ratio of the tangent of the angle of incidence to the tangent of the angle of reflection.

Example.—*Two balls, $m_1 = 30$ lbs., $m_2 = 50$ lbs., impinge with the velocities $u_1 = 20$ and $u_2 = 25$ ft. per sec., making the angles with the line of impact $\alpha_1 = 21° 35'$ and $\alpha_2 = 65° 20'$. Find the velocities after impact if the bodies are non-elastic.*

ANS. $u_1 \sin \alpha_1 = 7.357$ ft. per sec., $u_2 \sin \alpha_2 = 22.719$ ft. per sec.,
$u_1 \cos \alpha_1 = 18.598$ " " " $u_2 \cos \alpha_2 = 10.433$ " " "

Hence

$$v_1 = 18.598 - (18.598 - 10.433)\frac{50}{80} = 13.495 \text{ ft. per sec.,}$$

$$v_2 = 10.433 + (18.598 - 10.433)\frac{30}{80} = 13.495 \text{ " " "}$$

The resulting velocities are then

$$w_1 = \sqrt{13.495^2 + 7.357^2} = 15.37 \text{ ft. per sec.,}$$
$$w_2 = \sqrt{13.495^2 + 22.719^2} = 26.42 \text{ " " "}$$

making the angles β_1 and β_2 with the line of impact given by

$$\tan \beta_1 = \frac{7.357}{13.495}, \quad \text{or} \quad \beta_1 = 28° 36',$$

$$\tan \beta_2 = \frac{22.719}{13.495}, \quad \text{or} \quad \beta_2 = 59° 17'.$$

Friction of Oblique Central Impact.—The pressure between the colliding bodies gives rise to friction. If P is the pressure due to impact, F the friction and μ the coefficient of friction, we have

$$F = \mu P.$$

Let the mass of the impinging body be m_1, and the initial and final velocity along the line of impact be u_1 and v_1, and t be the very short time of impact. Then we have for the impulse (page 255)

$$Ft = m_1(u_1 - v_1), \quad \text{or} \quad F = \frac{m_1(u_1 - v_1)}{t}.$$

Hence the friction is

$$F = \frac{\mu m_1(u_1 - v_1)}{t}, \quad \text{or} \quad \frac{Ft}{m_1} = \mu(u_1 - v_1).$$

That is, *the impulse of the friction divided by the mass is equal to μ times the change of velocity along the line of impact; or the change of velocity due to friction, at right angles to the line of impact, is equal to μ times the change of velocity along the line of impact.*

This change of velocity is always a retardation, since friction is a retarding force.

Thus if a mass m_1 falls vertically with a velocity u_1 upon a horizontal sled of mass m_2 moving horizontally with the velocity u_2, and if the velocity u_1 is entirely lost by the collision, we have for the friction

$$F = \frac{\mu m_1 u_1}{t}.$$

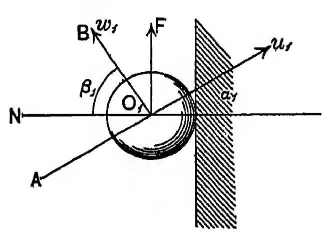
But the retarding force during the time t for both masses in contact, if v is the common velocity after impact, is also

$$F = \frac{(m_1 + m_2)(u_2 - v)}{t}. \qquad (1)$$

Hence we have for the change of velocity of the sled

$$u_2 - v = \frac{\mu m_1 u_1}{m_1 + m_2}, \quad \text{or} \quad v = u_2 - \frac{\mu m_1 u_1}{m_1 + m_2}. \qquad (2)$$

If a body of mass m_1 strikes an immovable mass with a velocity u_1 at an angle α_1, we have from equations (7), page 349, for the change of velocity along the line of impact, since $u_2 = 0$ and $m_2 = \infty$, $l_2 = \infty$ and $H_2 = 0$,



$$u_1 \cos \alpha_1 - v_1 = u_1 \cos \alpha_1(1 + e_2).$$

Hence the change of velocity due to friction is

$$\mu u_1 \cos \alpha_1(1 + e_2),$$

and after impact the component $u_1 \sin \alpha_1$ becomes

$$u_1 \sin \alpha_1 - \mu u_1 \cos \alpha_1(1 + e_2) = [\sin \alpha_1 - \mu \cos \alpha_1(1 + e_2)]u_1. \qquad (3)$$

For perfectly elastic bodies $e_2 = 1$ and (3) becomes

$$(\sin \alpha_1 - 2\mu \cos \alpha_1)\mu_1,$$

and for non-elastic bodies $e_2 = 0$ and (3) becomes

$$(\sin \alpha_1 - \mu \cos \alpha_1)u_1.$$

The friction often causes bodies to turn about an axis through the centre of mass, or if before impact rotation exists, that motion is changed.

Thus let r be the radius of a sphere, ω_1 its initial and ω its final angular velocity during the time t of impact.

Then the moment of the friction is $I \cdot \frac{\omega - \omega_1}{t}$ (page 322). But the moment of the friction is also $\frac{m_1 \mu u_1 \cos \alpha_1(1 + e_2)r}{t}$. Hence

$$\omega - \omega_1 = \frac{\mu m_1 u_1 \cos \alpha_1(1 + e_2)r}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) gives the change of angular velocity.

Example.—*A billiard-ball strikes the cushion with a velocity $u_1 = 15$ ft. per sec., the angle of incidence being $\alpha_1 = 45°$. If $e_2 = 0.55$ and the coefficient of friction is $\mu = 0.2$, find the motion after impact.*

ANS The velocity after impact along the line of impact is

$$v_1 = -e_2 u_1 \cos \alpha_1 = - 0.55 \times 15 \cos 45° = -5.833 \text{ ft. per sec.}$$

The velocity parallel to the cushion is

$$u_1 \sin \alpha_1 - \mu u_1 \cos \alpha_1(1 + e_2) = 7.319 \text{ ft. per sec.}$$

Hence the angle of reflection β_1 is given by

$$\tan \beta_1 = \frac{7.319}{5.833}, \quad \text{or} \quad \beta_1 = 51° \, 27',$$

and the velocity after impact is

$$w_1 = \frac{5.833}{\cos 51° \, 27'} = 9.36 \text{ ft. per sec.}$$

The ball also acquires the angular velocity ω_z about a vertical axis through the centre of mass. Since $I = \frac{2}{5}m_1 r^2$, where m_1 is the mass and r the radius of the ball:

$$\omega_z = \frac{0.2 m_1 \times 15 \cos 45° \times 1.55 r}{\frac{2}{5}m_1 r^2} = \frac{8.22}{r} \text{ radians per sec.}$$

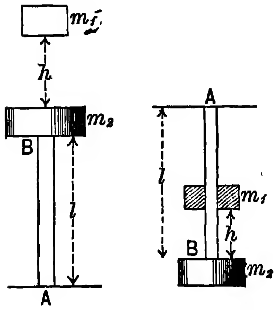
The ball also has angular velocity ω_x about a horizontal axis through the centre of mass at right angles to w_1 given by

$$\omega_x = \frac{w_1}{r} = \frac{9.36}{r} \text{ radians per sec.}$$

The resultant angular velocity is then

$$\omega = \sqrt{\omega_x^2 + \omega_z^2} = \frac{12.45}{r} \text{ radians per sec.}$$

about an axis through the centre of mass in a vertical plane at right angles to w_1, making an angle with the vertical whose tangent is $\frac{\omega_x}{\omega_z} = \frac{9.36}{8.22} = 1.138$, or $48° \, 42'$.



Strength and Impact.—Let the mass m_1, moving with the velocity u_1, impinge on the mass m_2 which is supported by the rod AB of uniform cross-section A and length l. Let $\bar{v}$ be the velocity of both masses during impact. Then, by conservation of momentum,

$$m_1 u_1 = (m_1 + m_2)\bar{v}, \quad \text{or} \quad \bar{v} = \frac{m_1 u_1}{m_1 + m_2},$$

and the work in foot-pounds necessary to bring the combined masses to rest is

$$\text{work} = \frac{(m_1 + m_2)}{2g}\bar{v}^2 = \frac{u_1^2}{2g} \cdot \frac{m_1^2}{m_1 + m_2} = \frac{m_1^2 h}{m_1 + m_2}, \quad . \quad (1)$$

where $\frac{u_1^2}{2g} = h$ is the height of fall of m_1.

This work is equal to the work of stretching or compressing the rod. If F is the force of impact at the end of the compression or stretch, λ, $\frac{F}{2}$ is the average force and $\frac{F}{2} \cdot \lambda$ is the work. But, from page 346, within the limit of elasticity we have

$$F = \frac{EA\lambda}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where E is the coefficient of elasticity. Hence

$$\frac{1}{2}F\lambda = \frac{EA\lambda^2}{2l} = \frac{m_1^2 h}{m_1 + m_2}, \quad \text{or} \quad \lambda = \sqrt{\frac{m_1^2}{m_1 + m_2} \cdot \frac{2lh}{EA}}. \quad . \quad . \quad . \quad . \quad (3)$$

From (3) we can find the strain λ of the rod caused by the impact. Let the rod be strained up to the elastic limit unit stress S_e. Then we have from (2), by putting $F = S_e A$,

$$\lambda = \frac{S_e l}{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and hence, from (3),

$$\frac{S_e^2}{2E} \cdot Al = \frac{m_1^2 h}{m_1 + m_2}.$$

But Al is the volume of the rod V. The velocity of impact

$$u_1 = \sqrt{2gh},$$

which is necessary to strain the rod up to the limit of elasticity, is then given by

$$h = \frac{m_1 + m_2}{m_1^2} \cdot \frac{S_e^2}{2E} \cdot V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The quantity $\frac{S_e^2}{2E}$ is called the *coefficient of resilience* (page 516). We see from (5) that the greater the volume or mass of the rod, the greater the blow it can bear. Hence the mass of bodies subjected to impact should be made as great as possible.

Since m_1 and m_2 fall during impact through the distance λ, we have more correctly

$$\text{work} = \frac{m_1^2 h}{m_1 + m_2} + (m_1 + m_2)\lambda,$$

and hence, instead of (5), we have

$$h = \frac{m_1 + m_2}{m_1^2} \cdot \frac{S_e^2}{2E} \cdot V - \frac{(m_1 + m_2)^2}{m_1^2} \cdot \frac{S_e l}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

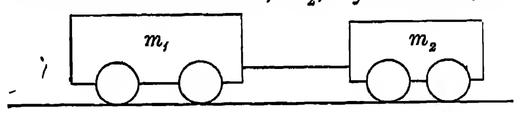
If, finally, we wish to take into account the mass m_3 of the rod, we have, since its centre of mass moves through the distance $\frac{1}{2}\lambda$,

$$\text{work} = \frac{m_1^2 h}{m_1 + m_2 + m_3} + \left(m_1 + m_2 + \frac{1}{2} m_3\right)\lambda,$$

and hence, instead of (5), we have

$$h = \frac{m_1 + m_2 + m_3}{m_1^2} \cdot \frac{S_e^2}{2E} \cdot V - \frac{\left(m_1 + m_2 + m_3\right)\left(m_1 + m_2 + \frac{1}{2} m_3\right)}{m_1^2} \cdot \frac{S_e l}{E}. \quad . \quad (7)$$

If a mass m_1 moving with a velocity u_1 puts in motion another mass, m_2, by means of a chain or rope, we have in the same way for the velocity of both bodies during impact



$$\overline{v} = \frac{m_1 u_1}{m_1 + m_2},$$

and the work in foot-pounds expended in stretching the chain is

$$\text{work} = \frac{m_1 u_1^2}{2g} - \frac{1}{2g}(m_1 + m_2)\overline{v}^2 = \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{u_1^2}{2g} = \frac{m_1 m_2}{m_1 + m_2} \cdot h,$$

where $h = \frac{u_1^2}{2g}$ is the height due to the velocity u_1.

We have then, if the chain is stretched to the elastic limit unit stress S_e,

$$\frac{S_e^2}{2E} \cdot Al = \frac{m_1 m_2}{m_1 + m_2} \cdot h,$$

where A is the cross-section and l the length of the chain. Hence

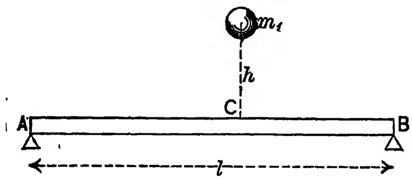
$$h = \frac{u_1^2}{2g} = \frac{m_1 + m_2}{m_1 m_2} \cdot \frac{S_e^2}{2E} \cdot Al. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Example.—*The two opposite suspension rods of a suspension bridge support a constant load of 5000 pounds, which is increased by 6000 pounds by a passing load. The coefficient of resilience $\frac{S_e^2}{2E}$ for wrought iron is 7 pounds per square inch (page 516). The length of rod is 200 inches, and cross-section 1.5 square inches. Find the height of fall to stretch the rods to the limit of elasticity.*

$$\text{ANS. } h = \frac{(5000 + 6000) \times 7 \times 200 \times 1.5 \times 2}{6000^2} = 1.28 \text{ inches.}$$

If then a cart of 6000 pounds should pass over an obstacle of 1.3 inches high, and drop, the rods are in danger of being stretched beyond the elastic limit.

Impact of Beams.—Let a mass m_1 fall from a height h upon a beam AB of uniform cross-section A and span l, supported at the ends.



Let δ be the density of the beam, and V its volume. Then

$$V = Al,$$

and the mass m_2 of the beam is

$$m_2 = \delta Al = \delta V.$$

Let the mass m_1 strike the beam at the centre C, and let the greatest velocity at the centre be v_c, and at any other point of the beam be v.

Then the work in foot-pounds necessary to bring the combined masses to rest is

$$\text{work} = \frac{m_1 v_c^2}{2g} + \int_0^l \frac{\delta A dx}{2g} \cdot v^2.$$

If y is the deflection at any point, and $\varDelta$ the deflection at the centre, we have

$$v = \frac{y}{\varDelta} v_c.$$

We have then

$$\text{work} = \frac{m_1 v_c^2}{2g} + \frac{\delta A v_c^2}{2g} \int_0^l \frac{y^2 dx}{\varDelta^2}.$$

Now if P is the pressure at the centre, we have, from page 543,

$$\varDelta = \frac{Pl^3}{48EI}, \quad \text{and} \quad y = \frac{P}{12EI}\left(x^3 - \frac{3}{4} l^2 x\right),$$

where E is the coefficient of elasticity, I the moment of inertia of the cross-section of the beam relative to the horizontal axis through the centre of mass of the cross-section at right angles to the length of the beam, and x is the distance of any point from the left end. We have then

$$\frac{y}{\varDelta} = \frac{4}{l^3}\left(x^3 - \frac{3}{4} l^2 x\right),$$

and substituting in the expression for the work and integrating, we have

$$\text{work} = \frac{m_1 v_c^2}{2g} + \frac{17 \delta Al}{35} \cdot \frac{v_c^2}{2g}.$$

The distance through which the point C moves in the short time t is $\frac{v_c}{2} \cdot t$. If we divide the work by this distance, we have the force which brings the bodies to rest. This force should equal $\frac{m_1 u_1}{gt}$. Hence

$$m_1 u_1 = m_1 v_c + \frac{17 \delta Al}{35} \cdot v_c, \quad \text{or} \quad v_c = \frac{m_1 u_1}{m_1 + \frac{17 \delta Al}{35}}.$$

If we insert this value of v_c, we have for the work of bringing the combined masses to rest

$$\text{work} = \frac{m_1^2 u_1^2}{2g\left(m_1 + \frac{17\delta A l}{35}\right)} = \frac{m_1^2 h}{m_1 + \frac{17\delta A l}{35}}.$$

But this work is also equal to $\frac{1}{2}P\Delta$. If the elastic limit is reached, we have, from page 500,

$$P = \frac{4S_e I}{vl} \quad \text{and} \quad \Delta = \frac{S_e l^2}{12vE},$$

where S_e is the elastic limit unit stress, and v is the distance of the most remote fibre of the cross-section from the neutral axis.

Hence

$$\frac{1}{2}P\Delta = \frac{S_e^2 I l}{6v^2 E} = \frac{m_1^2 h}{m_1 + \frac{17\delta A l}{35}}, \quad \text{or} \quad h = \frac{S_e^2}{2E} \cdot \frac{Il}{3v^2} \cdot \frac{m_1 + \frac{17\delta A l}{35}}{m_1^2}.$$

Since $\delta A l$ is the mass m_2 of the beam,

$$h = \frac{S_e^2}{2E} \cdot \frac{Il}{3v^2} \cdot \frac{m_1 + \frac{17}{35}m_2}{m_1^2}.$$

If, for instance, the cross-section of the beam is a rectangle of breadth b and depth d, we have $I = \frac{1}{12}bd^3$ and $v = \frac{1}{2}d$. Hence for this case

$$h = \frac{S_e^2}{2E} \cdot \frac{bdl}{9} \cdot \frac{m_1 + \frac{17}{35}m_2}{m_1^2}.$$

Example.—*Find the height from which a mass of 200 lbs. must fall in order that, striking the centre of a plate of cast iron 36 inches long, 12 inches wide and 3 inches thick, supported at both ends, it may bend it to the elastic limit.*

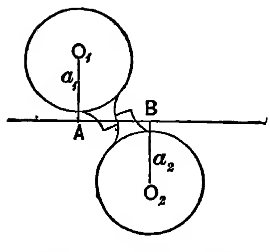
ANS. If we take the coefficient of resilience (page 516),

$$\frac{S_e^2}{2E} = 1.2 \text{ pounds per square inch},$$

we have, since δ for cast iron is about 0.259 lbs. per cubic inch, $m_2 = 12 \times 3 \times 36 \times 0.259 = 335.7$ lbs. Hence for height of fall

$$h = \frac{1.2 \times 12 \times 3 \times 36}{9} \cdot \frac{200 + \frac{17}{35}335.7}{40000} = 1.57 \text{ inches}.$$

Impact of Rotating Bodies.—Let two bodies of mass m_1 and m_2 rotate about fixed axes at O_1 and O_2 and impinge, and let AB be the line of impact. Let the normals $O_1A = a_1$ and $O_2B = a_2$.



Then (page 322) we can reduce the masses m_1 and m_2 to the equivalent masses at A and B

$$\frac{m_1\kappa_1'^2}{a_1^2} \quad \text{and} \quad \frac{m_2\kappa_2'^2}{a_2^2},$$

where κ_1' and κ_2' are the radii of gyration relative to O_1 and O_2. If then we substitute these masses in the place of m_1 and m_2 in the equations for central impact (page 349), we have for bodies of the same material

$$\left.\begin{aligned} v_1 &= u_1 - (u_1 - u_2)(1 + e)\frac{m_2\kappa_2'^2a_1^2}{m_1\kappa_1'^2a_2^2 + m_2\kappa_2'^2a_1^2}, \\ v_2 &= u_2 + (u_1 - u_2)(1 + e)\frac{m_1\kappa_1'^2a_2^2}{m_1\kappa_1'^2a_2^2 + m_2\kappa_2'^2a_1^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where u_1 and u_2 are the velocities at A and B before and v_1, v_2 after impact, and e the modulus of elasticity.

If ϵ_1 and ϵ_2 are the angular velocities before and ω_1, ω_2 after impact, we have, taking counter-clockwise rotation as positive, the origins at O_1 and O_2, and a_1, a_2 as coinciding with the axes of Y for each origin,

$$a_1\epsilon_1 = -u_1, \qquad a_2\epsilon_2 = -u_2, \qquad a_1\omega_1 = -v_1, \qquad a_2\omega_2 = -v_2.$$

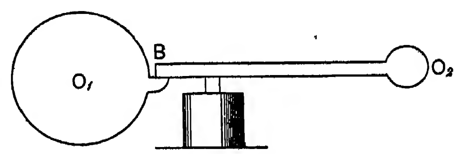
Hence

$$\left.\begin{aligned} \omega_1 &= \epsilon_1 - a_1(a_1\epsilon_1 - a_2\epsilon_2)(1 + e)\frac{m_2\kappa_2'^2}{m_1\kappa_1'^2a_2^2 + m_2\kappa_2'^2a_1^2}, \\ \omega_2 &= \epsilon_2 + a_2(a_1\epsilon_1 - a_2\epsilon_2)(1 + e)\frac{m_1k_1'^2}{m_1\kappa_1'^2a_2^2 + m_2\kappa_2'^2a_1^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

Examples.—*The moment of inertia of the shaft O_1B relative to its axis of rotation at O_1 is $m_1\kappa_1'^2 = 40000$ lb.-ft.^{2}, and that of the trip-hammer BO_2 relative to its axis of rotation at O_2 is $m_2\kappa_2'^2 = 150000$ lb.-ft.^{2}. The arm O_1B of the shaft is $a_1 = 2$ ft., and that of the hammer BO_2 is $a_2 = 6$ ft. The angular velocity of the shaft before impact is $\epsilon_1 = 1.05$ radians per sec. Find the velocity after impact and the loss of energy at each impact, supposing both bodies inelastic.*

ANS. The angular velocity of the shaft after impact is, since $e = 0$, $\epsilon_2 = 0$,

$$\omega_1 = 1.05 - \frac{4 \times 1.05 \times 150000}{40000 \times 36 + 150000 \times 4} = 0.741 \text{ radians per sec.}$$



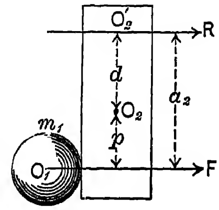
The angular velocity of the hammer after impact is

$$\omega_2 = \frac{6 \times 2 \times 1.05}{51} = 0.247 \text{ radians per sec.}$$

The loss of energy at each impact is, in foot-pounds,

$$\frac{m_1\kappa_1'^2}{2g}\epsilon_1^2 - \frac{m_1\kappa_1'^2}{2g}\omega_1^2 - \frac{m_2\kappa_2'^2}{2g}\omega_2^2 = 201.63 \text{ foot-pounds,}$$

Impact of an Oscillating Body.—If a body of mass m_1 impinges upon a body of mass m_2 which is suspended from an axis at O_2', the equations of the preceding article apply if we put in equations (1) m_1 in place of $\frac{m_1\kappa_1'^2}{a_1^2}$, and $-a_2\epsilon_2$ in place of u_2, and $-a_2\omega_2$ in place of v_2. We have then for the velocity of the mass m_1 after impact, taking clockwise rotation as positive, origin at O_2', and $O_2'O_2$ as axis of Y, if κ_1' and κ_2' are the radii of gyration for axes at O_1 and O_2,



$$v_1 = u_1 - (u_1 + a_2\epsilon_2)(1 + e)\frac{m_2\kappa_2'^2}{m_1a_2^2 + m_2\kappa_2'^2}, \quad . \quad . \quad . \quad (1)$$

and for the angular velocity of the mass m_2 after impact

$$\omega_2 = \epsilon_2 - a_2(u_1 + a_2\epsilon_2)(1 + e)\frac{m_1}{m_1a_2^2 + m_2\kappa_2'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the mass m_2 were at rest before impact, we have $\epsilon_2 = 0$ and

$$v_1 = u_1 - u_1(1 + e)\frac{m_2\kappa_2'^2}{m_1a_2^2 + m_2\kappa_2'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\omega_2 = - \quad u_1(1 + e)\frac{m_1a_2}{m_1a_2^2 + m_2\kappa_1'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If m_1 is at rest and the oscillating body impinges on it, we have $u_1 = 0$, and hence

$$v_1 = - \epsilon_2(1 + e)\frac{m_2\kappa_2'^2a_2}{m_1a_2^2 + m_2\kappa_2'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\omega_2 = \epsilon_2\left[1 - (1 + e)\frac{m_1a_2^2}{m_1a_2^2 + m_2\kappa_2'^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The angular velocity ω_2 in the first case, equation (4), or the velocity v_1 of m_1 in the second case, equation (5), is a maximum when

$$\frac{a_2}{m_1a_2^2 + m_2\kappa_2'^2}$$

is a maximum, or when

$$m_1a_2 + \frac{m_2\kappa_2'^2}{a_2}$$

is a minimum. Putting the first differential coefficient equal to zero, we have for the value of a_2 when the maximum velocity is imparted

$$a_2 = \kappa_2'\sqrt{\frac{m_2}{m_1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence the maximum velocity imparted to the oscillating body m_2 when at rest and struck by m_1 is given by

$$\omega_2 = -(1+e)\frac{u_1}{2\kappa_2'}\sqrt{\frac{m_1}{m_2}} = -(1+e)\frac{u_1}{2a_2}, \qquad (8)$$

and the maximum velocity imparted to m_1 when at rest and struck by m_2 is

$$v_1 = -\frac{1}{2}\kappa_1'\epsilon_2(1+e)\sqrt{\frac{m_2}{m_1}} = -(1+e)\frac{\epsilon_2 a_2}{2}. \qquad (9)$$

REACTION OF THE AXIS.—Let the force of impact be F, and the reaction of the axis be R. Then, from page 339,

$$R = F\left(\frac{a_2 d}{\kappa_2'^2} - 1\right). \qquad (10)$$

If we give to a_2 its value from (7), we have for the reaction of the axis when the maximum velocity is imparted

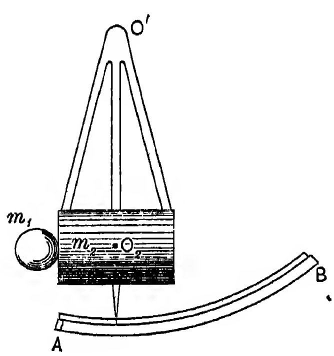
$$R = F\left(\frac{d}{\kappa_2'}\sqrt{\frac{m_2}{m_1}} - 1\right). \qquad (11)$$

The centre of percussion (page 339) is at the distance

$$a_2 = \frac{\kappa_2'^2}{d} = \frac{I_2'}{m_2 d}$$

from the axis. If the impact takes place at this distance, there is no reaction of the axis.

Ballistic Pendulum.—The ballistic pendulum consists of a large mass m_2 hung from a horizontal axis O'. It is set in oscillation by a cannon-ball shot against it, and is used to determine the velocity of the ball. In order to render the impact inelastic, the mass m_2 consists of a box filled with sand or clay, so that the ball enters the mass and oscillates with it.



In order to determine the velocity of the ball, the angle of oscillation is measured by a pointer directly below the centre of mass O, which moves on a graduated arc AB.

Let m_1 be the mass of the ball. Then, from equation (4) of the preceding article, making $e = 0$, we have for the angular velocity after impact

$$\omega_2 = -\frac{m_1 a_2 u_1'}{m_1 a_2^2 + m_2 \kappa_2'^2}, \qquad (1)$$

where κ_2' is the radius of gyration of the pendulum relative to the axis at O', and a_2 is the distance of the point of impact below this axis.

Let l be the length of the equivalent simple pendulum which oscillates in the same time as the ballistic pendulum, and let the angle measured on the arc AB be θ.

We have then for the simple pendulum (page 336) the angular acceleration

$$\alpha = \frac{g \sin \theta}{l}.$$

If $O'O = d$ is the distance of the centre of mass O of the ballistic pendulum from O' we have (page 317)

$$(m_1 + m_2) gd \sin \theta = I'\alpha = (m_1 a_2^2 + m_2 \kappa_2'^2)\alpha,$$

or

$$\alpha = \frac{(m_1 + m_2) gd \sin \theta}{m_1 a_2^2 + m_2 \kappa_2'^2}.$$

Equating these two values of α, we obtain for the length of the equivalent simple pendulum

$$l = \frac{m_1 a_2^2 + m_2 \kappa_2'^2}{(m_1 + m_2)d}. \quad \ldots \ldots \quad (2)$$

The height of displacement is

$$h = l - l \cos \theta = 2l \sin^2 \frac{\theta}{2}.$$

Hence the velocity at the lowest point of swing is

$$v = \sqrt{2gh} = 2\sqrt{gl} \sin \frac{\theta}{2},$$

and the corresponding angular velocity is

$$\omega = \frac{v}{l} = 2\sqrt{\frac{g}{l}} \cdot \sin \frac{\theta}{2}.$$

Equating this to (1) and inserting the value of l from (2), we have for the velocity of the ball

$$u_1 = 2\left(\frac{m_1 + m_2}{m_1}\right)\frac{d}{a_2}\sqrt{gl} \cdot \sin \frac{\theta}{2}. \quad \ldots \ldots \quad (3)$$

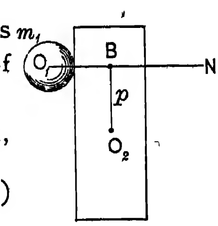
If the pendulum makes n vibrations per minute, the duration of a vibration is

$$t = \pi\sqrt{\frac{l}{g}} = \frac{60}{n}, \quad \text{and hence} \quad \sqrt{gl} = \frac{60g}{n\pi}.$$

Hence

$$u_1 = \frac{m_1 + m_2}{m_1} \cdot \frac{120gd}{n\pi a_2} \cdot \sin \frac{\theta}{2}. \quad \ldots \ldots \quad (4)$$

Eccentric Impact.—Let m_1, m_2 be the masses, O_1 and O_2 their centres of mass, BN the line of impact, and $BO_2 = p$ the distance from the line of impact to the centre of mass.



By conservation of momentum we have, so far as translation is concerned,

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2, \quad \ldots \ldots \quad (1)$$

where u_1 and u_2 are the initial and v_1, v_2 the final velocities of m_1 and m_2.

The mass m_2 reduced to B (page 322) is

$$\frac{m_2\kappa_2^2}{p^2},$$

where κ_2 is the radius of gyration of m_2 relative to axis at O_2, and the velocities of the point B before and after impact due to rotation about axis at O_2 are $-p\epsilon_2$ and $-p\omega_2$, or opposite in direction to u_1 when p, ϵ_2, ω_2 are positive.

We have then, so far as rotation is concerned,

$$m_1u_1 - \frac{m_2\kappa_2^2}{p^2}p\epsilon_2 = m_1v_1 - \frac{m_2\kappa_2^2}{p^2}\cdot p\omega_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the bodies are non-elastic, m_1 and the point B move together after impact and we have

$$v_1 = v_2 - p\omega_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating v_2 and ω_2 from (1) and (2) by means of (3) we have

$$u_1 - v_1 = \frac{m_2\kappa_2^2(u_1 - u_2 + p\epsilon_2)}{(m_1 + m_2)\kappa_2^2 + m_1p^2},$$

$$v_2 - u_2 = \frac{m_1\kappa_2^2(u_1 - u_2 + p\epsilon_2)}{(m_1 + m_2)\kappa_2^2 + m_1p^2},$$

$$\omega_2 - \epsilon_2 = -\frac{m_1p(u_1 - u_2 + p\epsilon_2)}{(m_1 + m_2)\kappa_2^2 + m_1p^2}.$$

If the bodies are perfectly elastic, these values are twice as great (page 345). If the bodies are imperfectly elastic and of the same material, these values are $(1 + e)$ times as great (page 349) where e is the modulus of elasticity.

We have then in general

$$v_1 = u_1 - (u_1 - u_2 + p\epsilon_2)\frac{(1 + e)m_2\kappa_2^2}{(m_1 + m_2)\kappa_2^2 + m_1p^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

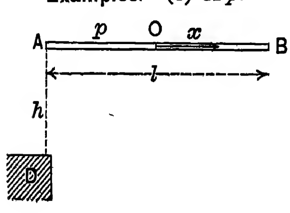
$$v_2 = u_2 + (u_1 - u_2 + p\epsilon_2)\frac{(1 + e)m_1\kappa_2^2}{(m_1 + m_2)\kappa_2^2 + m_1p^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\omega_2 = \epsilon_2 - (u_1 - u_2 + p\epsilon_2)\frac{(1 + e)m_1p}{(m_1 + m_2)\kappa_2^2 + m_1p^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These equations are general. If the impact is central, $p = 0$ and ω_2 in (6) is unchanged and equal to ϵ_2, while (4) and (5) reduce to equations (8), page 349.

If the bodies are perfectly elastic, $e = 1$. If non-elastic, $e = 0$. If m_2 moves towards m_1, u_2 is negative. If m_1 is initially at rest, $u_1 = 0$. If m_2 is initially at rest, $u_2 = 0$ and $\epsilon_2 = 0$. If m_1 is fixed, we can take $m_1 = \infty$ and $u_1 = 0$.

Examples.—(1) *A prismatic bar AB falls through a height h, retaining its horizontal position until one end strikes a fixed obstacle D. Find the motion after impact, considering the bodies non-elastic.*



ANS. Let m_2 be the mass of the bar, l its length, and $u_2 = -\sqrt{2gh}$ the velocity of the centre of mass O at the instant of impact. We have $e = 0$, $\epsilon_2 = 0$, $m_1 = \infty$, $u_1 = 0$. and hence the velocity of translation after impact is

$$v_2 = u_2 - u_2\frac{\kappa_2^2}{\kappa_2^2 + p^2} = \frac{u_2p^2}{\kappa_2^2 + p^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the angular velocity about O after impact is

$$\omega_2 = \frac{u_2 p}{\kappa_2^2 + p^2}. \qquad (2)$$

In the present case $\kappa_2^2 = \frac{l^2}{12}$, $p = +\frac{l}{2}$ if A strikes, and $p = -\frac{l}{2}$ if B strikes. Also, $u_2 = -\sqrt{2gh}$.

Hence in both cases of A or B striking

$$v_2 = -\frac{3}{4}\sqrt{2gh}.$$

If A strikes,

$$\omega_2 = -\frac{3}{2l}\sqrt{2gh}.$$

If B strikes,

$$\omega_2 = +\frac{3}{2l}\sqrt{2gh}.$$

The plus (+) sign denoting counter-clockwise and the minus (−) sign clockwise rotation.
We can obtain (1) and (2) directly as follows: We have (page 318)

$$I'\omega_2 = \text{moment of momentum} = m_2 u_2 p, \quad \text{or} \quad \omega_2 = \frac{m_2 u_2 p}{m_2(\kappa^2_2 + p^2)} = \frac{u_2 p}{\kappa_2^2 + p^2}.$$

Also, at the instant of impact

$$v_2 = p\omega_2.$$

Hence

$$v_2 = \frac{u_2 p^2}{\kappa_2^2 + p^2}.$$

The momentum after impact is then

$$m_2 v_2 = \frac{m_2 u_2 p^2}{\kappa_2^2 + p^2} = -\frac{3}{4} m_2 \sqrt{2gh}. \qquad (3)$$

The impulse is

$$m_2(v_2 - u_2) = -\frac{m_2 u_2 \kappa_2^2}{\kappa_2^2 + p^2} = \frac{1}{4} m_2 \sqrt{2gh}. \qquad (4)$$

The velocity at any point distant x from O after impact is

$$v_2 + x\omega_2 = \frac{u_2 p}{\kappa_2^2 + p^2}(p + x) = -\frac{p\sqrt{2gh}}{\kappa_2^2 + p^2}(p + x), \qquad (5)$$

where p and x are positive towards B and negative towards A. Hence for A striking

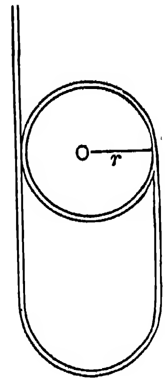
$$v_2 - x\omega_2 = -\frac{3\sqrt{2gh}}{2l}\left(\frac{l}{2} - x\right),$$

and for B striking

$$v_2 - x\omega_2 = -\frac{3\sqrt{2gh}}{2l}\left(\frac{l}{2} + x\right).$$

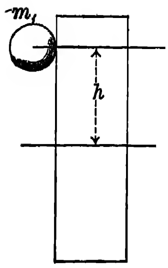
After impact the centre moves in the same vertical with the uniform acceleration g, while the angular velocity ω_2 remains unchanged.

(2) *An inextensible string is wound around a cylinder and has its free end attached to a fixed point. The cylinder falls through a height h, and at the instant of impact the string is vertical and tangent to the cylinder. Find the motion after impact.*



$$\text{ANS. } v = \frac{ur^2}{\kappa^2 + r^2} = \frac{2}{3}u = \frac{2}{3}\sqrt{2gh},$$

$$\omega = \frac{ur}{\kappa^2 + r^2} = \frac{2}{3}\frac{u}{r} = \frac{2}{3r}\sqrt{2gh}.$$



(3) *An iron ball of mass $m_1 = 65$ lbs. moving with a velocity of 36 ft. per sec. strikes a pine beam of uniform rectangular cross-section in the centre line of a side and at right angles, at a distance $p = 1\frac{3}{4}$ ft. above the centre of mass. The mass of the beam is 842.4 lbs., its length 5 ft. and breadth 2 ft. If the beam is at rest, find the motion after impact, considering the impact as non-elastic.*

ANS. The moment of inertia (page 38) is the same as for $\frac{m_2}{3}$ concentrated at a corner We have then

$$I = \frac{m}{3}\left[\left(\frac{5}{2}\right)^2 + \left(\frac{2}{2}\right)^2\right] = 2.416 m_2, \quad \text{or} \quad \kappa_2^2 = 2.416.$$

Hence the velocity of the ball after impact is

$$v_1 = u_1 - \frac{m_2\kappa_2^2 u_1}{(m_1 + m_2)\kappa_2^2 + m_1 p^2} = 36\left(1 - \frac{2035.2}{2192.3 + 65 \times 1.75}\right) = 5.364 \text{ ft. per sec.}$$

The velocity of the centre of mass of the beam after impact is

$$v_2 = \frac{m_1\kappa_2^2 u_1}{(m_1 + m_2)\kappa_2^2 + m_1 p^2} = 2.364 \text{ ft. per sec.}$$

The angular velocity of the beam after impact is

$$\omega_2 = -\frac{m_1 p u_1}{(m_1 + m_2)\kappa_2^2 + m_1 p^2} = -1.712 \text{ radians per sec.}$$

(4) *A ballistic pendulum weighing 30,000 lbs. is set in oscillation by a 6-lb. ball, and the angular displacement is 15°. If the distance d of the centre of mass from the axis is 5 ft., and the distance a_2 of the point of impact below the axis is 5.5 ft., and the number of oscillations per minute is $n = 40$, find the velocity of the ball.*

$$\text{ANS. } u_1 = \frac{3006}{6} \cdot \frac{120 \times 32.2 \times 5}{40 \times 3.1416 \times 5.5} \sin 7\tfrac{1}{2}^\circ = 1828 \text{ ft. per sec.}$$

CHAPTER VI.

ROTATION ABOUT A TRANSLATING AXIS.

Effective Forces—Rotation about a Translating Axis.—Equations (5), page 159, give the components of the acceleration for any particle of a rotating and translating body. If the axis of rotation passes through the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$. The co-ordinates x, y, z are taken from the centre of mass O.

If then we make these changes in equations (5), page 159, multiply each term by the mass m of the particle and sum up for all the particles, we have, since x, y, z are taken from the centre of mass O and hence $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, for the component effective forces for a body of mass $\overline{\mathrm{m}} = \Sigma m$ rotating about any translating axis

$$\Sigma mf_x = \overline{\mathrm{m}}\bar{f}_x, \qquad \Sigma mf_y = \overline{\mathrm{m}}\bar{f}_y, \qquad \Sigma mf_z = \overline{\mathrm{m}}\bar{f}_z. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence the effective force in any direction for a body rotating about a translating axis is the same as for a particle of mass equal to the mass of the body having the acceleration of the centre of mass in that direction.

If we take mass in lbs. and acceleration in ft.-per-sec. per sec., we have force in poundals. For force in pounds divide by g (page 171).

Moments of the Effective Forces—Translating Axis.—Equations (10), page 160, give the component moments of the acceleration for any particle of a rotating and translating body. If the axis of rotation passes through the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and if it does not change in direction, we have $\omega_x\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_x = 0$, $\omega_y\omega_z = 0$, $\omega_z\omega_x = 0$, $\omega_z\omega_y = 0$. If we make these changes in equations (10), page 160, multiply each term by the mass m of the particle and sum up for all the particles, we have, since x, y, z are taken from the centre of mass O and hence

$$\Sigma mx = 0, \qquad \Sigma my = 0, \qquad \Sigma mz = 0,$$
$$\Sigma m(y^2 + z^2) = I_x, \qquad \Sigma m(z^2 + x^2) = I_y, \qquad \Sigma m(x^2 + y^2) = I_z,$$

where I_x, I_y, I_z are the moments of inertia of the body for the axes of X, Y, Z through the centre of mass O,

$$\left.\begin{aligned} M'_{fx} &= \overline{\mathrm{m}}\bar{f}_z\bar{y} - \overline{\mathrm{m}}\bar{f}_y\bar{z} - \alpha_y\Sigma myx - \alpha_z\Sigma mzx + (\omega_z^2 - \omega_y^2)\Sigma mzy + I_x\alpha_x, \\ M'_{fy} &= \overline{\mathrm{m}}\bar{f}_x\bar{z} - \overline{\mathrm{m}}\bar{f}_z\bar{x} - \alpha_z\Sigma mzy - \alpha_x\Sigma mxy + (\omega_x^2 - \omega_z^2)\Sigma mxz + I_y\alpha_y, \\ M'_{fz} &= \overline{\mathrm{m}}\bar{f}_y\bar{x} - \overline{\mathrm{m}}\bar{f}_x\bar{y} - \alpha_x\Sigma mxz - \alpha_y\Sigma myz + (\omega_y^2 - \omega_x^2)\Sigma myx + I_z\alpha_z. \end{aligned}\right\} \qquad (2)$$

If the translating axis coincides with one of the co-ordinate axes, as, for instance, $O'X'$, we have $\omega_y = 0$, $\omega_z = 0$, $\alpha_y = 0$, $\alpha_z = 0$, and

$$\left.\begin{aligned} M'_{fx} &= \overline{\mathrm{m}}\bar{f}_z\bar{y} - \overline{\mathrm{m}}\bar{f}_y\bar{z} + I_x\alpha_x, \\ M'_{fy} &= \overline{\mathrm{m}}\bar{f}_x\bar{z} - \overline{\mathrm{m}}\bar{f}_z\bar{x} + \omega_x^2\Sigma mxz - \alpha_x\Sigma mxy, \\ M'_{fz} &= \overline{\mathrm{m}}\bar{f}_y\bar{x} - \overline{\mathrm{m}}\bar{f}_x\bar{y} - \omega_x^2\Sigma myx - \alpha_x\Sigma mxz. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since we can take any co-ordinate axes we please, let the co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$ be principal axes at O', the intersection with the axis of rotation of a plane through the centre of mass O at right angles to the axis. (Figure, page 153.) Then the parallel axes OX, OY, OZ at the centre of mass O will be principal axes, and we have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$, and equations (2) become

$$\left.\begin{aligned} M_{fx} &= \mathrm{m}\bar{f}_z\bar{y} - \mathrm{m}\bar{f}_y\bar{z} + I_x\alpha_x, \\ M_{fy} &= \mathrm{m}\bar{f}_x\bar{z} - \mathrm{m}\bar{f}_z\bar{x} + I_y\alpha_y, \\ M_{fz} &= \mathrm{m}\bar{f}_y\bar{x} - \mathrm{m}\bar{f}_x\bar{y} + I_z\alpha_z. \end{aligned}\right\} \quad \ldots \ldots \quad (4)$$

If the axis of rotation is a principal axis, let it coincide with $O'X'$, and we have in (4) $\alpha_y = 0$, $\alpha_z = 0$. If the axis passes through the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$ and $M_{fx} = I_x\alpha_x$, $M_{fy} = I_y\alpha_y$, $M_{fz} = I_z\alpha_z$.

If we take distance in feet and mass in lbs., these equations (2), (3), (4) give moments in poundal-feet. For pound-feet divide by g (page 171).

Momentum—Translating Axis.—Equations (4), page 154, give the component velocities for any particle of a rotating and translating body.

If we multiply each term by the mass m of the particle and sum up for all the particles, we have, since x, y, z are taken from the centre of mass O and hence $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$

$$\Sigma mv_x = \mathrm{m}\bar{v}_x, \qquad \Sigma mv_y = \mathrm{m}\bar{v}_y, \qquad \Sigma mv_z = \mathrm{m}\bar{v}_z. \quad \ldots \ldots \quad (5)$$

Hence the momentum in any direction for a body rotating about a translating axis is the same as for a particle of mass equal to the mass of the body having the velocity of the centre of mass in that direction.

Moment of Momentum—Translating Axis.—Equations (10), page 155, give the component moments of velocity for any particle of a rotating and translating body. If we multiply each term by the mass m of the particle and sum up for all the particles, we shall obtain the component moments of momentum for any co-ordinate axes we please.

Let us take these axes as principal axes at the point O', the intersection with the axis of rotation of a plane through the centre of mass O at right angles to the axis. (Figure, page 153.) Then the parallel axes OX, OY, OZ at the centre of mass O are principal axes, and we have

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0, \qquad \Sigma mx = 0, \qquad \Sigma my = 0, \qquad \Sigma mz = 0,$$

$$\Sigma m(y^2 + z^2) = I_x, \qquad \Sigma m(z^2 + x^2) = I_y, \qquad \Sigma m(x^2 + y^2) = I_z,$$

where I_x, I_y, I_z are the moments of inertia of the body for the parallel axes OX, OY, OZ. We have then, from equations (10), page 155,

$$\left.\begin{aligned} M'_{vx} &= \mathrm{m}\bar{v}_z\bar{y} - \mathrm{m}\bar{v}_y\bar{z} + I_x\omega_x, \\ M'_{vy} &= \mathrm{m}\bar{v}_x\bar{z} - \mathrm{m}\bar{v}_z\bar{x} + I_y\omega_y, \\ M'_{vz} &= \mathrm{m}\bar{v}_y\bar{x} - \mathrm{m}v_x\bar{y} + I_z\omega_z. \end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

If the axis passes through the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$ and $M_{vx} = I_x\omega_x$, $M_{vy} = I_y\omega_y$, $M_{vz} = I_z\omega_z$.

Pressures on Translating Axis—Permanent Axis.—We have just as on page 319 for fixed axis, and using the same notation,

$$\left.\begin{array}{c} R_x' + R_x'' + \Sigma F_x = \mathrm{m}\bar{f}_x, \qquad R_y' + R_y'' + \Sigma F_y = \mathrm{m}\bar{f}_y, \\ R_z' + R_z'' + \Sigma F_z = \mathrm{m}\bar{f}_z. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We have also

$$\left.\begin{array}{r} R_y'l_1 - R_y''l_2 + \Sigma F_y x - \Sigma F_x y = \mathrm{m}\bar{f}_y\bar{x} - \mathrm{m}\bar{f}_x\bar{y} - \omega_x^2\Sigma myx - \alpha_x\Sigma mxz, \\ -R_z'l_1 + R_z''l_2 + \Sigma F_x z - \Sigma F_z x = \mathrm{m}\bar{f}_x\bar{z} - \mathrm{m}\bar{f}_z\bar{x} + \omega_x^2\Sigma mxz - \alpha_x\Sigma mxy, \\ \Sigma F_z y - \Sigma F_y z = \mathrm{m}\bar{f}_z\bar{y} - \mathrm{m}\bar{f}_y\bar{z} + I_x\alpha_x. \end{array}\right\} \quad . \quad . \quad (8)$$

For a principal axis through the centre of mass we have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$ and $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and these equations become

$$R_y'l_1 - R_y''l_2 + \Sigma F_y x - \Sigma F_x y = 0, \qquad -R_z'l_1 + R_z''l_2 + \Sigma F_x z - \Sigma F_z x = 0,$$
$$\Sigma F_z y - \Sigma F_y z = I_x\alpha_x.$$

We see from these last equations that if we have always

$$M_z = \Sigma F_y x - \Sigma F_x y = 0, \qquad M_y = \Sigma F_x z - \Sigma F_z x = 0,$$

that is, if the moments M_z, M_y about the axes of Z and Y of the impressed forces F are always zero, there is no rotation of the axis, and hence even if this axis is unconstrained it will not change its direction. The axis is then a permanent axis of rotation.

Hence if a body rotates about a principal axis through the centre of mass and M_z, M_y are always zero, the axis of rotation is a permanent axis and will not change in direction even if the pressures on the axis are removed.

M_z and M_y are always zero when there are no impressed forces; when the impressed forces always reduce to a resultant force through the centre of mass, or to a resultant force through the centre of mass and a couple whose plane is at right angles to the axis; when the impressed forces all lie in a plane through the centre of mass at right angles to the axis, or reduce to a resultant force, or force and couple, in this plane.

Conservation of Moment of Momentum—Translating Axis.—We have from equations (6) for the component moments of momentum for a body rotating about a translating axis, taking the co-ordinate axes as principal axes at O',

$$\left.\begin{array}{l} M_{vx} = \mathrm{m}\bar{v}_z\bar{y} - \mathrm{m}\bar{v}_y\bar{z} + I_x\omega_x, \\ M_{vy} = \mathrm{m}\bar{v}_x\bar{z} - \mathrm{m}\bar{v}_z\bar{x} + I_y\omega_y, \\ M_{vz} = \mathrm{m}\bar{v}_y\bar{x} - \mathrm{m}\bar{v}_x\bar{y} + I_z\omega_z, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and from equations (4) for the component moments of the effective forces

$$\left.\begin{array}{l} M_{fx} = \mathrm{m}\bar{f}_z\bar{y} - \mathrm{m}\bar{f}_y\bar{z} + I_x\alpha_x, \\ M_{fy} = \mathrm{m}\bar{f}_x\bar{z} - \mathrm{m}\bar{f}_z\bar{x} + I_y\alpha_y, \\ M_{fz} = \mathrm{m}\bar{f}_y\bar{x} - \mathrm{m}\bar{f}_x\bar{y} + I_z\alpha_z. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If in equations (b) we have $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, and also $\bar{f}_x = 0$, $\bar{f}_y = 0$, $\bar{f}_z = 0$, we shall evidently have $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$, and ω_x, ω_y, ω_z, $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ constant in

equations (a). But since the motion of the centre of mass is the same as if all the impressed forces were applied to a particle of mass $\overline{m}$ at the centre of mass (page 299), when $\bar{f}_x = 0$, $\bar{f}_y = 0$, $\bar{f}_z = 0$ we must have the algebraic sum of the components of all the impressed forces in three rectangular directions equal to zero, or

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0, \qquad \Sigma F_z = 0.$$

Also, since by D'Alembert's principle the moment of the effective forces is equal to the moment of the impressed forces, when $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$ we have the moment of the impressed forces for the axis of rotation zero also. If we have at the same time

$$M_{fx} = 0, \quad M_{fy} = 0, \quad M_{fz} = 0, \quad \text{and} \quad \Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0,$$

the impressed forces then form a system of forces in static equilibrium, and from (b) we see that $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$.

Hence *if the impressed forces acting upon a body rotating about a translating axis form a system of forces in static equilibrium, the moment of momentum about that axis is constant.*

Kinetic Energy—Translating Axis.—The kinetic energy of a particle of mass m and velocity v is $\frac{1}{2}mv^2$. If a particle has the component velocities v_x, v_y, v_z, we have $v^2 = v_x^2 + v_y^2 + v_z^2$, and

$$\frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2).$$

Equations (4), page 154, give the component velocities for any particle of a translating and rotating body. If we square these component velocities, multiply each term by $\frac{1}{2}m$, sum up for all the particles and add, we shall have the kinetic energy for a rotating and translating body for any co-ordinate axes we please.

Let us take these axes as principal axes at the point O', the intersection with the axis of rotation of a plane through the centre of mass O at right angles to the axis. (Figure, page 153.) Then the parallel axes OX, OY, OZ at the centre of mass O are principal axes and we have

$$\Sigma mxy = 0, \quad \Sigma myz = 0, \quad \Sigma mzx = 0, \quad \Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0,$$
$$\Sigma m(y^2 + z^2) = I_x, \quad \Sigma m(z^2 + x^2) = I_y, \quad \Sigma m(x^2 + y^2) = I_z,$$

where I_x, I_y, I_z are the moments of inertia of the body for the parallel axes OX, OY, OZ.

We have then from equations (4), page 154, for the kinetic energy

$$\mathfrak{K} = \frac{1}{2}\overline{m}\bar{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2. \quad \quad (9)$$

If the axis of rotation coincides with one of the principal axes, as, for instance, $O'Z'$, we have $\omega_x = 0$, $\omega_y = 0$, and

$$\mathfrak{K} = \frac{1}{2}\overline{m}\bar{v}^2 + \frac{1}{2}I_z\omega_z^2. \quad \quad (10)$$

The kinetic energy is then the sum of the kinetic energy of translation and rotation. If we take mass in lbs., these equations give kinetic energy in foot-poundals. For foot-pounds divide by g (page 171).

Instantaneous Axis.—If a body rotates about a translating axis, the instantaneous axis is parallel to the translating axis and passes through a point whose co-ordinates are given by equations (15), page 157, for the instantaneous axis of rotation, and by equations (15), page 161, for the instantaneous axis of acceleration.

If we take the translating axis as the axis of OX, we have $\omega_y = 0$, $\omega_z = 0$, and for the instantaneous axis of rotation the co-ordinates from the centre of mass are

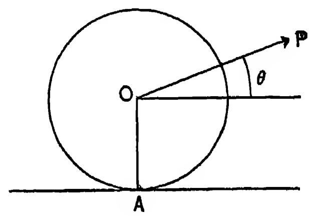
$$p_x = 0, \qquad p_y = -\frac{\overline{v}_z}{\omega_x}, \qquad p_z = \frac{\overline{v}_y}{\omega_x}.$$

We also have $\alpha_y = 0$, $\alpha_z = 0$, and the co-ordinates of the instantaneous axis of acceleration are

$$p_x = 0, \qquad p_y = -\frac{\overline{f}_z}{\alpha_x}, \qquad p_z = \frac{\overline{f}_y}{\alpha_x}.$$

Examples.—(1) *A circular disc of mass $\overline{\text{m}}$ and whose plane is vertical has a force of P pounds applied at the centre, and rolls upon a horizontal plane. Determine its motion.*

ANS. Let the force P make the angle θ with the horizontal and act in the plane of the disc. The force is Pg poundals, and the horizontal component $Pg \cos \theta$, and the vertical component $Pg \sin \theta$. Let r be the radius, and A the point of contact.



The moment of the impressed forces about A is $-Pg \cos \theta \,.\, r$. Let κ be the radius of gyration for axis through the centre of mass O at right angles to the disc, and I' the moment of inertia for parallax axis through A. Then $I' = \overline{\text{m}}\,(\kappa^2 + r^2)$, and we have the moment of the effective forces $I'\alpha$. By D'Alembert's principle,

$$-I'\alpha - Pg \cos \theta \,.\, r = 0, \quad \text{or} \quad \alpha = -\frac{Pgr \cos \theta}{\overline{\text{m}}(\kappa^2 + r^2)}.$$

The axis at A is the instantaneous axis. The acceleration at the centre O is then

$$\overline{f}_x = -r\alpha = \frac{Pgr^2 \cos \theta}{\overline{\text{m}}(\kappa^2 + r^2)}.$$

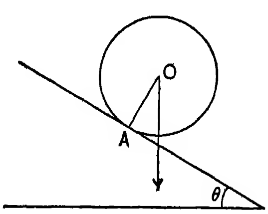
For a disc, $\kappa^2 = \frac{r^2}{2}$, hence

$$\alpha = -\frac{2Pg \cos \theta}{3\overline{\text{m}}r}, \qquad \overline{f}_x = \frac{2Pg \cos \theta}{3\overline{\text{m}}}.$$

Both angular and linear accelerations are then constant if P is constant.

(2) *A disc of mass $\overline{\text{m}}$ whose plane is vertical rolls (without sliding) down a rigid inclined plane. Determine its motion.*

ANS. Let the radius be r, the radius of gyration about an axis through the centre of mass O perpendicular to the plane of the disc be κ, and θ the inclination of the plane, and A the point of contact.



Then the weight is $\overline{\text{m}}g$, the force parallel to the plane is $\overline{\text{m}}g \sin \theta$, and its moment about A, $-\overline{\text{m}}gr \sin \theta$. We have then, as in the preceding example,

$$I'\alpha = \overline{\text{m}}(\kappa^2 + r^2)\alpha = -\overline{\text{m}}gr \sin \theta, \quad \text{or} \quad \alpha = -\frac{gr \sin \theta}{\kappa^2 + r^2}.$$

Also, since A is the instantaneous axis, the linear acceleration of the centre is

$$\overline{f}_x = -r\alpha = \frac{gr^2 \sin \theta}{\kappa^2 + r^2}.$$

Since $\kappa^2 = \frac{r^2}{2}$, we have

$$\alpha = -\frac{2g \sin \theta}{3r}, \qquad \overline{f}_x = \frac{2g \sin \theta}{3}.$$

Both linear, tangential and angular accelerations are constant and the velocity after any time may readily be determined.

(3) *Find the time a rigid homogeneous cylinder will take to roll from rest down a plane 20 ft. long and inclined 30° to the horizon, the axis of the cylinder being horizontal.*

ANS. *1.93* sec.

(4) *A rigid homogeneous circular disk of mass m and radius r, whose plane is vertical, moves in contact with a smooth inclined plane whose angle is θ. From a point in the same vertical plane as the disc, and at a distance from the inclined plane equal to the diameter of the disc, a string is carried parallel to the inclined plane and is wrapped round the edge of the disc, and its end is fixed in the circumference. Find the tension T in the string, the linear acceleration $\bar{f}$ of the centre, and the angular acceleration α of the disc.*

ANS. $T = \dfrac{mg\kappa^2 \sin\theta}{\kappa^2 + r^2} = \dfrac{mg \sin\theta}{3}$ poundals or $\dfrac{m \sin\theta}{3}$ lbs.;

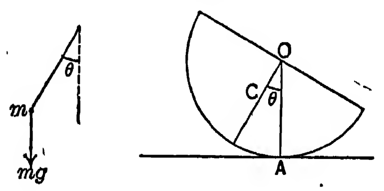
$$\bar{f} = \frac{gr^2 \sin\theta}{\kappa^2 + r^2} = \frac{2g \sin\theta}{3} \text{ ft.-per-sec. per sec.};$$

$$\alpha = \frac{gr \sin\theta}{\kappa^2 + r^2} = \frac{2g \sin\theta}{3r} \text{ radians-per-sec. per sec.}$$

(5) *A perfectly flexible and inextensible ribbon is coiled on the circumference of a homogeneous circular disc of radius r and mass m, and has its free end attached at a fixed point. A part of the ribbon is unrolled and vertical, and the disc is allowed to fall from rest by its own weight. Find the acceleration $\bar{f}$ of the centre and the angular acceleration α before the ribbon becomes wholly unrolled, and the distance s which the centre will descend in one second.*

ANS. $\alpha = \dfrac{2g}{3r}$, $\quad \bar{f} = \dfrac{2g}{3}$, $\quad s = \dfrac{1}{2}\bar{f}t^2 = \dfrac{g}{3}$.

(6) *A homogeneous hemisphere of radius r performs small oscillations on a rough horizontal plane. Find the periodic time.*



ANS. For the simple pendulum of length l and mass m we have

$$mg \times l \sin\theta = ml^2\alpha, \quad \text{or} \quad \alpha = \frac{g \sin\theta}{l}. \quad . \quad . \quad . \quad . \quad (1)$$

For the hemisphere let d be the distance OC from the centre of the hemisphere O to the centre of mass C, and let κ be the radius of gyration about an axis through the centre of mass C parallel to the instantaneous axis at A. Then the moment of inertia for the instantaneous axis at A is

$$I' = m[\kappa^2 + (d \sin\theta)^2 + (r - d\cos\theta)^2].$$

If θ is small, we may put $\sin^2\theta = 0$ and $\cos\theta = 1$, and we have

$$I' = m[\kappa^2 + (r - d)^2].$$

We have then

$$I'\alpha = mg \times d \sin\theta, \quad \text{or} \quad \alpha = \frac{gd \sin\theta}{\kappa^2 + (r-d)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating (1) and (2), we have for the length of the equivalent simple pendulum

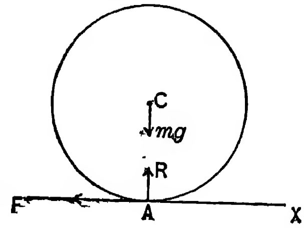
$$l = \frac{\kappa^2 + (r-d)^2}{d}.$$

The periodic time is then

$$t = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{\kappa^2 + (r-d)^2}{dg}}.$$

(7) *A homogeneous circular hoop moving in a vertical plane in contact with a rough horizontal surface has at a given instant an angular velocity opposite in direction to that which would enable it to roll in the direction of its translation at that instant. Determine its motion.*

ANS. Let m be the mass of the hoop. The forces acting on the hoop are its weight mg at the centre C, the upward pressure of the plane R at A, and the friction F opposite to the direction of translation AX.



The acceleration of the centre is then

$$\bar{f} = -\frac{F}{m},$$

and the angular acceleration upon the axis through C perpendicular to the plane of the hoop is, if κ is the radius of gyration for this axis,

$$\alpha = -\frac{Fr}{m\kappa^2}.$$

We have also

$$R - mg = 0, \quad \text{or} \quad R = mg.$$

If μ is the coefficient of sliding friction, we have

$$F = \mu mg.$$

Hence

$$\bar{f} = -\mu g, \qquad \alpha = -\frac{\mu g r}{\kappa^2}.$$

If $\bar{v}_1$ and ω_1 are the initial values of the linear and angular velocities, we have then for the linear and angular velocities after any time t

$$\bar{v} = \bar{v}_1 - \mu g t, \qquad \omega = \omega_1 - \frac{\mu g r t}{\kappa^2}.$$

If at any instant there is no slipping, we have at that instant the velocity at A zero, or

$$\bar{v} + r\omega = 0.$$

If we eliminate $\bar{v}$ and ω by means of this equation, we have then for the time after which slipping ceases

$$t = \frac{\kappa^2(\bar{v}_1 + r\omega_1)}{\mu g(\kappa^2 + r^2)}.$$

At this instant there is no tendency to slip and μ becomes zero, and hence at this instant $\bar{f} = 0$ and $\alpha = 0$. Hence after the time t the linear and angular velocities are constant and given by

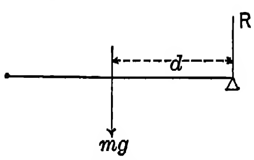
$$\bar{v} = \bar{v}_1 - \frac{\kappa^2(\bar{v}_1 + r\omega_1)}{\kappa^2 + r^2} = \frac{r(r\bar{v}_1 - \kappa^2\omega)}{\kappa^2 + r^2},$$

$$\omega = \omega_1 - \frac{r(\bar{v}_1 + r\omega_1)}{\kappa^2 + r^2} = \frac{\kappa^2\omega - r\bar{v}_1}{\kappa^2 + r^2}.$$

If $r\bar{v}_1 - \kappa^2\omega$ is negative, $\bar{v}$ is negative. Hence if ω_1 is positive and greater than $\frac{r\bar{v}_1}{\kappa^2}$, the translation of the hoop after the time t will be in the opposite direction to the initial translation.

(8) *A homogeneous beam is supported horizontally on two supports. Find where one of them must be placed in order that when the other is removed the instantaneous force exerted on the former may be equal to half the weight of the beam.*

ANS. Let d be the required distance of the permanent support from the centre of the beam, κ the radius of gyration of the beam about a normal axis through the centre, m the mass of the beam, α its angular acceleration, and R the reaction of the permanent support immediately after the removal of the other. Then



$$R = \frac{1}{2}mg,$$

and for the centre of the beam

$$m\kappa^2\alpha = Rd, \quad \text{or} \quad \alpha = \frac{Rd}{m\kappa^2} = \frac{gd}{2\kappa^2}.$$

For the end of the beam

$$m(\kappa^2 + d^2)\alpha = mgd, \quad \text{or} \quad \alpha = \frac{gd}{\kappa^2 + d^2}.$$

Equating these two values of α, we have $d = \kappa$.

(9) *A homogeneous circular cylinder of radius r, radius of gyration about the axis κ, rotating about its axis with angular velocity ω_1, is placed with its axis horizontal on a rough inclined plane (coefficient of friction μ, inclination θ, so that $\mu = \tan\theta$), the direction of rotation being that which it would have if the cylinder were rolling without sliding up the plane. Show that the axis of the cylinder will be stationary for a time $t = \dfrac{\kappa^2\omega_1}{\mu r g \cos\theta}$, at the end of which the angular velocity will be zero.*

(10) *A uniform square is supported in a vertical plane with one diagonal horizontal by two supports, one at each extremity of the diagonal. Show that the initial force on one support when the other is removed is equal to one fourth of the weight of the square.*

(11) *A uniform horizontal bar, suspended from any two points in its length by two parallel cords, is at rest. If one of the cords be cut, find the initial tension in the other.*

ANS. $T = \dfrac{Wl^2}{l^2 + 12d^2}$ where l is the length of the bar, d the distance from its centre of mass to the point of attachment of the uncut cord, and W is the weight of the bar.

(12) *A uniform beam of weight W rests with one end against a smooth vertical wall and the other on a smooth horizontal plane, the inclination to the horizon being θ. It is prevented from falling by a string attached to its lower end and to the wall. Find the force between the upper end and the wall when the string is cut.*

ANS. $\dfrac{1}{2} W \cot\theta$.

(13) *A sphere is laid upon a rough inclined plane of inclination θ. Show that it will not slide if the coefficient of friction is equal to or greater than $\dfrac{2}{7}\tan\theta$.*

(14) *A sphere of radius r whose centre of mass is not at its centre of figure is placed on a rough horizontal plane, coefficient of friction μ. Find whether it will slide or roll.*

ANS Let κ' be the radius of gyration of the sphere about the line through the point of contact at right angles to the plane of the centres of figure and mass. Then if the initial distance of the centre of mass from a vertical through the centre of figure is greater than $\dfrac{\mu\kappa'^2}{r}$, it will begin to slide; if less, to roll.

CHAPTER VII.

ROTATION ABOUT A FIXED POINT.

Effective Forces—Rotation about a Fixed Point.—Equations (5), page 159, give the components of the acceleration of any particle of a rotating and translating body.

For rotation only we should omit all terms containing $\bar{f}_x, \bar{f}_y, \bar{f}_z$. If we make these changes in equations (5), page 159, multiply each term by the mass m of the particle and sum up for all the particles, we have, since $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, for the component effective forces for a rotating body of mass $\mathrm{\bar{m}} = \Sigma m$ for any co-ordinate axes we please through the fixed point O'

$$\left.\begin{aligned}\Sigma mf_x &= \mathrm{\bar{m}}\bar{y}\omega_x\omega_y + \mathrm{\bar{m}}\bar{z}\omega_x\omega_z - \mathrm{\bar{m}}\bar{x}\omega_y^2 - \mathrm{\bar{m}}\bar{x}\omega_z^2 + \mathrm{\bar{m}}\bar{z}\alpha_y - \mathrm{\bar{m}}\bar{y}\alpha_z,\\ \Sigma mf_y &= \mathrm{\bar{m}}\bar{z}\,\omega_y\omega_z + \mathrm{\bar{m}}\bar{x}\,\omega_y\omega_x - \mathrm{\bar{m}}\bar{y}\,\omega_z^2 - \mathrm{\bar{m}}\bar{y}\,\omega_x^2 + \mathrm{\bar{m}}\bar{x}\alpha_z - \mathrm{\bar{m}}\bar{z}\,\alpha_x,\\ \Sigma mf_z &= \mathrm{\bar{m}}\bar{x}\,\omega_z\omega_x + \mathrm{\bar{m}}\bar{y}\,\omega_z\omega_y - \mathrm{\bar{m}}\bar{z}\,\omega_x^2 - \mathrm{\bar{m}}\bar{z}\,\omega_y^2 + \mathrm{\bar{m}}\bar{y}\,\alpha_x - \mathrm{\bar{m}}\bar{x}\alpha_y.\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

If any one of the co-ordinate axes is fixed, as, for instance, $O'Z'$ we have $\omega_z\omega_y = 0$, $\omega_z\omega_x = 0$, $\omega_x\omega_y = 0$, $\omega_y\omega_x = 0$. For $O'Y'$ fixed we have $\omega_y\omega_x = 0$, $\omega_y\omega_z = 0$, $\omega_x\omega_z = 0$, $\omega_z\omega_x = 0$. For $O'X'$ fixed we have $\omega_x\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_z = 0$, $\omega_z\omega_y = 0$. For all the co-ordinate axes fixed we have equations (3), page 316, for a fixed axis.

If the fixed point is at the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and hence $\Sigma mf_x = 0$, $\Sigma mf_y = 0$, $\Sigma mf_z = 0$. That is, if the centre of mass is fixed, the components of the effective forces are zero.

If we take distance in feet and mass in lbs., equations (1) give force in poundals. For force in pounds divide by g (page 171).

Moment of Effective Forces—Rotation about a Fixed Point.—Equations (10), page 160, give the component moments of the acceleration of any particle of a rotating and translating body.

For rotation only we should omit all terms containing $\bar{f}_x, \bar{f}_y, \bar{f}_z$, and put x', y', z' in place of $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$. If we make these changes in equations (10), page 160, multiply each term by the mass m of the particle and sum up for all the particles, we obtain the component moments of the effective forces. In making the summation we have

$$\Sigma m(y'^2 + z'^2) = I'_x, \qquad \Sigma m(z'^2 + x'^2) = I'_y, \qquad \Sigma m(x'^2 + y'^2) = I'_z,$$

where I'_x, I'_y, I'_z are the moments of inertia of the body for the co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$. We also have

$$\Sigma mx'^2 = I'_{yz}, \qquad \Sigma my'^2 = I'_{zx}, \ \Sigma mz'^2 = I'_{xy},$$

where I'_{xy}, I'_{yz}, I'_{zx} are the moments of inertia of the body for the co-ordinate planes $X'Y'$, $Y'Z'$, $Z'X'$.

We have then

$$\left.\begin{aligned}
M'_{fx} &= \omega_z\omega_x\Sigma my'x' - \omega_y\omega_x\Sigma mz'x' - \alpha_y\Sigma my'x' - \alpha_z\Sigma mz'x' + (\omega_z^2 - \omega_y^2)\Sigma mz'y' \\
&\qquad + I'_{zx}\omega_z\omega_y - I'_{xy}\omega_y\omega_z + I'_x\alpha_x, \\
M'_{fy} &= \omega_x\omega_y\Sigma mz'y' - \omega_z\omega_y\Sigma mx'y' - \alpha_z\Sigma mz'y' - \alpha_x\Sigma mx'y' + (\omega_x^2 - \omega_z^2)\Sigma mx'z' \\
&\qquad + I'_{xy}\omega_x\omega_z - I'_{yz}\omega_z\omega_x + I'_y\alpha_y, \\
M'_{fz} &= \omega_y\omega_z\Sigma mx'z' - \omega_x\omega_z\Sigma my'z' - \alpha_x\Sigma mx'z' - \alpha_y\Sigma my'z' + (\omega_y^2 + \omega_x^2)\Sigma my'x' \\
&\qquad + I'_{yz}\omega_y\omega_x - I'_{zx}\omega_x\omega_y + I'_z\alpha_z.
\end{aligned}\right\} \quad . \quad . \quad (2)$$

Since we can take any co-ordinate axes we please, let us take them principal axes at the fixed point O'. Then we have $\Sigma mx'y' = 0$, $\Sigma my'z' = 0$, $\Sigma mz'x' = 0$, and equations (2) become

$$\left.\begin{aligned}
M'_{fx} &= I'_{zx}\omega_z\omega_y - I'_{xy}\omega_y\omega_z + I'_x\alpha_x, \\
M'_{fy} &= I'_{xy}\omega_x\omega_z - I'_{yz}\omega_z\omega_x + I'_y\alpha_y, \\
M'_{fz} &= I'_{yz}\omega_y\omega_x - I'_{zx}\omega_x\omega_y + I'_z\alpha_z.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If any co-ordinate axis is fixed, as, for instance, $O'Z'$, we have $\omega_z\omega_x = 0$, $\omega_z\omega_y = 0$, $\omega_x\omega_y = 0$, $\omega_y\omega_x = 0$. If $O'Y'$ is fixed, we have $\omega_y\omega_x = 0$, $\omega_y\omega_z = 0$, $\omega_x\omega_z = 0$, $\omega_z\omega_x = 0$. If $O'X'$ is fixed, we have $\omega_x\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_z = 0$, $\omega_z\omega_y = 0$. If all are fixed, equations (2) and (3) become equations (4) and (6), page 317, for fixed axis.

If the co-ordinate axes are not fixed, we have $\omega_z\omega_y = \omega_y\omega_z$, $\omega_x\omega_z = \omega_z\omega_x$, $\omega_y\omega_x = \omega_x\omega_y$; and since we have

$$I'_{zx} - I'_{xy} = \Sigma m(y'^2 - z'^2) = I'_z - I'_y, \qquad I'_{xy} - I'_{yz} = \Sigma m(z'^2 - x'^2) = I'_x - I'_z,$$
$$I'_{yz} - I'_{zx} = \Sigma m(x'^2 - y'^2) = I'_y - I'_x,$$

equations (3) become

$$\left.\begin{aligned}
M'_{fx} &= (I'_z - I'_y)\omega_z\omega_y + I'_x\alpha_x, \\
M'_{fy} &= (I'_x - I'_z)\omega_x\omega_z + I'_y\alpha_y, \\
M'_{fz} &= (I'_y - I'_x)\omega_y\omega_x + I'_z\alpha_z.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we take distance in feet and mass in lbs., all these equations give moments in poundal-feet. For pound-feet divide by g (page 171).

Momentum—Rotation about Fixed Point.—From equations (2), page 154, we have the component velocities for any particle of a rotating body,

$$v_x = (\bar{z} + z)\omega_y - (\bar{y} + y)\omega_z, \qquad v_y = (\bar{x} + x)\omega_z - (\bar{z} + z)\omega_x, \qquad v_z = (\bar{y} + y)\omega_x - (\bar{x} + x)\omega_y.$$

If we multiply by the mass m of the particle and sum up, we have, since $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$ for the components of the momentum

$$\left.\begin{aligned}
\Sigma mv_x &= \mathrm{m}\bar{z}\,\omega_y - \mathrm{m}\bar{y}\,\omega_z, \\
\Sigma mv_y &= \mathrm{m}\bar{x}\,\omega_z - \mathrm{m}\bar{z}\,\omega_x, \\
\Sigma mv_z &= \mathrm{m}\bar{y}\,\omega_x - \mathrm{m}\bar{x}\,\omega_y.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For axis through the centre of mass we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and hence $\Sigma mv_x = 0$, $\Sigma mv_y = 0$, $\Sigma mv_z = 0$.

Hence the momentum for a body rotating about a fixed point is the same as for a particle of equal mass at the centre of mass.

Moment of Momentum—Rotation about Fixed Point.—Equations (10), page 155, give the component moments of velocity for any particle of a rotating and translating body. If the origin O' is a fixed point, we should omit terms containing $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$, and $\bar{x}$, $\bar{y}$, $\bar{z}$, and put x', y', z' in place of x, y, z.

If we make these changes in equations (10), page 155, multiply each term by the mass m of the particle and sum up for all the particles, we shall obtain the component moments of momentum for any co-ordinate axes we please.

Let us take these axes as principal axes at O'. Then we have $\Sigma mx'y' = 0$, $\Sigma my'z' = 0$, $\Sigma mz'x' = 0$. We have then from equations (10), page 155, since $\Sigma m(y'^2 + z'^2) = I'_x$, $\Sigma m(z'^2 + x'^2) = I'_y$, $\Sigma m(x'^2 + y'^2) = I'_z$,

$$M'_{vx} = I'_x\omega_x, \qquad M'_{vy} = I'_y\omega_y, \qquad M'_{vz} = I'_z\omega_z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Pressure on Fixed Point.—Let the component pressures at the fixed point be R_x, R_y, R_z. Let the components of all other impressed forces be ΣF_x, ΣF_y, ΣF_z. We have then from equations (1), by D'Alembert's principle,

$$\left.\begin{aligned} R_x + \Sigma F_x &= \text{m}\bar{y}\omega_x\omega_y + \text{m}\bar{z}\omega_x\omega_z - \text{m}\bar{x}\omega_y^2 - \text{m}\bar{x}\omega_z^2 + \text{m}\bar{z}\,\alpha_y - \text{m}\bar{y}\alpha_z, \\ R_y + \Sigma F_y &= \text{m}\bar{z}\,\omega_y\,\omega_z + \text{m}\bar{x}\omega_y\omega_x - \text{m}\bar{y}\omega_z^2 - \text{m}\bar{y}\omega_x^2 + \text{m}\bar{x}\alpha_z - \text{m}\bar{z}\,\alpha_x, \\ R_z + \Sigma F_z &= \text{m}\bar{x}\omega_z\,\omega_x + \text{m}\bar{y}\,\omega_z\omega_y - \text{m}\bar{z}\,\omega_x^2 - \text{m}\bar{z}\,\omega_y^2 + \text{m}\bar{y}\alpha_x - \text{m}\bar{x}\alpha_y. \end{aligned}\right\} \quad . \quad (7)$$

Equations (7) give R_x, R_y, R_z. If the centre of mass is the fixed point, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and hence

$$R_x + \Sigma F_x = 0, \qquad R_y + \Sigma F_y = 0, \qquad R_z + \Sigma F_z = 0;$$

that is, when the centre of mass is fixed, the pressure on the fixed point is the same as if there were no rotation, and is found by the conditions of static equilibrium.

Conservation of Moment of Momentum—Rotation about Fixed Point.—We have from equations (6) for the component moments of momentum for a body rotating about a fixed point O', taking the co-ordinate axes as principal axes at O',

$$M'_{vx} = I_x'\,\omega_x, \qquad M'_{vy} = I_y'\,\omega_y, \qquad M'_{vz} = I_z'\,\omega_z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and from equations (4) for the component moments of the effective forces

$$\left.\begin{aligned} M_{fx} &= (I_z' - I_y')\omega_z\,\omega_y + I_x'\alpha_x, \\ M_{fy} &= (I_x' - I_z')\omega_x\omega_z + I_y'\alpha_y, \\ M_{fz} &= (I_y' - I_x')\omega_y\,\omega_x + I_z'\alpha_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If in equations (b) the co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$ are fixed (figure, page 153), we have $\omega_z\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_x = 0$, and the axis of rotation is fixed. If also we have $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, we have $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$, and ω_x, ω_y, ω_z in equations (a) are constant. But since, by D'Alembert's principle, the moment of the effective forces is equal to the moment of the impressed forces, we have the moment of the impressed forces zero.

Hence *if the moment of the impressed forces about any fixed axis through the fixed point is always zero, the moment of momentum of the body about that axis is constant, and if the moment of inertia about that axis does not change, then the angular velocity about that axis is also constant.*

Invariable Axis.—If the moment of the impressed forces relative to the fixed point O' is always zero, the resultant must either be zero or always pass through O'. In such case the co-ordinate axes do not change in direction, and we have $\omega_z\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_x = 0$, and also $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$. Hence, from equations (b), $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, and ω_x, ω_y, ω_z in equations (a) are constant.

We have then the moment of momentum

$$I'\omega = \sqrt{I_x'^2\omega_x^2 + I_y'^2\omega_y^2 + I_z'^2\omega_z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

about an axis through the fixed point whose direction cosines are

$$\cos\alpha = \frac{I_x'\omega_x}{I'\omega}, \qquad \cos\beta = \frac{I_y'\omega_y}{I'\omega}, \qquad \cos\gamma = \frac{I_z'\omega_z}{I'\omega}, \quad . \quad . \quad . \quad . \quad (d)$$

and this axis is then *invariable in direction.*

Hence *when the moment of the impressed forces relative to the fixed point is always zero, or if there are no impressed forces, and a body rotates about any axis through the fixed point at any instant, there will be a certain axis through the fixed point for which the moment of momentum is constant. This axis is invariable in direction, and its direction cosines are given by equations* (d). *The moment of momentum about this axis is given by equations* (c).

Kinetic Energy—Rotation about a Fixed Point.—The kinetic energy of a particle of mass m and velocity v is $\frac{1}{2}mv^2$. If a particle has the component velocities v_x, v_y, v_z, we have $v^2 = v_x^2 + v_y^2 + v_z^2$ and

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2.$$

From page 154 we have for the component velocities for any particle of a rotating body

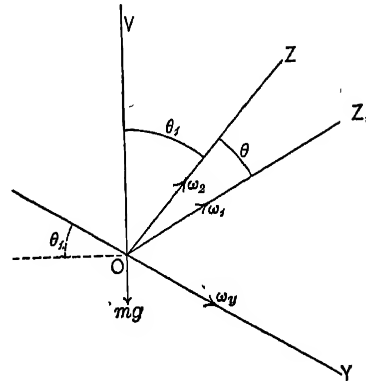
$$v_x = z'\omega_y - y'\omega_z, \qquad v_y = x'\omega_z - z'\omega_x, \qquad v_z = y'\omega_x - x'\omega_y,$$

where x', y', z' are to be taken from the fixed point O' as origin. If we square these component velocities, multiply each term by $\frac{1}{2}m$, sum up for all the particles and add, we shall have the kinetic energy for a rotating body for any co-ordinate axes we please. Let us take these axes as principal axes at O'. Then we have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. We have then, since $\Sigma m(y'^2 + z'^2) = I_x'$, $\Sigma m(z'^2 + x'^2) = I_y'$, $\Sigma m(x'^2 + y'^2) = I_z'$, for the kinetic energy

$$\mathfrak{K} = \frac{1}{2}I_x'\omega_x^2 + \frac{1}{2}I_y'\omega_y^2 + \frac{1}{2}I_z'\omega_z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we take mass in lbs., this equation gives kinetic energy in foot-poundals. For foot-pound divide by g.

Examples.—(1) *A circular disc supported at its centre of mass O rotates about the principal axis OZ at right angles to the plane of the disc with an angular velocity $\omega_z = \sqrt{3}$ radians per second. The plane of the disc makes an angle $\theta_1 = 30°$ with the horizontal. If now an angular velocity $\omega_y = 1$ radian per second is given to the disc about a principal axis OY in the plane of the disc, find the motion.*



ANS. From equations (4), page 376, we have for the component moments of the effective forces

$$M_{fx} = (I_z - I_y)\omega_z\omega_y + I_x\alpha_x,$$

$$M_{fy} = (I_x - I_z)\omega_x\omega_z + I_y\alpha_y,$$

$$M_{fz} = (I_y - I_x)\omega_y\omega_x + I_z\alpha_z.$$

By D'Alembert's principle the moments of the effective forces are equal to the moments of the impressed forces. In the present case the only impressed force is the weight mg acting at O and the equal and opposite reaction R of the support at O. The component moments of the impressed forces are therefore zero, and hence $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$. We have also $\omega_x = 0$ and $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, and the angular velocities ω_y, ω_z are therefore constant.

By the principle of page 378, we have in this case an invariable axis of rotation OZ_1 fixed in space for which the moment of momentum is constant.

Let I_1 be the moment of inertia for this invariable axis, and $\frac{d\psi}{dt}$ be the angular velocity about it, so that $I_1\frac{d\psi}{dt}$ is the moment of momentum. Since $I_z = 2I_y$ for the disc, we have

$$I_1\frac{d\psi}{dt} = \sqrt{I_z^2\omega_z^2 + I_y^2\omega_y^2} = I_y\sqrt{4\omega_z^2 + \omega_y^2}. \qquad (1)$$

If θ is the angle ZOZ_1, we have

$$\cos\theta = \frac{I_z\omega_z}{I_1\frac{d\psi}{dt}} = \frac{2\omega_z}{\sqrt{4\omega_z^2 + \omega_y^2}}, \qquad \sin\theta = \frac{I_y\omega_y}{I_1\frac{d\psi}{dt}} = \frac{\omega_y}{\sqrt{4\omega_z^2 + \omega_y^2}}. \qquad (2)$$

From page 35,

$$I_1 = I_z\cos^2\theta + I_y\sin^2\theta = I_y(1 + \cos^2\theta). \qquad (3)$$

and therefore, from (1) and (2),

$$\frac{d\psi}{dt} = \frac{\sqrt{4\omega_z^2 + \omega_y^2}}{1 + \cos^2\theta} = \frac{(4\omega_z^2 + \omega_y^2)^{\frac{3}{2}}}{8\omega_z^2 + \omega_y^2}. \qquad (4)$$

Equations (2) give the angle ZOZ_1 of OZ with the invariable axis OZ_1, and the motion of the disc is the same as if the axis OZ fixed to the disc were to rotate about OZ_1, always making the angle θ with it, with the angular velocity $\frac{d\psi}{dt}$ given by (4).

Inserting numerical values, we have

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{13}} = 0.95958, \quad \text{or} \quad \theta = 16° 21',$$

$$\frac{d\psi}{dt} = \frac{(13)^{\frac{3}{2}}}{25} = 1.87 \text{ radians per sec.}$$

2D SOLUTION.—From page 168 we have Euler's geometric equations,

$$0 = \omega_x = \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi,$$

$$\omega_y = \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi,$$

$$\omega_z = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt},$$

where $\frac{d\psi}{dt}$ is the angular velocity about OZ_1, $\frac{d\theta}{dt}$ the angular velocity of OZ about the line of nodes ON (see figure, page 167), $\frac{d\phi}{dt}$ the angular velocity of OY relative to ON.

From the first two of these equations, eliminating $\frac{d\theta}{dt}$, we obtain

$$\frac{d\psi}{dt} = \frac{\omega_y \sin\phi}{\sin\theta}, \qquad \frac{d\theta}{dt} = \omega_y \cos\phi, \qquad \frac{d\phi}{dt} = \omega_z - \frac{\omega_y \sin\phi\cos\theta}{\sin\theta}.$$

We have also the moment of momentum for the invariable axis,

$$I_1 \frac{d\psi}{dt} = \sqrt{I_z^2\omega_z^2 + I_y^2\omega_y^2} = I_y\sqrt{4\omega_z^2 + \omega_y^2},$$

and, as before,

$$\cos\theta = \frac{2\omega_z}{\sqrt{4\omega_z^2 + \omega_y^2}}, \qquad \sin\theta = \frac{\omega_y}{\sqrt{4\omega_z^2 + \omega_y^2}}, \qquad I_1 = I_y(1 + \cos^2\theta).$$

We have then $\sin\phi = \frac{1}{1 + \cos^2\theta}$, and

$$\frac{d\psi}{dt} = \frac{\omega_y}{\sin\theta(1 + \cos^2\theta)}, \qquad \frac{d\theta}{dt} = \frac{\omega_y\cos\theta}{1 + \cos^2\theta}\sqrt{2 + \cos^2\theta}, \qquad \frac{d\phi}{dt} = \omega_z - \frac{\omega_y\cos\theta}{\sin\theta(1 + \cos^2\theta)},$$

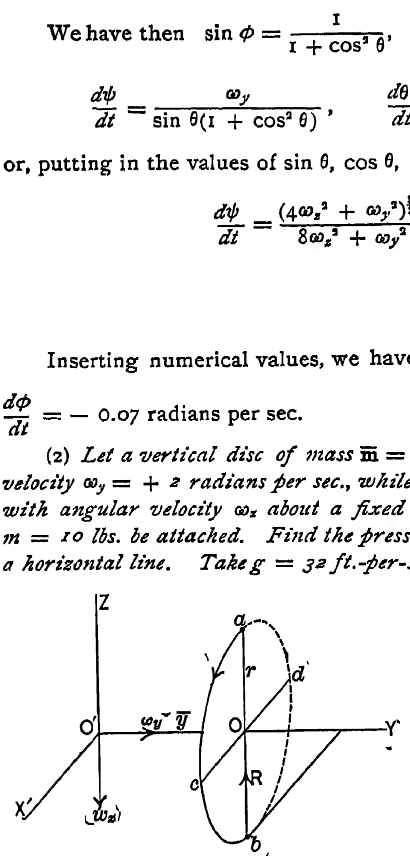
or, putting in the values of $\sin\theta$, $\cos\theta$,

$$\frac{d\psi}{dt} = \frac{(4\omega_z^2 + \omega_y^2)^{\frac{3}{2}}}{8\omega_z^2 + \omega_y^2} \qquad \frac{d\theta}{dt} = \frac{2\omega_z\omega_y\sqrt{4\omega_z^2 + \omega_y^2}\sqrt{12\omega_z^2 + 2\omega_y^2}}{8\omega_z^2 + \omega_y^2},$$

$$\frac{d\phi}{dt} = -\omega_z\frac{\omega_y^2}{8\omega_z^2 + \omega_y^2}.$$

Inserting numerical values, we have $\theta = 16° 21'$, $\frac{d\psi}{dt} = 1.87$ radians per sec., $\frac{d\theta}{dt} = 3.08$ radians per sec., $\frac{d\phi}{dt} = -0.07$ radians per sec.

(2) *Let a vertical disc of mass* $\overline{m} = 40$ *lbs. and radius* $r = 3$ *ft. roll on a horizontal plane with angular velocity* $\omega_y = +2$ *radians per sec., while its centre O describes a horizontal circle of radius* $O'O = \bar{y} = 9$ *feet with angular velocity* ω_z *about a fixed vertical axis* $O'Z'$. *At each extremity of a diameter ab let a mass* $m = 10$ *lbs. be attached. Find the pressure on the horizontal plane when these masses are in a vertical and in a horizontal line. Take* $g = 32$ *ft.-per-sec. per sec.*



ANS. Let O' be the origin, $O'Z'$ the fixed axis, and take ω_y in the direction $O'Y'$. The axes $O'X'$, $O'Y'$, $O'Z'$ are principal axes. We have $\omega_x = 0$ and $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$. Also, since $O'Z'$ is fixed, $\omega_y\omega_x = 0$, $\omega_x\omega_y = 0$, $\omega_z\omega_x = 0$, $\omega_z\omega_y = 0$.

From equations (3), page 376, we have then

$$M_{fx} = -I'_{xy}\omega_y\omega_z,$$

and since the disc rolls,

$$r\omega_y = -\bar{y}\omega_z, \quad \text{or} \quad \omega_z = -\frac{r\omega_y}{\bar{y}}.$$

Now I'_{xy}, or the moment of inertia relative to the plane $X'Y'$, is the same as the moment of inertia for the horizontal axis cd through the centre of mass. Hence if θ is the angle of Oa with the vertical,

$$I'_{xy} = \frac{\overline{m}r^2}{4} + 2mr^2\cos^2\theta.$$

Let R be the vertical pressure of the horizontal plane at b. Then, by D'Alembert's principle,

$$R\bar{y} - (\overline{m} + 2m)g\bar{y} = M_{fx} = -\left(\frac{\overline{m}r^2}{4} + 2mr^2\cos^2\theta\right)\omega_y\omega_z.$$

Substituting the value of ω_z, we have

$$R = (\overline{\text{m}} + 2m)g + \frac{r\omega_y^2}{\overline{y}^2}\left(\frac{\overline{\text{m}}r^2}{4} + 2mr^2\cos^2\theta\right).$$

If we take distance in feet and mass in lbs, this equation gives R in poundals. For R in pounds we divide by g and obtain

$$R = \overline{\text{m}} + 2m + \frac{r\omega_y^2}{g\overline{y}^2}\left(\frac{\overline{\text{m}}r^2}{4} + 2mr^2\cos^2\theta\right). \quad \ldots \ldots \ldots \ldots \quad (1)$$

We see from (1) that the pressure R is greater than the total weight $\overline{\text{m}} + 2m$. The greatest possible pressure for given r and $\overline{y}$ is when $\cos\theta = 1$ or $\theta = 0$, that is, when the masses m are at a and b in the figure. The least possible pressure is when $\cos\theta = 0$, or $\theta = 90°$, that is, when the masses m are at c and d in the figure.

In the first case, when the masses m are in the vertical line ab, we have $\theta = 0$ and

$$R = 60 + \frac{1}{216}(90 + 180) = 61.25 \text{ pounds.}$$

In the second case, when the masses are in the horizontal line cd, we have $\theta = 90$ and

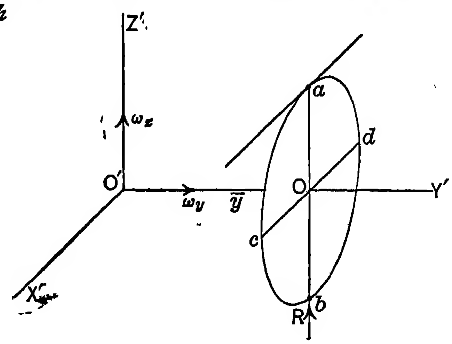
$$R = 60 + \frac{90}{216} = 60.4166 \text{ pounds.}$$

If the masses m are removed, we have in all positions

$$R = 60 + \frac{90}{216} = 60.4166 \text{ pounds.}$$

We see, then, that for a disc rolling as in the example the pressure on the horizontal plane is greater than the weight of the disc.

(3) *In the preceding example let the horizontal plane on which the disc rolls be above instead of below the disc. Find the force R necessary to keep the disc in contact with the plane.*



ANS. In this case we have

$$r\omega_y = \overline{y}\omega_z, \quad \text{or} \quad \omega_z = +\frac{r\omega_y}{\overline{y}},$$

and, as before,

$$I'_{xy} = \frac{\overline{\text{m}}r^2}{4} + 2mr^2\cos^2\theta,$$

and

$$M_{fx} = -I'_{xy}\omega_y\omega_z.$$

We have then, as before,

$$R\overline{y} - (\overline{\text{m}} + 2m)g\overline{y} = M_{fx} = -\left(\frac{\overline{\text{m}}r^2}{4} + 2mr^2\cos^2\theta\right)\omega_y\omega_z.$$

Substituting the value of ω_z, we have

$$R = (\overline{\text{m}} + 2m)g - \frac{r\omega_y^2}{\overline{y}^2}\left(\frac{\overline{\text{m}}r^2}{4} + 2mr^2\cos^2\theta\right),$$

or for R in pounds

$$R = (\overline{\text{m}} + 2m) - \frac{r\omega_y^2}{g\overline{y}^2}\left(\frac{\overline{\text{m}}r^2}{4} + 2mr^2\cos^2\theta\right). \quad \ldots \ldots \ldots \ldots \quad (1)$$

We see in this case that the force R is *less* than the total weight $(\overline{\text{m}} + 2m)$.

If the masses m are vertical at a and b, we have $\theta = 0$ and

$$R = 60 - \frac{1}{216}(90 + 180) = 58.75 \text{ pounds.}$$

If the masses m are horizontal at c and d, we have $\theta = 90$ and

$$R = 60 - \frac{90}{216} = 59.3833 \text{ pounds.}$$

If in (1) we put $R = 0$, we have for the value of ω_y for which *no force R will be required*

$$\omega_y = \sqrt{\frac{(\bar{m} + 2m)g\bar{y}^2}{r\left(\frac{\bar{m}r^2}{4} + 2mr^2\cos^2\theta\right)}}. \quad \text{. . . . .} \quad (2)$$

If the masses m are removed, we have

$$\omega_y = \sqrt{\frac{4g\bar{y}^2}{r^2}} = 19.6 \text{ radians per sec.}$$

If the disc rolls with this angular velocity, R will be zero, that is, the disc is self-supporting and will roll round on the horizontal plane above it without any force R being necessary to keep it in contact.

If ω_y is greater than this, the disc will press against the horizontal plane above it. If ω_y is less than this, a force R will be necessary to keep the disc in position.

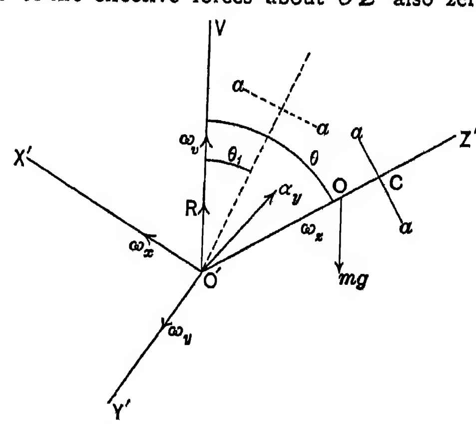
(4) *Discuss the action of the Gyroscope and Spinning Top.*

DESCRIPTION.—The gyroscope consists of a disc aa which is set in rotation about an axis in the direction of the diameter of the ring R. This ring is attached to the rod S, which passes through a sleeve at h. This sleeve is pivoted in the fork g so that S can rotate in a vertical plane, and the fork g is pivoted at f so that this rod can rotate horizontally. A sliding counterweight G can be so adjusted that the centre of mass of the apparatus can be made to lie on the same side of the standard as the disc, on the opposite side, or directly over the standard.

SOLUTION.—Let $O'Z'$ be the axis of the disc, ω_z the angular velocity about this axis, the rotation being clockwise as we look from O' to C, the centre of the disc. Take the co-ordinate axes $O'X$, $O'Y'$, $O'Z'$ principal axes at the fixed point O'.

Let the counterweight be adjusted so that the centre of mass O is on the same side of the standard as the disc. Let $\bar{m}$ be the mass of the apparatus, and $\bar{m}g$ its weight acting at the centre of mass O. The impressed forces are then $\bar{m}g$ and the reaction R of the standard at O'. These always pass through the axis $O'Z'$. Hence the moment about the axis $O'Z'$ of the impressed forces is always zero, and by D'Alembert's principle we have $M_{fz} = 0$, or the moment of the effective forces about $O'Z'$ also zero. From equations (4), page 376, we have then

$$M_{fz} = 0 = (I_y' - I_x')\omega_y\omega_x + I_z'\alpha_z.$$



In the case of a circular disc I_y' is equal to I_x', and hence $I_z'\alpha_z = 0$, or $\alpha_z = 0$. Hence the angular velocity ω_z about the axis $O'Z'$ is constant whatever the position of the axis.

Let the axis $O'C$ of the disc make the initial angle θ_1 with the vertical $O'V$, and the disc be allowed to fall under the action of gravity to the position in the figure where the axis $O'C$ makes the angle θ with the vertical. The angular acceleration α_y will be always at right angles to the axis $O'C$ of the disc in the direction shown in the figure if O falls, in the opposite direction if O rises. Hence the axis $O'C$ rotates about $O'V$ clockwise as we look from O' to V if O falls so that the angular velocity ω_v about $O'V$ is as represented in the figure.

Just as in the preceding example, then, there is a certain value of ω_v for which the disc is self-supporting, and when that value is obtained there is no farther fall.

If the counterweight is so adjusted that the centre of mass O is on the opposite side of the standard from the disc, the axis $O'C$ of the disc will move upwards, the angular acceleration α and angular velocity ω_v will be opposite in direction to that shown in the figure, and the axis $O'C$ then rotates about $O'V'$ in a clockwise direction as we look from V to O'.

If the counterweight is so adjusted that the centre of mass is over the standard, the axis $O'C$ of the disc neither rises nor falls, there is no angular acceleration α and the axis $O'C$ does not turn about $O'V$.

Let the centre of mass O be as in the figure, and suppose the axis $O'C$ of the disc has fallen from the initial angle θ_1 till it makes the angle θ with $O'V$.

Take the co-ordinate axes $O'X'$, $O'Y'$, $O'Z'$ fixed in the body, and let the angular velocities about these axes be ω_x, ω_y and ω_z.

The moment of the impressed forces $\overline{m}g$ and R is always zero about the fixed axis $O'V$, and by conservation of momentum (page 390) the moment of momentum about this axis is constant for all positions of the body.

In the initial position the moment of momentum about $O'V$ is $I_z'\omega_z \cos\theta_1$, since ω_x, ω_y for this position are zero. In the final position the moment of momentum about $O'V$ is

$$I_z'\omega_z \cos\theta + I_y'\omega_y \cos VO'Y' + I_x'\omega_x \cos VOX'.$$

If we insert the values of the cosines from equations (5), page 168, we have then

$$I_z'\omega_z \cos\theta_1 = I_z'\omega_z \cos\theta + I_y'\omega_y \sin\theta \sin\phi - I_x'\omega_x \sin\theta\cos\phi, \quad . \quad . \quad . \quad (1)$$

where (page 167) ϕ is the angle of $O'Y'$ with the line of nodes.

The initial kinetic energy is $\frac{1}{2}I_z'\omega_z^2$ (page 378), and the initial potential energy relative to a horizontal plane at a distance $\bar{y}$ below O', if $\bar{y}$ is the distance $O'O$ of the centre of mass, is $\overline{m}g(\bar{y}+\bar{y}\cos\theta_1)$ when O is on the same side of the standard as the disk, and $\overline{m}g(\bar{y}-\bar{y}\cos\theta_1)$ when O is on the opposite side of the standard from the disc. The total initial energy is then

$$\mathfrak{E}_1 = \frac{1}{2}I_z'\omega_z^2 + \overline{m}g(\bar{y} \pm \bar{y}\cos\theta_1).$$

The final kinetic energy is (page 378)

$$\frac{1}{2}I_x'\omega_x^2 + \frac{1}{2}I_y'\omega_y^2 + \frac{1}{2}I_z'\omega_z^2,$$

and the final potential energy is $\overline{m}g(\bar{y} \pm \bar{y}\cos\theta)$. The total final energy is then

$$\mathfrak{E} = \frac{1}{2}I_x'\omega_x^2 + \frac{1}{2}I_y'\omega_y^2 + \frac{1}{2}I_z'\omega_z^2 + \overline{m}g(\bar{y} \pm \bar{y}\cos\theta).$$

If we disregard friction, we have, by the principle of conservation of energy (page 304), $\mathfrak{E} = \mathfrak{E}_1$, or

$$\frac{1}{2}I_x'\omega_x^2 + \frac{1}{2}I_y'\omega_y^2 = \pm\,\overline{m}g\bar{y}(\cos\theta_1 - \cos\theta), \quad . \quad . \quad . \quad (2)$$

where the (+) sign is to be taken for centre of mass O on same side of standard as the disc, and the (—) sign when it is on the opposite side.

We have also, from Euler's geometric equations (page 168),

$$\left.\begin{aligned} \omega_x &= \frac{d\theta}{dt}\cdot\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi, \\ \omega_y &= \frac{d\theta}{dt}\cos\phi + \frac{d\phi}{dt}\sin\theta\sin\phi, \\ \omega_z &= \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

where $\frac{d\psi}{dt}$ is the angular velocity ω_v about $O'V$, or the angular velocity of precession; $\frac{d\theta}{dt}$ is the angular velocity of $O'Z'$ about the line of nodes (see figure, page 167), or the angular velocity of nutation; $\frac{d\phi}{dt}$ is the angular velocity of $O'Y$ relative to the line of nodes.

Squaring and adding the first two of equations (3), we have

$$\omega_x^2 + \omega_y^2 = \frac{d\theta^2}{dt^2} + \frac{d\psi^2}{dt^2} \sin^2 \theta.$$

Substituting this in (2), we have, since for the disc $I_x' = I_y'$,

$$I_x' \frac{d\theta^2}{dt^2} + I_x' \frac{d\psi^2}{dt^2} \sin^2 \theta = \pm 2 \overline{m} g \overline{y} \, (\cos \theta_1 - \cos \theta). \qquad (4)$$

Substituting the values of ω_x, ω_y from (3) in (1), we have

$$I_x' \frac{d\psi}{dt} \sin^2 \theta = I_z' \omega_z (\cos \theta_1 - \cos \theta). \qquad (5)$$

We have also, from the last of equations (3),

$$\frac{d\phi}{dt} + \frac{d\psi}{dt} \cos \theta = \omega_z. \qquad (6)$$

Equations (4), (5), (6) are the differential equations for the gyroscope. When $\theta = \theta_1$, or at the beginning of motion, we have

$$\frac{d\psi}{dt} = 0, \qquad \frac{d\theta}{dt} = 0, \qquad \frac{d\phi}{dt} = \omega_z.$$

From (5) we have the angular velocity ω_v about $O'V$, or the angular velocity of precession,

$$\omega_v = \frac{d\psi}{dt} = \frac{I_z' \omega_z}{I_x'} \cdot \frac{\cos \theta_1 - \cos \theta}{\sin^2 \theta}, \qquad (7)$$

and substituting this in (4) we have for the angular velocity of $O'Z'$ about the line of nodes, or the angular velocity of nutation,

$$\frac{d\theta}{dt} = \sqrt{(\cos \theta_1 - \cos \theta) \left[\pm \frac{2 \overline{m} g \overline{y}}{I_x'} - \frac{I_z'^2 \omega_z^2}{I_x'^2 \sin^2 \theta} (\cos \theta_1 - \cos \theta) \right]}, \qquad (8)$$

where the (+) sign is taken for centre of mass O on same side of standard as disc, and the (—) sign when it is on the opposite side.

From (8), taking the (+) sign, we see that for the centre of mass O on same side of standard as the disc, $\frac{d\theta}{dt}$ is imaginary when θ is less than θ_1. Taking the (—) sign, we see that for the centre of mass O on the opposite side of standard from disc, $\frac{d\theta}{dt}$ is imaginary when θ is greater than θ_1. Hence the centre of mass O *always falls from its initial position and can never rise above it.*

From (7), then, if ω_z is positive, that is, if the rotation of the disc when looking from O' to Z' is clockwise, $\frac{d\psi}{dt}$ is positive, or the rotation about $O'V$ looking from O' to V is clockwise if the centre of mass O is on the same side of standard as the disc.

If O is on the opposite side and ω_z is positive, $\frac{d\psi}{dt}$ is negative. If the centre of mass is over the standard, $\overline{y} = 0$ and the angle θ_1 remains unchanged. Hence $\cos \theta_1 - \cos \theta = 0$, and $\frac{d\theta}{dt} = 0$, $\frac{d\psi}{dt} = 0$. The axis $O'Z'$ then remains stationary.

These conclusions can be verified by the apparatus (figure, page 382) by shifting the counterweight.

If we put $\frac{d\theta}{dt} = 0$ in (8), we obtain the maximum and minimum values of θ. We have $\frac{d\theta}{dt} = 0$ when

$\theta = \theta_1$, and this is the minimum value of θ, for we have just seen that O always falls, and hence θ cannot be less than θ_1. We also have $\frac{d\theta}{dt} = 0$ and θ a maximum when

$$\cos\theta_1 - \cos\theta = \pm 2\overline{m}g\bar{y}\cdot\frac{I'_x \sin^2\theta}{I_z'^2\omega_z^2}. \quad (9)$$

If we insert this value in (7), we have for the maximum value of ω_v

$$\text{max. } \omega_v = \pm\frac{2\overline{m}g\bar{y}}{I'_z\omega_z}. \quad (10)$$

The minimum value of ω_v is when $\theta = \theta_1$, or min. $\omega_v = 0$. We see from (10) that the maximum value of ω_v is independent of θ_1 and θ, or of the initial and final positions of the disc.

From (9) we obtain for the maximum value of θ

$$\cos\theta_{\text{max.}} = \pm\frac{I'_z\omega_z^2}{4\overline{m}g\bar{y}I'_x} + \sqrt{1 - \frac{I_z'^2\omega_z^2\cos\theta_1}{\pm 2\overline{m}g\bar{y}I'_x} + \left(\frac{I_z'^2\omega_z^2}{4\overline{m}g\bar{y}I'_x}\right)^2}. \quad (11)$$

Let us put for brevity and convenience the quantity

$$\frac{I_z'^2\omega_z^2}{4\overline{m}g\bar{y}I'_x} = \beta^2.$$

Then we can write equation (11)

$$\cos\theta_{\text{max.}} = \pm\beta^2 + \sqrt{1 \mp 2\beta^2\cos\theta_1 + \beta^4}, \quad (12)$$

where the upper signs are for O on same side of standard as disc, and the lower signs for O on the opposite side from disc.

We see, from (12), that the maximum value of θ depends upon β, and that this maximum value can be $0°$ or $180°$, that is, $\cos\theta_{\text{max.}}$ can be $+1$ or -1 only when $\beta = 0$. But β can be zero only when $\omega_z = 0$. Hence any velocity of rotation ω_z of the disc, however small, *is sufficient to prevent the axis $O'Z'$ from reaching the vertical.* The self-sustaining power of the gyroscope is thus proved.

From (12) we have

$$\cos\theta_1 - \cos\theta_{\text{max.}} = \pm\frac{\sin^2\theta_{\text{max.}}}{2\beta^2}. \quad (13)$$

If β or ω_z is very great, $\cos\theta_1 - \cos\theta_{\text{max.}}$ is very small. Hence, by increasing the value of ω_z, $\theta_{\text{max.}} - \theta_1$ can be made less than any assignable quantity. This proves the apparently paradoxical result that the disc when rotating rapidly and set at any angle θ_1 with the vertical does not *visibly rise or fall.* But we see, from (7), that for $\theta = \theta_1$, $\frac{d\psi}{dt}$ is zero, and hence the disc *must rise or fall* in order to generate rotation about $O'V$. If ω_z is great, this rise or fall is, however, very small and may not be visible.

Let λ be the length of the simple pendulum which would oscillate about $O'X'$ or $O'Y'$ in the same time as the apparatus, when ω_z is zero. Then (page 337) we have

$$\lambda = \frac{I'_x}{\overline{m}\bar{y}},$$

and equation (8) becomes

$$\sin^2\theta\frac{d\theta^2}{dt^2} = \frac{2g}{\lambda}\Big[\pm\sin^2\theta - 2\beta^2(\cos\theta_1 - \cos\theta)\Big](\cos\theta_1 - \cos\theta), \quad (14)$$

and equation (7) becomes

$$\sin^2\theta\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}}(\cos\theta_1 - \cos\theta). \quad (15)$$

and equation (10) becomes

$$\text{max. } \omega_v = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}}. \quad (16)$$

The centre of mass then oscillates up and down between the minimum and maximum values of θ_1 and max. θ as given by (12), while the angular velocity of the centre of mass about $O'V$ varies from $\frac{d\psi}{dt} = 0$, when the axis is in its initial position, to the maximum value given by (16).

The complete solution of the problem requires the integration of the differential equations (4), (5) and (6). This necessitates the use of elliptic functions. If, however, we *assume* that the velocity of rotation of the disc ω_x is very great, and hence $\cos\theta_1 - \cos\theta$ max. or $\theta - \theta_1$ very minute, we may obtain integrals of (4) and (5) which will express the motion with all requisite accuracy.

Let us then assume ω_x or β very large and $\theta - \theta_1$ very small, the centre of mass O being on the same side of O' as the disc.

Let $\theta - \theta_1 = u$, or

$$\theta = \theta_1 + u, \qquad d\theta = du,$$

where u is a very small angle.

Then we have

$$\sin\theta = \sin\theta_1 \cos u + \cos\theta_1 \sin u,$$

$$\cos\theta = \cos\theta_1 \cos u - \sin\theta_1 \sin u.$$

Also by series, since u is a very small angle, neglecting higher powers of u than the square,

$$\sin u = u, \qquad \cos u = 1 - \frac{u^2}{2}.$$

Substituting, we have

$$\sin\theta = \sin\theta_1\left(1 - \frac{u^2}{2}\right) + u\cos\theta_1,$$

$$\cos\theta = \cos\theta_1\left(1 - \frac{u^2}{2}\right) - u\sin\theta_1.$$

Hence, neglecting higher powers of u than the square,

$$\sin^2\theta = \sin^2\theta_1 - u^2\sin^2\theta_1 + u^2\cos^2\theta_1 + 2u\sin\theta_1\cos\theta_1,$$

$$\cos\theta_1 - \cos\theta = u\sin\theta_1 + \frac{1}{2}u^2\cos\theta_1, \quad \dots\dots\dots \quad (17)$$

and therefore

$$\frac{\cos\theta_1 - \cos\theta}{\sin^2\theta} = \frac{u\sin\theta_1 + \frac{1}{2}u^2\cos\theta_1}{\sin^2\theta_1 + 2u\sin\theta_1\cos\theta_1 + u^2\cos^2\theta_1 - u^2\sin^2\theta_1}, \quad \dots\dots \quad (18)$$

$$\frac{(\cos\theta_1 - \cos\theta)^2}{\sin^2\theta} = \frac{u^2\sin^2\theta_1}{\sin^2\theta_1 + 2u\sin\theta_1\cos\theta_1 + u^2\cos^2\theta_1 - u^2\sin^2\theta_1} = u^2. \quad \dots\dots \quad (19)$$

Inserting (17) and (19) in (14), we obtain

$$\sqrt{\frac{g}{\lambda}}\,.\,dt = \frac{du}{\sqrt{2u\sin\theta_1 + u^2(\cos\theta_1 - 4\beta^2)}}.$$

Since β or ω_x has been assumed very great, $\cos\theta_1$ may be neglected in comparison with $4\beta^2$, and we have

$$\sqrt{\frac{g}{\lambda}}\,.\,dt = \frac{du}{\sqrt{2u\sin\theta_1 - 4\beta^2u^2}} = \frac{1}{2\beta}\cdot\frac{du}{\sqrt{2u\frac{\sin\theta_1}{4\beta^2} - u^2}}. \quad \dots\dots \quad (20)$$

Integrating, since when $t = 0$, $u = \theta - \theta_1 = 0$,

$$\sqrt{\frac{g}{\lambda}}\,.\,t = \frac{1}{2\beta}\,\text{versin}^{-1}\frac{4\beta^2}{\sin\theta_1}\,.\,u = \frac{1}{2\beta}\cos^{-1}\left(1 - \frac{4\beta^2u}{\sin\theta_1}\right). \quad \dots\dots \quad (21)$$

Hence

$$u = \frac{\sin\theta_1}{4\beta^2}\left[1 - \cos\left(2\beta\sqrt{\frac{g}{\lambda}}\,.\,t\right)\right]; \quad \dots\dots \quad (22)$$

or, since $\cos 2A = 1 - 2\sin^2 A$,

$$u = \frac{1}{2\beta^2}\sin\theta_1 \sin^2\left(\beta\sqrt{\frac{g}{\lambda}}\,.\,t\right). \quad (23)$$

We have from (18), neglecting the *square as well as higher powers of u* (which may be done without sensible error owing to the minuteness of u, though it could not be done in the foregoing values of dt and t, since β^2 is great when u is small),

$$\frac{\cos\theta_1 - \cos\theta}{\sin^2\theta} = \frac{u\sin\theta_1}{\sin^2\theta_1 + 2u\sin\theta_1\cos\theta_1}.$$

The greatest possible value of $\sin\theta_1\cos\theta_1$ is for $\theta_1 = 45°$, or

$$\sin\theta_1\cos\theta_1 = \frac{1}{2}.$$

Since u is very small, we have then, neglecting $2u\sin\theta_1\cos\theta_1$,

$$\frac{\cos\theta_1 - \cos\theta}{\sin^2\theta} = \frac{u}{\sin\theta_1};$$

and substituting in (15), we obtain

$$\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}}\,.\,\frac{u}{\sin\theta_1}.$$

Inserting the value of u from (23), we have

$$\frac{d\psi}{dt} = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}}\sin^2\left(\beta\sqrt{\frac{g}{\lambda}}\,.\,t\right). \quad (24)$$

Integrating, since $d\psi = 0$ when $t = 0$, we obtain

$$\psi = \frac{1}{2\beta}\sqrt{\frac{g}{\lambda}}\,.\,t - \frac{1}{4\beta^2}\sin\left(2\beta\sqrt{\frac{g}{\lambda}}\,.\,t\right). \quad (25)$$

Equations (23), (24), (25) give with all requisite accuracy the vertical angular depression of OZ', $u = \theta - \theta_1$, the horizontal angular velocity $\frac{d\psi}{dt}$, and the horizontal angle ψ at the end of any time t, provided ω_x is very great.

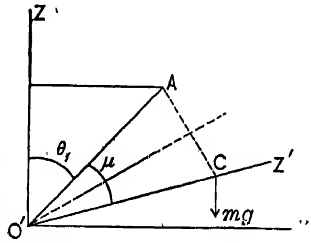
Referring to (20), we see that *it is the differential equation of a cycloid generated by a circle whose angular diameter is* $\frac{\sin\theta_1}{2\beta^2}$ (page 140).

When, starting from $t = 0$ (and therefore $u = 0$, $\frac{d\psi}{dt}$ and $\psi = 0$), u has its greatest value, we have, from (21), (22), (24), (25)

$$t = \frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}}, \qquad u = \frac{\sin\theta_1}{2\beta^2}, \qquad \frac{d\psi}{dt} = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}}, \qquad \psi = \frac{\pi}{4\beta^2}.$$

After the expiration of the time $t = \frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}}$ we have

$$u = 0, \qquad \frac{d\psi}{dt} = 0, \qquad \psi = \frac{\pi}{2\beta^2},$$



and $O'Z'$ has regained its original elevation and the horizontal velocity is zero.

The axis $O'Z'$ then moves as if it were the element of a right circular cone $AO'Z'$, the angle $AO'Z'$ being equal to u, which rolls on the cone $ZO'A$, the angle $ZO'A$ being equal to θ_1.

(5) *A layer of dust of uniform depth d, d being small compared to the radius of the earth, is formed on the earth by the fall of meteors reaching the earth from all directions. Consider the earth as a homogeneous sphere of radius r and density Δ, and let δ be the density of the layer. Find the change in length of the day.*

ANS. Let ω_1 be the angular velocity before and ω after the layer is formed, and I_1 the moment of inertia

of the earth and I that of the layer. Since there are no forces in the system except the mutual action of the particles by the principle of conservation of moment of momentum (page 376), we have

$$I_1\omega_1 = (I_1 + I)\omega, \quad \text{or} \quad \omega = \frac{1}{1 + \frac{I}{I_1}}\omega_1.$$

The mass of the earth is $\frac{4}{3}\Delta\pi r^3$. The moment of inertia of the earth is then

$$I_1 = \frac{2r^2}{5} \cdot \frac{4}{3}\Delta\pi r^3 = \frac{8}{15}\Delta\pi r^5.$$

The moment of inertia of the dust-layer is

$$I = \frac{8}{15}\delta\pi(r + d)^5 - \frac{8}{15}\delta\pi r^5.$$

Hence

$$\frac{I}{I_1} = \frac{\delta(r + d)^5 - \delta r^5}{\Delta r^5}.$$

Expanding, and neglecting $\frac{d^2}{r^2}$ and all higher powers, we have

$$\frac{I}{I_1} = \frac{5\delta d}{\Delta r}.$$

Therefore

$$\omega = \frac{1}{1 + \frac{5\delta d}{\Delta r}}\omega_1.$$

If the density of the earth is taken at 5.5 as compared to water, and the density of the dust is taken at 2, and $d = \frac{1}{20}r$, we have

$$\omega = \frac{1}{1 + \frac{10}{5.5 \times 20}}\omega_1 = \frac{11}{12}\omega_1.$$

The length of the day in this case would be $\frac{11}{12}$ of 24 hours, or only 22 hours.

(6) *If the earth gradually contracted by radiation of heat, so as to be always similar to itself as regards its physical constitution and form, show that when every radius vector has contracted an nth part of its length where $\frac{1}{n}$ is a small fraction, the angular velocity has increased a 2nth part of its value.*

ANS. Let m be the mass of the earth, r_1 its initial and r its final radius, and I_1 its initial and I its final moment of inertia.

Then

$$I_1 = \frac{2}{5}mr_1^2, \qquad I = \frac{2}{5}mr^2,$$

and

$$I_1\omega_1 = I\omega, \quad \text{or} \quad \omega = \frac{I_1}{I}\omega_1 = \frac{r_1^2}{r^2}.$$

But $r = r_1 - nr_1 = r_1(1 - n)$. Hence

$$\omega = \frac{r_1^2}{r_1^2(1 - n)^2}\omega_1 = \frac{1}{(1 - n)^2}\omega_1.$$

Expanding, and neglecting n^2 and higher powers,

$$\omega = \frac{1}{1 - 2n}\omega_1 = (1 + 2n)\omega_1.$$

(7) *If two railway trains each of mass m were to move in opposite directions from the poles along a meridian, and arrive at the equator at the same time, show that the angular velocity of the earth would be decreased by* $\frac{5m}{E}$ *of itself, where E is the mass of the earth.*

ANS. Let I be the moment of inertia of the earth, and ω_1 and ω the angular velocities before and after. Then $I=\frac{2}{5}Er^2$, where r is the radius of the earth, and we have

$$I\omega_1 = I\omega + 2mr^2\omega, \quad \text{or} \quad \frac{2}{5}Er^2\omega_1 = \frac{2}{5}Er^2\omega + 2mr^2\omega.$$

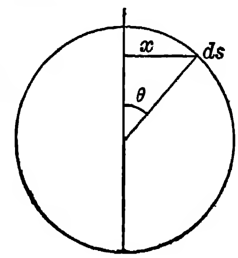
Hence, neglecting $\frac{m^2}{E^2}$ and higher powers,

$$\omega = \frac{E\omega_1}{E+5m} = \left(1-\frac{5m}{E}\right)\omega_1.$$

(8) *Suppose a mass of ice m to melt from the polar regions for 20 degrees round each pole to the extent of about a foot thick, or enough to give* $1\frac{1}{10}$ *ft. over those areas, which spread over the whole globe would raise the sea-level by only some such undiscoverable difference as three fourths of an inch. Show that this would diminish the time of the earth's rotation by one tenth of a second per year.*

ANS. Let θ be the angle from the pole, and δ the density of the ice. Then the mass m is

$$m = 4\pi\delta r^2(1-\cos\theta),$$



where r is the radius of the earth. We have $rd\theta = ds$ and the mass of a strip is $2\pi\delta x ds$. But $x = r\sin\theta$, hence the mass of a strip is $2\pi\delta r^2\sin\theta\, d\theta$. The moment of inertia is then

$$I = 2\int_\theta^0 2\pi\delta r^4\sin^3\theta\, d\theta = \frac{4\pi\delta r^4}{3}\cos\theta(1-\cos\theta)(1+\cos\theta),$$

or, substituting the value of m,

$$I = \frac{mr^2}{3}\cos\theta(1+\cos\theta).$$

If E is the mass of the earth, the moment of inertia of the earth is $I=\frac{2Er^2}{5}$. If ω_1 is the angular velocity before and ω after melting, we have, by the principle of conservation of moment of momentum,

$$\frac{2Er^2}{5}\omega_1 + \frac{mr^2}{3}\cos\theta(1+\cos\theta)\omega_1 = \frac{2Er^2}{5}\omega,$$

or

$$\frac{\omega_1-\omega}{\omega} = -\frac{5m}{6E}\cos\theta(1+\cos\theta).$$

The final angular velocity ω is then greater than the initial, and the time of rotation is diminished. The difference of time for one day is

$$\frac{2\pi}{\omega_1} - \frac{2\pi}{\omega}.$$

Substituting numerical values, the difference of time is easily found.

CHAPTER VIII.

ROTATION AND TRANSLATION.

Effective Forces — Rotation and Translation.—Equations (5), page 159, give the components of the acceleration of any particle of a rotating and translating body. For origin at the centre of mass we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$. If we make these changes, multiply each term by the mass m of the particle and sum up for all the particles, we have, since $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, for the component effective forces for a rotating and translating body of mass $\overline{\mathrm{m}} = \Sigma m$, for any co-ordinate axes we please,

$$\Sigma mf_x = \overline{\mathrm{m}}\bar{f}_x, \qquad \Sigma mf_y = \overline{\mathrm{m}}\bar{f}_y, \qquad \Sigma mf_z = \overline{\mathrm{m}}\bar{f}_z, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z$ are the component accelerations of the centre of mass.

That is, the effective force in any direction is the same as for a particle of mass equal to the mass of the body having the acceleration in that direction of the centre of mass.

Moments of Effective Forces—Rotation and Translation.—Equations (10), page 160, give the component moments of the acceleration of any particle of a rotating and translating body. If we multiply each term by the mass m of the particle and sum up for all the particles, we shall obtain the component moments of the effective forces for any co-ordinate axes we please. Let us take these axes as principal axes at the origin O'. Then we have $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, also $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$.

We also have

$$\Sigma m(y^2 + z^2) = I_x, \qquad \Sigma m(z^2 + x^2) = I_y, \qquad \Sigma m(x^2 + y^2) = I_z$$

where I_x, I_y, I_z are the moments of inertia of the body for the co-ordinate axes OX, OY, OZ.

We also have

$$\Sigma mx^2 = I_{yz}, \qquad \Sigma my^2 = I_{zx}, \qquad \Sigma mx^2 = I_{xy},$$

where I_{xy}, I_{yz}, I_{zx} are the moments of inertia of the body for the co-ordinate planes XY, YZ, ZX.

We have then from equations (10), page 160, for the component moments of the effective forces for principal co-ordinate axes through the origin O'

$$\left.\begin{aligned} M'_{fx} &= \overline{\mathrm{m}}\bar{f}_z\bar{y} - \overline{\mathrm{m}}\bar{f}_y\bar{z} + I_{zx}\omega_z\omega_y - I_{xy}\omega_y\omega_z + I_x\alpha_x, \\ M'_{fy} &= \overline{\mathrm{m}}\bar{f}_x\bar{z} - \overline{\mathrm{m}}\bar{f}_z\bar{x} + I_{xy}\omega_x\omega_z - I_{yz}\omega_z\omega_x + I_y\alpha_y, \\ M'_{fz} &= \overline{\mathrm{m}}\bar{f}_y\bar{x} - \overline{\mathrm{m}}\bar{f}_x\bar{y} + I_{yz}\omega_y\omega_x - I_{zx}\omega_x\omega_y + I_z\alpha_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the co-ordinates of the centre of mass and $\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z$ are the component accelerations of the centre of mass.

If any co-ordinate axis, as $O'X'$, is fixed, we have $\omega_x\omega_y = 0$, $\omega_x\omega_z = 0$; if $O'Y'$ is fixed, we have $\omega_y\omega_x = 0$, $\omega_y\omega_z = 0$; if $O'Z'$ is fixed, we have $\omega_z\omega_x = 0$, $\omega_z\omega_y = 0$.

If the co-ordinate axes are not fixed, we have $\omega_x\omega_y = \omega_y\omega_x$, $\omega_y\omega_z = \omega_z\omega_y$, $\omega_z\omega_x = \omega_x\omega_z$, and since

$$I_{zx} - I_{xy} = \Sigma m(y^2 - z^2) = I_z - I_y, \qquad I_{xy} - I_{yz} = \Sigma m(z^2 - x^2) = I_x - I_z,$$
$$I_{yz} - I_{zx} = \Sigma m(x^2 - y^2) = I_y - I_x,$$

equations (2) reduce to

$$\left.\begin{aligned} M'_{fx} &= \overline{m}\bar{f}_z\bar{y} - \overline{m}\bar{f}_y\bar{z} + (I_z - I_y)\omega_z\omega_y + I_x\alpha_x, \\ M'_{xy} &= \overline{m}\bar{f}_x\bar{z} - \overline{m}\bar{f}_z\bar{x} + (I_x - I_z)\omega_x\omega_z + I_y\alpha_y, \\ M'_{fz} &= \overline{m}\bar{f}_y\bar{x} - \overline{m}\bar{f}_x\bar{y} + (I_y - I_x)\omega_y\omega_x + I_z\alpha_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we take distance in feet and mass in lbs., these equations give moments in poundal-feet. For pound-feet divide by g.

Momentum—Rotation and Translation.—Equations (4), page 154, give the component velocities for any particle of a rotating and translating body. If we multiply each term by the mass m of the particle, and sum up for all the particles, we have, since $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, for the components of the momentum for a rotating and translating body of mass $\overline{m} = \Sigma m$, for any co-ordinate axes we please,

$$\Sigma mv_x = \overline{m}\bar{v}_x, \qquad \Sigma mv_y = \overline{m}\bar{v}_y, \qquad \Sigma mv_z = \overline{m}\bar{v}_z, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where v_x, v_y, v_z are the component velocities of the centre of mass.

That is, the momentum in any direction is the same as for a particle of mass equal to the mass of the body having the velocity in that direction of the centre of mass.

Moment of Momentum—Rotation and Translation.—Equations (10), page 155, give the component moments of velocity for any particle of a rotating and translating body. If we multiply each term by the mass m of the particle and sum up for all the particles, we shall obtain the component moments of momentum for any co-ordinate axes we please. We have then from equations (10), page 155, since $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, and $\Sigma m(y^2 + z^2) = I_x$, $\Sigma m(z^2 + x^2) = I_y$, $\Sigma m(x^2 + y^2) = I_z$, for the component moments of momentum for a rotating and translating body of mass $\overline{m} = \Sigma m$

$$\left.\begin{aligned} M_{vx} &= \overline{m}\bar{v}_z\bar{y} - \overline{m}\bar{v}_y\bar{z} + I_x\omega_x, \\ M_{vy} &= \overline{m}\bar{v}_x\bar{z} - \overline{m}\bar{v}_z\bar{x} + I_y\omega_y, \\ M_{vz} &= \overline{m}\bar{v}_y\bar{x} - \overline{m}\bar{v}_x\bar{y} + I_z\omega_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Conservation of Moment of Momentum—Rotation and Translation.—We have from equations (4) for the component moments of momentum for a body rotating about a translating axis through the centre of mass, since $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$,

$$M_{vx} = I_x\omega_x, \qquad M_{vy} = I_y\omega_y, \qquad M_{vz} = I_z\omega_z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and from equations (3) for the component moments of the effective forces

$$\left.\begin{aligned} M_{fx} &= (I_z - I_y)\omega_z\omega_y + I_x\alpha_x, \\ M_{fy} &= (I_x - I_z)\omega_x\omega_z + I_y\alpha_y, \\ M_{fz} &= (I_y - I_x)\omega_y\omega_x + I_z\alpha_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If in equations (b) the co-ordinate axes OX, OY, OZ (figure, page 153) are fixed, we have $\omega_z\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_x = 0$, and the axis of rotation is fixed. If also we have $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$, we have $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$, and ω_x, ω_y, ω_z in equations (a) are constant. But since, by D'Alembert's principle, the moment of the effective forces is equal to the moment of the impressed forces, we have the moment of the impressed forces zero.

Hence *if the moment of the impressed forces about any fixed axis through the centre of mass is always zero, the moment of momentum of the body about that axis does not change.*

Invariable Axis.—If the moment of the impressed forces relative to the centre of mass O is always zero, the resultant must either be zero or always pass through O. In such case the co-ordinate axes do not change in direction and we have $\omega_z\omega_y = 0$, $\omega_x\omega_z = 0$, $\omega_y\omega_x = 0$, and also $M_{fx} = 0$, $M_{fy} = 0$, $M_{fz} = 0$. Hence, from equations (b), $\alpha_x = 0$, $\alpha_y = 0$, $\alpha_z = 0$ and ω_x, ω_y, ω_z in equations (a) are constant.

We have then the moment of momentum

$$I\omega = \sqrt{I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

about an axis through the centre of mass whose direction cosines are

$$\cos\alpha = \frac{I_x\omega_x}{I\omega}, \qquad \cos\beta = \frac{I_y\omega_y}{I\omega}, \qquad \cos\gamma = \frac{I_z\omega_z}{I\omega}, \quad . \quad . \quad . \quad . \quad . \quad (d)$$

and this axis is then invariable in direction.

Hence *if the moment of the impressed forces relative to the centre of mass is always zero, or if there are no impressed forces, there will be a certain axis through the centre of mass for which the moment of momentum is constant. This axis is invariable in direction, and its direction cosines are given by equations* (d). *The moment of momentum about this axis is given by equations* (c).

Kinetic Energy—Rotation and Translation.—The kinetic energy of a particle of mass m and velocity v is $\frac{1}{2}mv^2$. If a particle has the component velocities v_x, v_y, v_z, we have $v^2 = v_x^2 + v_y^2 + v_z^2$, and

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2.$$

Equations (4), page 154, give the component velocities for any particle of a rotating and translating body. If we square these component velocities, multiply each term by $\frac{1}{2}m$, sum up for all the particles and add, we shall have the kinetic energy for a rotating and translating body for any co-ordinate axes we please. Let us take these axes as principal axes at the centre of mass. Then we have $\Sigma mx = 0$, $\Sigma my = 0$, $\Sigma mz = 0$, and also $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. We have also $\Sigma m(y^2 + z^2) = I_x$, $\Sigma m(z^2 + x^2) = I_y$, $\Sigma m(x^2 + y^2) = I_z$.

We have then for the kinetic energy of a rotating and translating body

$$\mathfrak{K} = \frac{1}{2}\mathrm{m}\bar{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we take mass in lbs., this equation gives kinetic energy in foot-poundals. For foot-pounds divide by g.

Spontaneous Axis of Rotation.—The axis through the centre of mass about which a body rotates at any instant is called the *spontaneous* axis of rotation. If $\omega_x, \omega_y, \omega_z$ are the component angular velocities for any co-ordinate axes through the centre of mass, the resultant angular velocity is

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2},$$

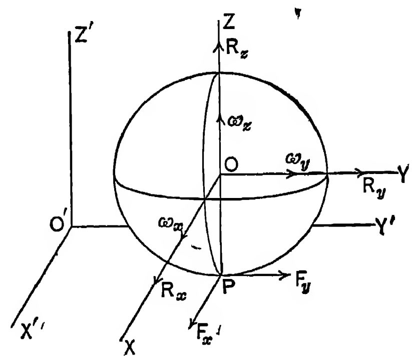
and the direction cosines of the axis are

$$\cos\alpha = \frac{\omega_x}{\omega}, \qquad \cos\beta = \frac{\omega_y}{\omega}, \qquad \cos\gamma = \frac{\omega_z}{\omega}.$$

Instantaneous Axis of Rotation.—The instantaneous axis is parallel at any instant to the spontaneous axis and passes through a point whose co-ordinates relative to the centre of mass are given by equations (15), page 156.

Examples.—(1) *Find the motion of a sphere rolling on a rough plane.*

ANS. Let the plane be the plane of $X'Y'$, and let the components of friction be F_x, F_y. All the *other* impressed forces can be reduced to a single resultant R at the centre of mass O and a couple which causes angular acceleration α about an axis through O.



Let R_x, R_y, R_z be the components of the resultant R, and α_x, α_y, α_z the components of the angular acceleration α.

Take the axes OX, OY, OZ through the centre of mass O parallel to $O'X'$, $O'Y'$, $O'Z'$.

We have from equations (1), page 390, the component effective forces, $\overline{m}\bar{f}_x$, $\overline{m}\bar{f}_y$, $\overline{m}\bar{f}_z$, and from equations (3), page 391, since $I_x = I_y = I_z$, for the moment of the effective forces about OX, OY, OZ, $M_{fx} = I_x\alpha_x$, $M_{fy} = I_y\alpha_y$, $M_{fz} = I_z\alpha_z$.

By D'Alembert's principle we have then

$$\left.\begin{array}{lll} \overline{m}\bar{f}_x = F_x + R_x, & \overline{m}\bar{f}_y = F_y + R_y, & \overline{m}\bar{f}_z = R_z, \\ I_x\alpha_x = F_y r, & I_y\alpha_y = -F_x r, & I_z\alpha_z = 0. \end{array}\right\} \quad (1)$$

But

$$\bar{f}_x = \bar{z}\alpha_y - \bar{y}\alpha_z, \qquad \bar{f}_y = \bar{x}\alpha_z - \bar{z}\alpha_x, \qquad \bar{f}_z = \bar{y}\alpha_x - \bar{x}\alpha_y;$$

and taking the origin at the point of contact P,

$$\bar{x} = 0, \qquad \bar{y} = 0, \qquad \bar{z} = +r.$$

We have then $\bar{f}_x = r\alpha_y$, $\bar{f}_y = -r\alpha_x$, $\bar{f}_z = 0$, or $\alpha_y = \frac{\bar{f}_x}{r}$, $\alpha_x = -\frac{\bar{f}_y}{r}$, $\alpha_z = 0$.

Substituting in equations (1), we obtain

$$\bar{f}_x = \frac{r^2}{\overline{m}r^2 + I_y} \cdot R_x, \qquad \bar{f}_y = \frac{r^2}{\overline{m}r^2 + I_x} \cdot R_y, \qquad \bar{f}_z = \frac{R_z}{\overline{m}}.$$

Or, since $I_x = I_y = \overline{m}\kappa^2$, where κ is the radius of gyration,

$$\overline{m}\bar{f}_x = \frac{r^2}{r^2 + \kappa^2} \cdot R_x, \qquad \overline{m}\bar{f}_y = \frac{r^2}{r^2 + \kappa^2} \cdot R_y, \qquad \overline{m}\bar{f}_z = R_z. \quad \ldots\ldots \quad (2)$$

Now, $\kappa^2 = \frac{2}{5}r^2$ and $r^2 + \kappa^2 = \frac{7}{5}r^2$. Hence these are the same equations as for a particle of equal mass acted upon by $\frac{r^2}{r^2+\kappa^2} = \frac{5}{7}$ of the acting forces.

Hence *the motion of the centre of mass of a homogeneous sphere rolling on a rough plane under the action of any forces is the same as for a particle of the same mass if all the impressed forces except friction are reduced to* $\frac{5}{7}$ *of their former value.*

Now $\sqrt{\bar{f}_x^2 + \bar{f}_y^2}$ is the resultant horizontal acceleration $\bar{f}_h$ in the plane $X'Y'$, and $\sqrt{R_x^2 + R_y^2}$ is the resultant horizontal impressed force H.

Hence, from (2),

$$\mathrm{\bar{m}}\sqrt{\bar{f}_x^2 + \bar{f}_y^2} = \frac{5}{7}\sqrt{R_x^2 + R_y^2}, \quad \text{or} \quad \mathrm{\bar{m}}\bar{f}_h = \frac{5}{7}H.$$

If R_z is the normal force and μ the coefficient of friction, then μR_z is the friction, and hence

$$\mathrm{\bar{m}}\bar{f}_h = H - \mu R_z.$$

Hence

$$\frac{5}{7}H = H - \mu R_z, \quad \text{or} \quad \mu = \frac{2H}{7R_z}.$$

If then the coefficient of friction is equal to or greater than $\frac{2H}{7R_z}$, the sphere *will roll without sliding.*

(2) *A sphere is placed on an inclined plane sufficiently rough to prevent sliding, and a velocity in any direction is communicated to it. Show that the path of the centre is a parabola. If v is the initial horizontal velocity of the centre, and α the inclination of the plane, show that the latus rectum will be* $\frac{14}{5}\frac{v^2}{g \sin \alpha}$.

ANS. The acceleration down the plane, from the principle of the preceding example, is $\frac{5}{7}g \sin \alpha$. If the initial velocity v makes an angle θ with the line of slope, the velocity down the plane is $v \cos \theta$, and at right angles $v \sin \theta$. There is no acceleration at right angles. The distance y passed over at right angles in the time t is then

$$y = vt \sin \theta,$$

and the distance x down the plane is

$$x = vt \cos \theta + \frac{5}{14}gt^2 \sin \alpha.$$

Eliminating t, we have for the equation of the curve

$$x = \frac{y}{\tan \theta} + \frac{5}{14}\frac{g \sin \alpha}{v^2 \sin^2 \theta}y^2.$$

This is the equation of a parabola. If the initial velocity is horizontal, $\theta = 90°$, $\sin \theta = 1$, $\tan \theta = \infty$, and we have

$$y^2 = \frac{14}{5}\frac{v^2}{g \sin \alpha}x.$$

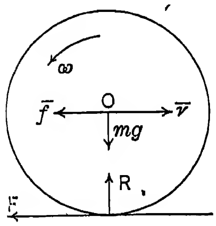
(3) *Show, as in example (1), that the motion of the centre of mass of a homogeneous disc rolling on a rough plane under the action of any forces is the same as for a particle of the same mass, if all the forces except friction are reduced to two thirds of their former value; also that the disc will roll without sliding if the coefficient of friction is equal to or greater than* $\frac{1}{3}\cdot\frac{H}{R_z}$.

(4) *Show in the same way that for a circular hoop the motion of the centre of mass is the same as for a particle of the same mass, if all the forces except friction are reduced to one half their former value; also that the hoop will roll without sliding if the coefficient of friction is equal to or greater than* $\frac{1}{2}\frac{H}{R_z}$.

(5) *Also show in the same way for a spherical shell that the forces are reduced to three fifths their former value, and that the shell will roll without sliding if the coefficient of friction is equal to or greater than* $\frac{2}{5}\frac{H}{R_z}$.

(6) *A hoop moving in a vertical plane in contact with a rough horizontal surface has at a given instant an angular velocity opposite in direction to that which would enable it to roll in the direction of its translation at that instant. Find the motion.*

ANS. We have the effective force $\overline{\text{m}}\bar{f}$ and the moment of the effective force $I\alpha$, where I is the moment of inertia for axis through centre of mass O at right angles to the plane of the hoop.



The impressed forces are the weight mg, the equal and opposite pressure R and the friction $F = \mu mg$ acting opposite to the direction of motion. We have then, from D'Alembert's principle,

$$\overline{\text{m}}\bar{f} = -\mu mg, \quad \text{or} \quad \bar{f} = -\mu g;$$

$$I\alpha = -\mu mgr, \quad \text{or} \quad \alpha = -\frac{\mu mgr}{I} = -\frac{\mu gr}{\kappa^2},$$

where κ is the radius of gyration.

Hence the linear and angular accelerations are constant.

Let $\bar{v}_1$ and ω_1 be the initial values of the linear and angular velocities and $\bar{v}$, ω the final values at any time t, and we have

$$\left.\begin{aligned} \bar{v} &= \bar{v}_1 - \mu gt, \\ \omega &= \omega_1 - \frac{\mu gr}{\kappa^2}t. \end{aligned}\right\} \quad \ldots \quad (1)$$

At the instant when slipping ceases we have $\bar{v} + r\omega = 0$.

Eliminating $\bar{v}$ and ω, we have for the time after which slipping ceases

$$t = \frac{\kappa^2(\bar{v}_1 + r\omega_1)}{\mu g(\kappa^2 + r^2)}, \quad \ldots \quad (2)$$

where μ is the coefficient of friction for slipping. For the linear and angular velocity at the instant when slipping ceases we have then

$$\bar{v} = \frac{r(r\bar{v}_1 - \kappa^2\omega_1)}{\kappa^2 + r^2}, \qquad \omega = \frac{\kappa^2\omega_1 - r\bar{v}_1}{\kappa^2 + r^2}. \quad \ldots \quad (3)$$

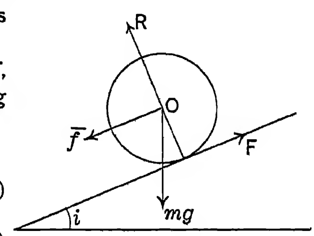
For any interval of time less than t as given by (2) equations (1) give $\bar{v}$ and ω, when μ is the coefficient for slipping. For any interval of time greater than this equations (1) also give $\bar{v}$ and ω, but μ is the coefficient for rolling friction.

In equations (3), if $r\bar{v}_1 - \kappa^2\omega_1$ is negative, $\bar{v}$ is negative. Hence if ω_1 is greater than $\frac{r\bar{v}_1}{\kappa^2}$, the hoop at the instant sliding ceases will have translation opposite in direction to the initial translation.

Equations (1), (2), (3) hold for a sphere or cylinder also. We have only to substitute the value of κ^2 in each case.

(7) *A disc whose plane is vertical rolls without sliding down an inclined plane. Find its motion.*

ANS. The effective force is $\overline{\text{m}}\bar{f}$ parallel to the plane and the moment of the effective force $I\alpha$, where I is the moment of inertia for axis through the centre of mass O at right angles to the plane of the disc.



The impressed forces are the weight mg, the normal pressure $R = mg\cos i$, where i is the angle of inclination, and the friction $F = \mu R = \mu mg\cos i$ acting opposite to the motion.

By D'Alembert's principle we have then

$$\overline{\text{m}}\bar{f} = \overline{\text{m}}g\sin i - \mu mg\cos i, \quad \text{or} \quad \bar{f} = g\sin i - \mu g\cos i, \quad \ldots \quad (1)$$

$$I\alpha = -Fr = -\mu mrg\cos i, \quad \text{or} \quad \alpha = -\frac{\mu mrg\cos i}{I} = -\frac{\mu rg\cos i}{\kappa^2}, \quad (2)$$

where κ is the radius of gyration.

Now the condition for rolling without sliding is $\bar{f} + r\alpha = 0$. From (2), then, we have

$$\bar{f} = \frac{\mu r^2 g\cos i}{\kappa^2}, \quad \text{or} \quad \mu = \frac{\bar{f}\kappa^2}{gr^2\cos i}.$$

Substituting this value of μ in (1), we have

$$\bar{f} = \frac{gr^2 \sin i}{\kappa^2 + r^2}, \quad \text{or} \quad \alpha = -\frac{gr \sin i}{\kappa^2 + r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (3) give the linear and angular accelerations. We see that they are both constant. We have then

$$\bar{v} = \bar{v}_1 + \frac{gr^2 \sin i}{\kappa^2 + r^2} . t, \qquad \omega = \omega_1 - \frac{gr \sin i}{\kappa^2 + r^2} . t,$$

$$s = \bar{v}_1 t + \frac{gr^2 \sin i}{2(\kappa^2 + r^2)} . t, \qquad \theta = \omega_1 t - \frac{gr \sin i}{2(\kappa^2 + r^2)} . t^2.$$

The equations hold for a sphere or cylinder also. We have only to substitute the value of κ^2 in each case.

(8) *Find the time a cylinder will take to roll from rest down a plane 20 ft. long, inclined 30°, the axis of the cylinder being horizontal.*

ANS. We have $\bar{v}_1 = 0$ and $\kappa^2 = \frac{r^2}{2}$. Hence, since $\sin i = \frac{1}{2}$ and taking $g = 32$,

$$t^2 = \frac{40(\kappa^2 + r^2.}{gr^2 \sin i} = 3.75, \quad \text{or} \quad t = 1.93 \text{ sec.}$$

STATICS OF RIGID BODIES.

CHAPTER I.

FRAMED STRUCTURES.

Stress.—We have seen (page 174) that the exertion of force upon a body or particle is only one side of the complete phenomenon which consists of the simultaneous action of equal and opposite forces between two bodies or particles. We call these internal forces STRESSES. Stress, then, is a force internal to the body or system considered, while the term force is reserved for external action.

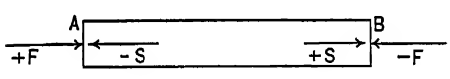
We speak, then, of the force *on* a body or particle, and the stress *in* a body or *between* two bodies or particles.

Tensile Stress and Force.—Let a body AB be acted upon by two equal and opposite forces $+F_1 - F$ in the same straight line. These forces act to stretch the body, and are therefore called *tensile* forces.

If there is equilibrium, we must have at any two sections A and B two equal and opposite *stresses*, $+S$, $-S$, which resist the extension. These are called TENSILE STRESSES. The tensile forces act away from each other, tending to pull A and B apart. The tensile stresses act towards each other, tending to draw A and B together. We have thus a force F and an equal and opposite stress in equilibrium at A, and a force F and an equal and opposite stress in equilibrium at B.

We see also that if the stresses are tensile, they act *away from* the ends A and B.

Compressive Stress and Force.—In the same way, if the forces are reversed in direction, we have COMPRESSIVE FORCE and COMPRESSIVE STRESS. The compressive forces act to make A and B move towards each other, and the compressive stresses to move them apart.



We see also that if the stresses are compressive, they *act towards* the ends A and B.

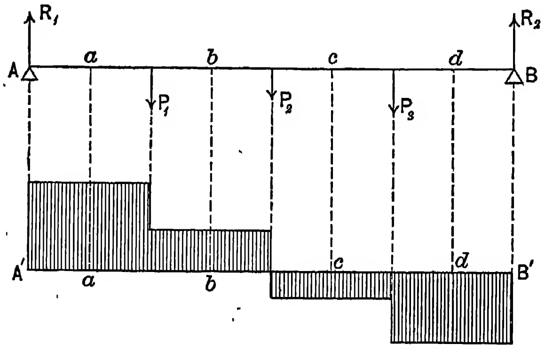
Shearing Force and Stress.—The algebraic sum of the components parallel to a section of all the external forces on the right of that section tends to make that section slide upon the consecutive section on the left. The algebraic sum of the components parallel to a section of all the external forces on the left of a section tends to make that section slide upon the consecutive section on the right.

For reasons to be given later (page 402) we always take the algebraic sum of the components parallel to a section of all the external forces *on the left* of that section, and call this algebraic sum the shearing force for that section.

We define, then, the *shearing force* for any section as the algebraic sum of the components parallel to that section *of all the external forces on the left.*

It is resisted by the shearing *stress* or resistance of the section to sliding on the consecutive section on the right. This shearing stress is equal and opposite to the shearing force. Since for equilibrium the algebraic sum on one side must be equal and opposite to that on the other, we can define the *shearing stress* for any section as the algebraic sum of

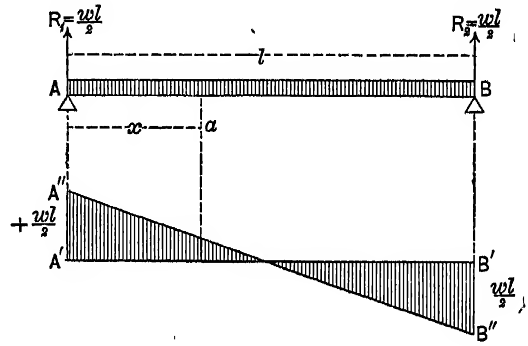
the components parallel to that section of all the external forces *on the right*. Thus in the case of a horizontal beam AB acted upon by vertical loads P_1, P_2, P_3 and the vertical upward reactions R_1 and R_2, the shearing force for any section at a between the left end and P_1 is by definition $+R_1$. For any section at b between P_1 and P_2 the shearing force is $+R_1-P_1$. For any section at c between P_2 and P_3 the shearing force is $+R_1-P_1-P_2$. For any section at d between P_3 and the right end the shearing force is $+R_1-P_1-P_2-P_3$. The ordinates of the shaded area thus give the shearing force to scale. The shearing stress is equal and opposite.



If we have a load of w per unit of length, uniformly distributed, we have for the shearing force for any section distant x from the left end

$$\textit{shearing force} = \frac{wl}{2} - wx,$$

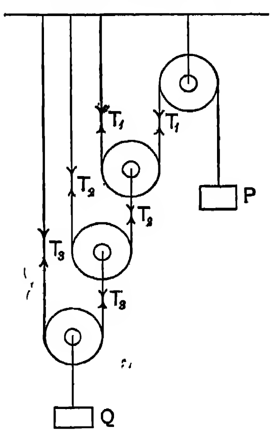
which is the equation to the straight line $A''B''$ passing through the centre of the span. The ordinate to this line at any section a gives the shearing force to scale. The shearing stress is equal and opposite. The shearing force at the centre is zero.



Equilibrium of a System of Bodies.—If a number of rigid bodies are connected by strings, rods, joints, etc., and the system is in equilibrium, each body of the system must be in equilibrium, and the conditions of equilibrium apply to each body as well as to the whole system.

EXAMPLES.—(1) *In the system of pulleys shown find the relation between P and Q for equilibrium, disregarding friction and rigidity of ropes.*

ANS. The tensile stress in a continuous string with two equal and opposite forces at the ends must be the same at every point of the string.



Let T_1, T_2, T_3, etc., be the tensile stresses as shown, and m the mass of each pulley.

Then we have for equilibrium in gravitation measure

$$T_1 = P,$$

$$2T_1 - T_2 - m = 0, \qquad T_2 = 2P - m,$$

$$2T_2 - T_3 - m = 0, \qquad T_3 = 2^2P - (2^2 - 1)m,$$

$$2T_3 - T_4 - m = 0, \qquad T_4 = 2^3P - (2^3 - 1)m,$$

or in general, if n is the number of *movable* pulleys,

$$T_n = 2^{n-1}P - (2^{n-1} - 1)m.$$

But for the last pulley we have

$$2T_n - Q - m = 0.$$

Hence we obtain in general

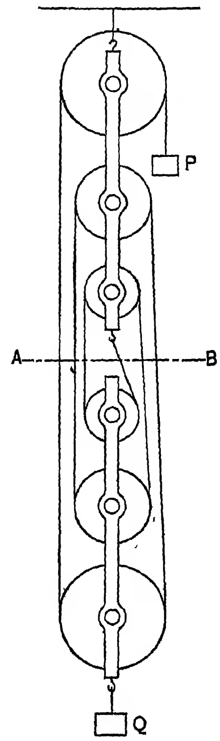
$$Q = 2^nP - (2^n - 1)m, \quad \text{or} \quad P = \frac{Q + (2^n - 1)m}{2^n}.$$

[Student should compare solution by virtual work, example (6), page 218.]

(2) *In the system of pulleys shown find the relation between P and Q for equilibrium, disregarding friction and rigidity of ropes.*

[Student should compare solution by virtual work, example (7), page 218.]

ANS. The tensile stress in the rope is $T = P$ at every point. If, then, n is the number of times the rope would be cut by a plane AB between the two blocks, and m is the mass of a block, we have

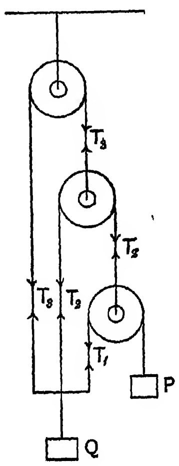


$$nP = Q + m, \quad \text{or} \quad P = \frac{Q + m}{n}.$$

In the figure $n = 6$.

(3) *In the system of pulleys shown find the relation between P and Q for equilibrium, disregarding friction and rigidity of ropes.*

ANS. We have, if m is the mass of each pulley,



$$T_1 = P,$$

$$T_2 - 2T_1 - m = 0, \qquad T_2 = 2P + m,$$

$$T_3 - 2T_2 - m = 0, \qquad T_3 = 4P + 3m,$$

$$\text{etc.} \qquad T_4 = 8P + 7m,$$

$$\cdots\cdots$$

$$T_n = 2^{n-1}P + (2^{n-1} - 1)m,$$

where n is the number of pulleys. Also we have

$$T_1 + T_2 + T_3 + \ldots T_n = Q.$$

Substituting the values of T_1, T_2, etc.,

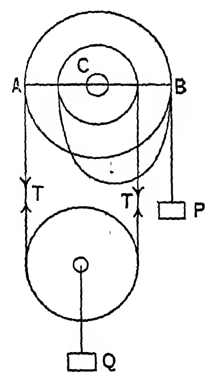
$$P[1 + 2 + 4 + \ldots 2^{n-1}] + m[1 + 3 + 7 + \ldots (2^{n-1} - 1)] = Q.$$

Hence

$$P(2^n - 1) + m(2^n - 1) - nm = Q, \quad \text{or} \quad P = \frac{Q + nm - m(2^n - 1)}{(2^n - 1)}.$$

[Student should solve by virtual work.]

(4) *In the differential pulley shown in the figure, an endless chain passes over a fixed pulley A, then under a movable pulley to which the mass Q is hung, and then over another fixed pulley B a little smaller than A. The two pulleys, A and B, are in one piece and obliged to turn together. The two ends of the chain are joined. The force P is applied as shown. To prevent the chain from slipping, there are cavities in the circumferences of the pulleys into which the links of the chain fit. Find the relation between P and Q for equilibrium.*



ANS. We have $2T - Q = 0$, or $T = \frac{Q}{2}$. Let a be the radius of A, and b the radius of B. Then, taking moments about C,

$$Ta - Tb - Pa = 0,$$

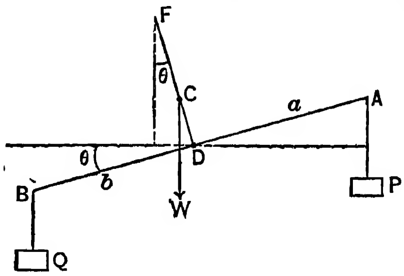
or, inserting $T = \frac{Q}{2}$,

$$P = \frac{Q(a - b)}{2a}.$$

By taking a and b nearly equal we can have P very small.

(5) *The requisites of a good balance are as follows: 1st. It should be "true," that is, when loaded with equal masses the beam should be horizontal. 2d. It should be "sensitive," that is, when the masses differ by a small amount the angle of the beam with the horizontal should be large. 3d. It should be "stable," that is, when moved from equilibrium it should return quickly. Show the conditions necessary for these requisites.*

ANS. Let the masses be P and Q, and W the weight in pounds of the balance acting at its centre of mass C. Let the fulcrum be at F, above C, and draw FD perpendicular to the beam AB at D.



Let $AD = a$, $BD = b$, $FC = d$, $FD = h$, and θ the angle of the beam with the horizontal.

Then for equilibrium, taking moments about F, we have

$$Q(b \cos \theta - h \sin \theta) - P(a \cos \theta + h \sin \theta) - Wd \sin \theta = 0.$$

Hence

$$\tan \theta = \frac{Qb - Pa}{(Q + P)h + Wd}. \qquad (1)$$

1. If the balance is "true," we must have $\theta = 0$ when $P = Q$. We see, from (1), that to satisfy this requisite *the arms must be equal.* We have then for a true balance $a = b$ and

$$\tan \theta = \frac{(Q - P)a}{(Q + P)h + Wd}. \qquad (2)$$

2. If the balance is to be "sensitive," θ must be large when $Q - P$ is small. We see from (2) that to satisfy this requisite h and d must be small relative to the length of arm a. The sensitiveness is increased, therefore, by either increasing the length of arm or by bringing the centre of mass and point of suspension nearer the beam.

3. If the balance is to be "stable," or to return quickly when disturbed, the moment $Wd \sin \theta$ of W about the fulcrum must be large. Hence stability is increased by moving the point of suspension away from the beam.

The conditions, then, for stability and sensitiveness are at variance. Stability is gained at the expense of sensitiveness, and *vice versa.*

In scientific measurements, where great accuracy is required, stability is sacrificed to obtain great sensitiveness. The balance recovers slowly from a disturbance, and time is required for it to come to rest. For ordinary commercial purposes, where it is desirable to save time, sensitiveness is sacrificed to stability.

Framed Structures.—A framed structure is a collection of straight MEMBERS joined together at the ends so as to make a rigid frame.

The simplest rigid frame is obviously a triangle, because that is the only figure whose shape cannot be altered without changing the lengths of its sides. All rigid frames must consist, therefore, of a combination of triangles.

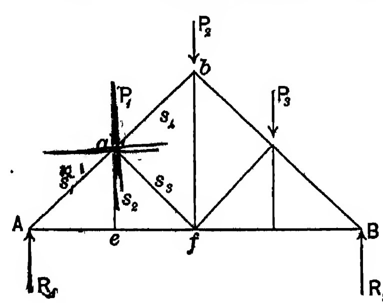
Any point where two or more members meet is called an APEX of the frame.

Determination of the Stresses in a Framed Structure.—We have external forces acting upon the frame, and tensile or compressive stresses in the members. If the frame is in equilibrium, the stresses and external forces form a system in equilibrium.

Also, since equilibrium must exist at every apex of the frame, all the forces and stresses at any apex form a system of concurring forces in equilibrium. Tensile stress acts away from an apex, compressive stress towards an apex (page 397).

We have then two methods of solution:

1. METHOD BY RESOLUTION OF FORCES.—In the figure we have a frame acted upon by known external forces P_1, P_2, P_3, and the upward pressures of the supports R_1 and R_2 at A and B.



Take any apex, as a. At this apex we have P_1 and the stresses in aA, ae, af and ab, forming a system of concurring forces in equilibrium.

If we denote these stresses by S_1, S_2, S_3, S_4, the angles of the members with the horizontal by α_1, α_2, α_3, α_4, and with the vertical by β_1, β_2, β_3, β_4, we have then for equilibrium, if α_5 is the angle of P_1 with the horizontal, and β_5 with the vertical,

$$S_1 \cos \alpha_1 + S_2 \cos \alpha_2 + S_3 \cos \alpha_3 + S_4 \cos \alpha_4 + P_1 \cos \alpha_5 = 0,$$
$$S_1 \cos \beta_1 + S_2 \cos \beta_2 + S_3 \cos \beta_3 + S_4 \cos \beta_4 + P_1 \cos \beta_5 = 0,$$

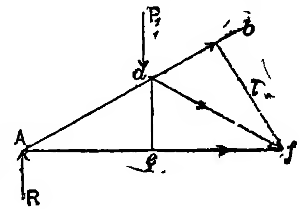
or in general, for any apex,

$$\left.\begin{array}{l}\Sigma S \cos \alpha + \Sigma P \cos \alpha = 0, \\ \Sigma S \cos \beta + \Sigma P \cos \beta = 0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since we have thus two equations of condition, this method can be applied at any apex *where there are not more than two unknown stresses*.

Components to the right and upward are positive, to the left and downward negative. If then a stress comes out with a minus sign, it denotes that the stress acts towards the apex and is compressive. If it comes out with a plus sign, it denotes that the stress acts away from the apex and is tensile (page 397).

2. METHOD BY MOMENTS.—Suppose the frame divided into two parts. Then the stresses belonging to the cut members must evidently hold in equilibrium the external forces on either side of the section.



Thus, in the figure, if we suppose *ab*, *af*, *ef* cut, the stresses of these members must hold in equilibrium R and P_1. The algebraic sum of the moments relative to any point must then be zero.

If then we wish to find the stress in *ab*, we can take the centre of moments at *f*. The moments of the stresses in *af* and *ef* will then be zero, and if l is the lever-arm for *ab*, we have

$$- ab \,.\, l + \Sigma \text{ moments of external forces on left of section} = 0.$$

Observe that the moment for the stress in *ab* is to be taken with the sign indicated by the arrow.

So if we wish to find *ef*, we should take moments about *a*, thus eliminating the moments of *ab* and *af*.

Or if we wish to find *af*, we should take moments about *A*, thus eliminating the moments of *ab* and *ef*.

The method, then, is in general as follows: Divide the frame and consider only one portion, as, for instance, the *left-hand* portion.

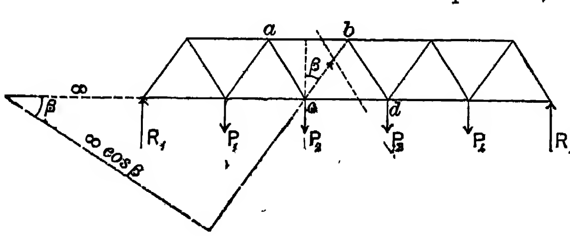
Place arrows on each cut member pointing *towards the section*.

To find the stress in any one of the cut members take the intersection of the other members cut as a point of moments. This will eliminate these members.

Place the algebraic sum of the moments of all forces and stresses for this point of moments equal to zero.

This method is general and can always be applied to find the stress in any member when all the cut members whose stresses are unknown, except the one whose stress is desired, meet in a point.

If two of the cut members are parallel, their intersection is at an infinite distance, but the method still applies.



Thus if we wish to find the stress in *cb*, we take a section cutting *ab*, *bc* and *cd*. The intersection of *ab* and *cd* is at an infinite distance. We therefore have the lever-arm for *cb*, $\infty \cos \beta$, where β is the angle of *cb* with the vertical. Hence

$$R_1\infty - P_1\infty - P_2\infty + cb \times \infty \cos \beta = 0, \quad \text{or} \quad cb \cos \beta + (R_1 - P_1 - P_2) = 0,$$

or

$$cb = -(R_1 - P_1 - P_2) \sec \beta.$$

The algebraic sum of the external forces $(R_1 - P_1 - P_2)$ is the SHEARING FORCE (page 397). For horizontal chords and vertical forces we have then the *vertical component of the stress in a brace in equilibrium with the shearing force.*

It is evident that if the shearing force is upwards or positive, the stress in cb will be compression and negative. If then we take the compression as negative and tension as positive, the sign will denote the character of the stress. It is for this reason that we have taken shearing force as the algebraic sum of all the forces *on the left* (page 397).

Superfluous Members.—We see from equations (1), page 401, that we have two equations of condition for equilibrium at any apex. If then at any apex there are more than two members whose stresses are necessarily unknown, the frame has superfluous members.

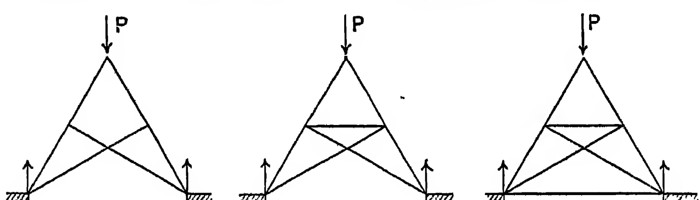
Criterion for Superfluous Members.—The simplest rigid frame is a triangle, and all rigid frames must consist of a combination of triangles.

Any one member fixes the position of two apices. Every other apex after the first two requires two members to fix its position. If then n is the number of apices, $2(n - 2)$ will be the number of members, lacking one. Let m be the number of members. Then if there are no superfluous members, we must have

$$m - 1 = 2(n - 2), \quad \text{or} \quad m = 2n - 3.$$

If m is less than $2n - 3$, there are not members enough. If m is greater than $2n - 3$, there are superfluous members.

EXAMPLES.—(1) *In the cases of the three frames represented, show that in the first case there are not enough members and the frame is not rigid; in the second case the frame is rigid and there are no superfluous members; in the third case there is one superfluous member.*



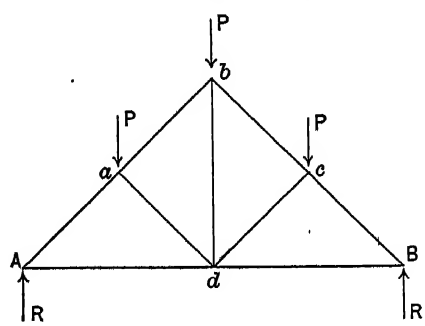
ANS. From our criterion we should have

$$m = 2n - 3.$$

But we have in the first case $n = 6$. Hence m should be 9. But m is only 8, or one less than the necessary number.

In the second case $n = 6$ and $m = 9$, as should be. In the third case $n = 6$ and $m = 10$, or one more than necessary.

(2) *A roof-truss has a span of 48 feet and a centre height of 18 feet. Each rafter is divided into two equal parts, and the lower tie into two equal parts, and the bracing is as shown in the figure. Find the stresses in the members for a load of 800 pounds at each upper apex.*



ANS. The reaction at each end is $R = 1200$ pounds. We find it as follows:

Take B as a point of moments. Then we have

$$R \times 48 - 800 \times 36 - 800 \times 24 - 800 \times 12 = 0, \quad \text{or}$$
$$R = +1200.$$

1st METHOD. For the different members we have for the cosines of the angles with the horizontal ($\cos \alpha$) and vertical ($\cos \beta$)

	Aa	ab	Ad	ad	bd
$\cos \alpha =$	0.8	0.8	1	0.8	0
$\cos \beta =$	0.6	0.6	0	0.6	1

At apex A we have then

$$+1200 + Aa \times 0.6 = 0, \quad \therefore \quad Aa = -2000 \text{ pounds.}$$
$$aA \times 0.8 + Ad = 0 \quad \therefore \quad Ad = +1600$$

The minus sign denotes stress towards the apex, or compression; the plus sign stress away from the apex, or tension.

At apex a we have

$$-Aa \times 0.6 - 800 + ab \times 0.6 - ad \times 0.6 = 0,$$
$$-Aa \times 0.8 + ab \times 0.8 + ad \times 0.8 = 0.$$

Inserting the value of Aa already found and solving for ab and ad, we have

$$ab = -1333\tfrac{1}{3}, \qquad ad = -666\tfrac{2}{3}.$$

At apex b, since bc is evidently the same as ab, we can put

$$-2ab \times 0.6 - 800 - bd = 0, \quad \text{or} \quad bd = +800.$$

2d METHOD. The centre of moments for Aa and ab should be taken at d, for ad and bd at A, for Ad at a.

We have then for the lever-arms

	Aa	ab	Ad	ad	bd
Lever-arm	1.44	14.4	9	14.4	24

Hence we have

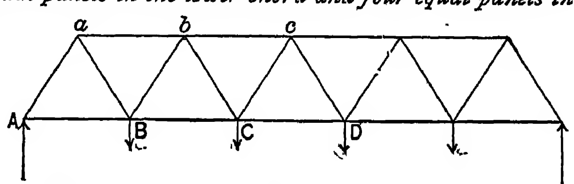
$$-Aa \times 14.4 - 1200 \times 24 = 0, \qquad Aa = -2000 \text{ pounds};$$
$$Ad \times 9 - 1200 \times 12 = 0, \qquad Ad = +1600 \text{ ''}$$
$$-ab \times 14.4 + 800 \times 12 - 1200 \times 24 = 0, \qquad ab = -1333\tfrac{1}{3} \text{ ''}$$
$$-ad \times 14.4 - 800 \times 12 = 0, \qquad ad = -666\tfrac{2}{3} \text{ ''}$$

For bd we have

$$bd \times 24 - 800 \times 12 + cd \times 14.4 = 0,$$

or, since $cd = ad = -666\tfrac{2}{3}$, $\qquad bd = +800$ pounds.

(3) *A bridge truss 100 ft. long is divided into five equal panels in the lower chord and four equal panels in the upper chord. The depth is constant and equal to 10 ft. The bracing is isosceles. Find the stresses for a load of 800 pounds at each lower apex.*



ANS. $AB = +1600$, $\quad BC = +4000$,

$CD = +4800$, $\quad ab = -3200$,

$bc = -4800$, $\quad Aa = -2262.4$, $\quad aB = +2262.4$, $\quad Bb = -1131.2$, $\quad bC = +1131.2$, $\quad Cc = 0$.

CHAPTER II.

GRAPHICAL STATICS. CONCURRING FORCES.

Graphical Statics.—While the solution of statical problems by computation and analytical methods is sometimes tedious and involved, they may often be solved with comparative ease and sufficient accuracy by graphic construction.

The solution of statical problems by graphic methods gives rise to GRAPHICAL STATICS. We shall consider only co-planar forces.

Concurring Co-planar Forces.—Let any number of co-planar forces F_1, F_2, F_3, F_4, etc., given in magnitude and direction, act at a point A, Fig. 1.

FIG. 1.

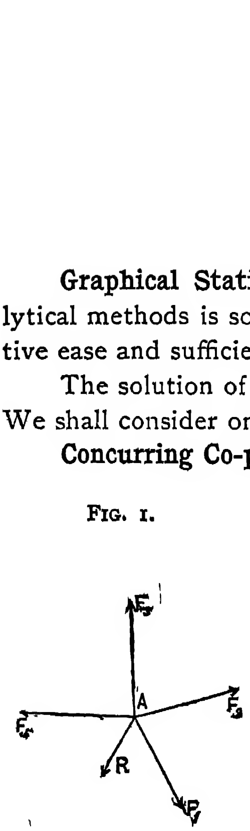


FIG. 2.

In Fig. 2, from any point 0, lay off to scale the line representative of F_1 from 0 to 1, then the line representative of F_2 from 1 to 2, then the line representative of F_3 from 2 to 3, then the line representative of F_4 from 3 to 4, and so on. The polygon 0 1 2 3 4 thus obtained we call the FORCE POLYGON.

If all these forces are in equilibrium, the algebraic sum of their horizontal and vertical components must be zero. But when this is the case, evidently 4 and 0, in Fig. 2, must coincide, or the *force polygon must close.* We have then the following principle:

If any number of concurring forces are in equilibrium, the force polygon is closed. If the force polygon is not closed, the line 0 4 necessary to make it close gives the magnitude and direction of the resultant R. If we consider this resultant acting at the point of application A in the direction from 4 to 0, obtained by following round the polygon in the direction of the forces, it will hold the forces at A in equilibrium. If taken as acting in the opposite direction at A, it will replace the forces.

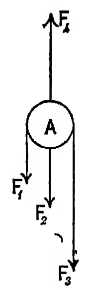
COR. 1. The order in which the forces are laid off in the force polygon is immaterial. Thus, in Fig. 2, if we had laid off 0 1, then the line representative of F_3 from 1 to 3′, and then the line representative of F_2, we should arrive at 3 just as before. By a similar change of two and two we can have any order we please.

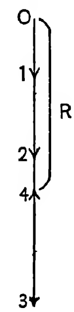
COR. 2. Any line in the force polygon, as 0 2, 0 3, or 1 3, is the resultant of the forces on either side. Thus 0 2 is the resultant of F_1 and F_2, and, acting in the direction from 2 to 0, holds F_1 and F_2 in equilibrium and replaces F_3, F_4 and R.

COR. 3. If the forces are all parallel, the force polygon becomes a straight line. Thus in Fig. 1, if the parallel forces F_1, F_2, F_3, F_4, etc., act at the point A, we have the force polygon Fig. 2, 0 1 2 3 4, and the closing line 4 0 is, as before, the resultant R and equal to the algebraic sum of the forces.

If taken as acting from 4 to 0, it will hold the forces at A in equilibrium. In the opposite direction it will replace the forces.

FIG. 1. FIG. 2.

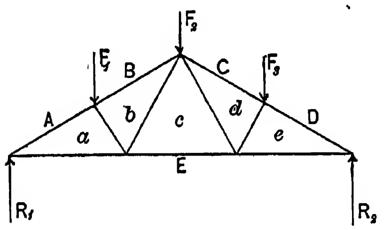




Notation for Framed Structures.—Let the figure represent a roof-truss composed of two rafters, a horizontal tie-rod and intermediate braces consisting of struts and ties.

The notation which we adopt in order to designate any member of a framed structure, or any force acting upon the structure, is as follows:

We place a letter in each of the triangular spaces into which the frame is divided by the members, and also a letter between any two forces. Any member or force is then denoted by the letters *on both sides of it.* Thus in the figure



AB denotes the force F_1, BC denotes the force F_2, CD denotes the force F_3, DE denotes the upward pressure of the right-hand support R_2, EA denotes the upward pressure of the left-hand support R_1. Also, Aa, Bb, Cd, De denote the portions of the rafters which have these letters on each side. The portions into which the lower tie is divided are in the same way Ea, Ec, Ee. The braces are ab, bc, cd, de.

The student should carefully adhere to this notation for the frame *whenever using the graphic method.*

Character of the Stresses.—The determination of the *kind* of stress in a member of a frame, whether tension or compression, is as important as the determination of the magnitude of the stress.

In the preceding figure, suppose we know the upward pressure at the left support R_1 or EA, and we wish to find the stresses in the members Ea and Aa, Fig. 1, which meet at the lower left-hand apex. If these stresses and R_1 are in equilibrium, they will make a closed polygon. If then we lay off EA in Fig. 2, upwards, equal to R_1, and then from A and E draw lines parallel to Aa and Ea in Fig. 1, and produce them till they intersect at a, Fig. 2, evidently the lines Aa and Ea in Fig. 2, taken to the same scale as EA, will give the magnitude of the stresses in Ea and Aa in Fig. 1.

FIG. 1.

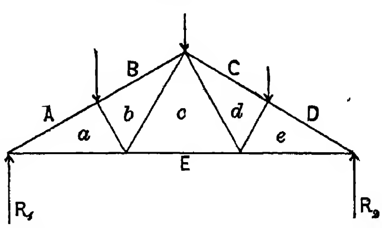
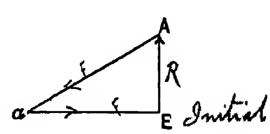


FIG. 2.



Thus *lines in the force polygon which have letters at each end give the stresses in those members of the frame denoted by the same letters at the sides.*

Now as to the character of these stresses, the directions Aa and aE in Fig. 2, obtained by following round in the known direction of R_1, are the directions for equilibrium (page 404).

Since we are considering the concurring forces acting at the left-hand apex, transfer these directions to Fig. 1, and we see that Aa acts towards the apex we are considering and thus resists compression, and aE acts away from it and therefore resists tension. The stress in Aa is therefore compressive (—), and in aE tensile (+).

In general, then, if we take any apex of the frame in Fig. 1, and consider the concurring forces acting at that apex as a system of concurring forces in equilibrium, we have the following rule:

Follow round the force polygon in Fig. 2 in the direction indicated by any one of these forces already known, and transfer the directions thus obtained for the stresses to the apex in Fig. 1 under consideration. If the stress in any member is thus found acting away from the apex, it is tension (+); *if towards the apex, it is compression* (−).

Application of Preceding Principles to a Frame.—Let Fig. 1 be a frame consisting of two rafters, a horizontal tie-rod and bracing as shown, carefully drawn to a scale of a certain number of feet to an inch. This we call the FRAME DIAGRAM.

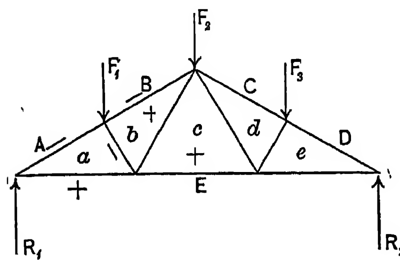


FIG. 1.

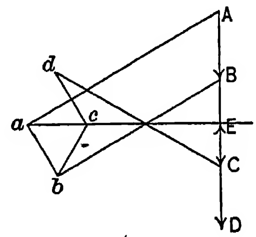


FIG. 2.

Let the forces F_1, F_2, F_3 act at the upper apices, and let the reactions or upward pressures of the supports be R_1 and R_2. Notate the frame and these forces as directed, so that $F_1 = AB$, $F_2 = BC$, $F_3 = CD$, $R_2 = DE$, $R_1 = EA$, while the members are Aa, Bb, Cd, De, Ee, Ec, Ea, ab, bc, cd, de.

The outer forces acting upon the frame cause stresses in the members. These outer forces must first be all known, or if any are unknown, they must first be found.

Lay off these outer forces AB, BC, CD, DE, EA in Fig. 2 to a scale of a certain number of pounds to an inch. Each force in Fig. 2, having letters *at its ends, is equal and parallel to those forces in Fig. 1 which have the same letters at the sides.*

The polygon formed by AB, BC, CD, DE, EA (in this case a straight line, Cor. 3, page 405) we have called the FORCE POLYGON.

If the frame is in equilibrium, this polygon *must always close*, that is, the outer forces acting upon the frame must be in equilibrium. If it does not close, these outer forces are not in equilibrium and the frame will move. That is, the frame itself, so far as its motion as a whole is considered, may be treated as a point.

Having thus drawn and notated the frame Fig. 1 and constructed the force polygon Fig. 2, we can find the stresses in the members. The forces and stresses *at each apex* must be in equilibrium, and therefore form a closed polygon.

Thus consider first the left-hand apex, Fig. 1. At this point we have the reaction EA and the stresses in Aa and Ea, constituting a system of concurring forces in equilibrium. But we already have EA laid off in Fig. 2. If then we draw Aa and Ea in Fig. 2 parallel to Aa and Ea in Fig. 1, and produce to intersection a, the polygon is closed and we have in Fig. 2 the stresses in Aa and Ea, to the same scale employed in laying off EA. Since EA acts upwards, if we follow round from E to A, and A to a, and a to E, in Fig. 2, and transfer the directions thus obtained for Aa and aE to the left-hand apex in Fig. 1, we have the stress in Aa towards this apex or compression (−), and the stress in aE away from the apex and therefore tension (+).

[The student should follow with his own sketch and mark each stress with its proper sign as he finds it.]

Let us now pass to the next upper apex, at F_1, Fig. 1. Here we have F_1 or AB and the stresses in Aa, ab and Bb in equilibrium. But we already have the stress in Aa and AB laid off in Fig. 2.

If then we draw from a and B in Fig. 2 lines parallel to ab and Bb in Fig. 1, and pro-

duce to intersection b, the polygon is closed and we have in Fig. 2 the stresses in ab and Bb. Since AB is known to act downward, we follow round in Fig. 2, from A to B, B to d, d to a, and a to A, and transfer the directions thus obtained to the apex at F_1, Fig. 1, under consideration. We thus obtain the stress in Bb towards the apex or compression, the stress in ba towards the apex or compression, and the stress in aA towards the apex or compression, *just as already found.*

Note that in the first case, when we were considering the apex at R_1, we found the stress in aA acting towards that apex. Now when we consider the apex at F_1 we find the stress in aA acting towards that apex—in both cases, then, compression (page 397).

Let us now consider the second lower apex, Fig. 1. We have here no outer force, but the stresses in Ea, ab, bc and cE must be in equilibrium and therefore form a closed polygon. But in Fig. 2 we have already found the stresses in Ea and ab. If then we draw from b a line parallel to bc in Fig. 1, and produce it to intersection c with Ea, the polygon closes, and we have in Fig. 2 the stresses in bc and cE. We have already found aE to be tension. It must therefore act away from the apex we are considering. We therefore follow round in Fig. 2, from E to a, a to b, b to c, and c to E, and transfer the directions thus found to the corresponding members in Fig. 1. We thus obtain the stress in Ea tension and the stress in ab compression as already found, and the stress in bc tension and in cE tension.

Let us now consider the top apex. We have here the force $F_2 = BC$, and the stresses in Bb, bc, cd, and dC, in equilibrium. But in Fig. 2 we have already laid off BC, and we have found the stresses in Bb and bc. If then we draw from c and C lines parallel to cd and Cd in Fig. 1, and produce to intersection d, the polygon closes and we have in Fig. 2 the stresses in cd and Cd. Since BC acts downwards, we follow round from B to C, C to d, d to c, c to b, and b to B. Transferring these directions to the corresponding members in Fig. 1, we obtain the stress in Cd compression and in dc tension, while the stress in cb is tension and in bB compression as already found.

We can thus go to each apex and find the stresses in every member.

The lines in Fig. 2 which thus give the stresses in the members constitute the STRESS DIAGRAM. *Each stress having letters at its ends in Fig. 2 is parallel to that member in Fig. 1 which has the same letters at its sides.*

Apparent Indetermination of Stresses.—It sometimes happens that a frame has no superfluous members, and yet in applying the graphic method we are unable to find any apex at which all the forces but two are known. In such case the difficulty may be overcome by taking out one or more of the members and replacing them by another member, and then applying the method until we find the stress in some member which *is not affected by the change.* Or we may find the stress in this member by the method of sections (page 401). Having found this stress, we can replace the members taken out and find the actual stresses.

Thus let Fig. 1 (page 408) be a frame* acted upon by the forces F_1, F_2, F_3, F_4, etc., and the reactions or upward pressures of the supports R_1, R_2.

Notate the frame and the forces by letters on each side as directed (page 405).

Then lay off to scale the outer forces in Fig. 2, thus forming the force polygon $ABCD \ldots HIA$. This polygon is a straight line in this case, because all the forces are parallel, and it must close, that is, the outer forces are in equilibrium.

We can now proceed to find the stresses as follows:

Consider first the left-hand apex, Fig. 1. At this point we have the reaction IA and the stresses in Aa and Ia constituting a system of concurring forces in equilibrium. But we already

* Disregard for the present the dotted member in Fig. 1.

have IA laid off in Fig. 2. If then we draw Aa and Ia in Fig. 2 parallel to Ia and Aa in Fig. 1, and produce to intersection a, the polygon is closed and we have in Fig. 2 the stresses in Aa and Ia to the same scale employed in laying off the forces. Since IA acts upwards, we

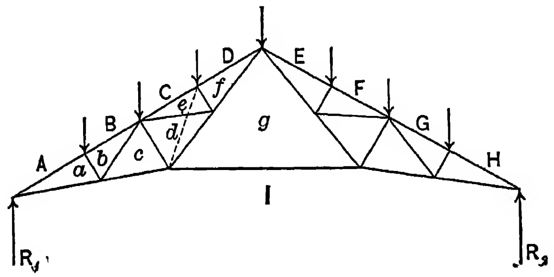


FIG. 1.

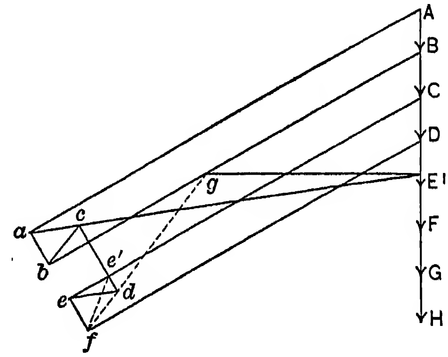


FIG. 2.

follow round from I to A, A to a, and a to I, in Fig. 2, and transfer the directions thus obtained for Aa and aI to the corresponding members in Fig. 1.

We have then the stress in Aa towards the apex we are considering or compression ($-$), and the stress in aI away from that apex or tension ($+$).

Considering now the next upper apex, we have here the force AB known, the stress in Aa already found, and the stresses in ab and Bb unknown. If then in Fig. 2 we draw ab and Bb, thus closing the polygon, we obtain the stresses in ab and Bb.

Since AB acts down, we follow round in Fig. 2 from A to B, B to b, b to a, and a back to A, and transfer the directions thus obtained to the corresponding members in Fig. 1. We have then the stress in Bb towards the apex we are considering or compression ($-$), the stress in ba towards that apex or compression ($-$), and the stress in aA also towards that apex or compression ($-$), just as we have already found it.

Note that when we were considering the apex at R_1, we found the stress in aA acting towards that apex. Now when we consider the apex at F_1 we find the stress in aA acting towards that apex. In both cases, then, compression (page 397).

We can now consider the next lower apex, where we have the stresses in Ia, ab, bc and cI in equilibrium. We already know Ia and ab, and if we draw in Fig. 2 bc and cI, we obtain the stresses in bc tension ($+$) and in cI tension ($+$).

Thus far there has been no difficulty in the application of the graphic method. But now we cannot consider the next upper or lower apex, because at each we have more than two unknown forces. If we should start at the right end, we should soon come to the same difficulty on the right side. Apparently we can go no farther.

The number of members is 27 (we disregard the dotted member in Fig. 1). The number of apices is 15. We have then, applying the criterion for superfluous members (page 402), $m = 2n - 3$. There are then no superfluous members.

If now we *remove* the two members de and ef and replace them by the dotted member $e'f$, where e' takes the place in the new notation of the two letters e and d, we have still a rigid frame with no superfluous members. For the number of members is now $m = 25$, and the number of apices is $n = 14$. We have then $m = 2n - 3$.

But this change has evidently *not affected the stress in the member Ig.* We can therefore now carry on the diagram until we find the stress in Ig, or we may compute the stress in Ig directly by the method of sections (page 401).

Thus if we now consider the apex at F_2, Fig. 1, we have at this point the stresses in the members Bb, bc, ce' and eC, and the force BC, all in equilibrium. We know BC, Bb and bc, and if we draw in Fig. 2 ce' and $e'C$, we obtain the stresses in $e'C$ compression and in ce compression.

We can then pass to the apex at F_3, Fig. 1, where we know all the forces except the stresses in Df and fe'. We draw then Df and fe' in Fig. 2, and obtain the stresses in Df compression and in fe' tension.

We can now pass to the next lower apex, where we have the stresses in Ic, ce' and $e'f$, and can therefore find fg and Ig. We draw then fg and Ig in Fig. 2, and obtain the stresses in fg and Ig tension.

We have thus found the stress in the member Ig, and since this is unchanged by the removal of the members de and ef, we can now replace those members and remove $e'f$.

We can now consider the second lower apex and find the stresses in cd and dg, and can then pass to the apex at F_3 and find the stresses in ef and Df, and so on. We can thus find the stress in every member of the frame, and there is no real indeterminateness.

Remarks upon the Method.—The method just illustrated we may call the "*graphic method by resolution of forces.*" The student will note that he must always know all but two of the forces concurring at any apex before he can consider that apex.

It is evident that if the frame is completely divided into two portions by cutting the members, the stresses which existed in the cut members before the section was made must hold in equilibrium the outer forces acting upon each portion of the frame (page 401).

This is at once made evident by Fig. 2, page 405.

Thus suppose a section cutting the members Bb, bc and cE, Fig. 1, and thus dividing the frame into two portions. We see from Fig. 2, page 406, that the stresses in the cut pieces make a closed polygon with EA and AB, the outer forces on the left-hand portion, or with BC, CD and DE, the outer forces on the right-hand portion.

If we solve the triangles in Fig. 2, page 406, we obtain algebraic expressions for the stresses identical with those obtained by the "algebraic method by resolution of forces" (page 400).

Thus, since the algebraic sum of the horizontal and vertical components of the forces acting at each apex must be zero, we have $+R_1 + Aa \cos a = 0$, or $Aa = -\dfrac{R_1}{\cos \alpha}$, where α is the angle of the rafter with the vertical. We get the same result at once from Fig. 2 by solving the triangle AaE. In the same way we have at once, from Fig. 2, $ab = -F_1 \cos \beta$, where β is the angle of ab with the vertical.

We see also from Fig. 2, page 406, other relations. Thus we see that the stress in ab will be the least possible when ab is perpendicular to the rafter. We also see at a glance how the stress in any member is affected by a change of inclination of the member.

Finally, the application of the method is equally simple no matter how irregular the frame may be.

If the frame is symmetrical with respect to the centre, and the forces F_1, F_3 in Fig. 2 (page 406) are equal, it is evident that the stresses in each half will be the same. We have then $Cd = Bb$, $cd = cb$, and so on.

Choice of Scales, etc.—In general the larger the frame is drawn in Fig. 1, the better, as it then gives more accurately the direction of the members composing it.

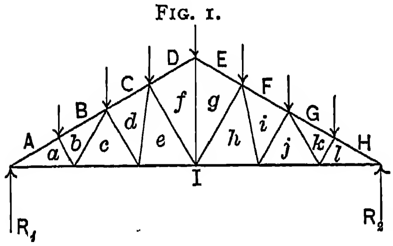
The force polygon Fig. 2, on the other hand, should be taken to no larger scale than consistent with scaling off the forces to the degree of accuracy required, so as to avoid the intersection of very long lines, where a slight deviation from true direction multiplies the

error. If an error of one twenty-fifth of an inch is considered the allowable limit, the scale should be so chosen that one twenty-fifth of an inch shall represent a small number of pounds, within the degree of accuracy required.

The stress polygon Fig. 2 should be completely finished and the signs for tension (+) and compression (−) placed on *the frame* for each member *as its stress is found*, to avoid confusion, before the stresses are taken off to scale. A good scale, dividers, straight-edge, triangle, and hard fine-pointed pencil are all the tools required. The work should be done with care, all lines drawn light, points of intersection accurately located and the frame properly notated to correspond with the force polygon. Care should be exercised to secure perfect parallelism in the lines of the frame and stress polygon. Some practice is necessary in order to obtain close results. It should be remembered that careful habits of manipulation, while they tend to give constantly increased skill and more accurate results, affect very slightly the rapidity and ease with which these results are obtained.

Examples.—(1) *A roof-truss has a span of 50 feet and rise of 12.5 feet. Each rafter is divided into four equal panels, and the lower horizontal tie into six equal panels. The bracing is as shown in the figure. A weight of 800 lbs. is sustained at each upper apex. Find the stresses.*

FIG. 1.



ANS. Draw the frame in Fig. 1 to a scale of, say, 12 feet to an inch, and notate it. Then construct the force polygon $ABC...HIA$, Fig. 2.

Note that R_2 or HI and R_1 or IA are equal and each 2800 lbs. The force polygon then closes as it should. We can take the scale of Fig. 2 as 3200 lbs. to an inch. Then an error of $\frac{1}{25}$ of an inch will be about 128 lbs.

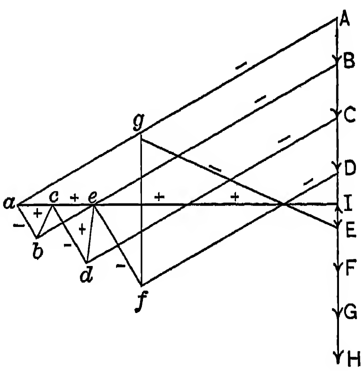


FIG. 2.

We can then find the stresses as shown in Fig. 2.

Aa	*Bb*	*Cd*	*Df*	*Ia*	*Ic*	*Ie*	*ab*
− 6280	− 5816	− 4700	− 3580	− 5624	+ 4832	+ 4024	− 720

bc	*cd*	*de*	*ef*	*fg*
+ 720	− 1060	+ 928	− 1452	+ 2400 lbs.

The accurate results as found by computation (page 402) are

− 6260 − 5813 − 4696 − 3577 + 5600 + 4802 + 4003 − 720

+ 720 − 1081 + 920 − 1443 + 2401 lbs.

It will be seen that the greatest error is only 30 lbs. The above results were actually obtained from the diagram, using the scales given.

(2) *Sketch the stress diagram for a roof-truss as shown in the following Fig. 1, equal forces acting at every upper and lower apex.*

ANS. The student should note that the reactions DE and GA are each equal to half the sum of the downward forces, or $2\frac{1}{2}$ forces.

We lay off then in Fig. 2 AB, BC, CD downwards. Then DE upwards equal to $2\frac{1}{2}$ forces. Then EF, FG downwards. Then GA upwards equal to $2\frac{1}{2}$ forces, and *closing the force polygon.*

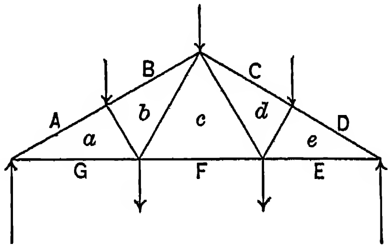


FIG. 1.

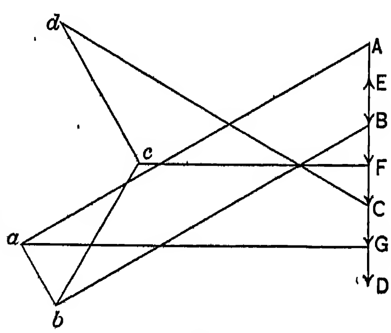


FIG. 2.

The stresses can now be found as always.

(3) *We give in the following figures a number of frames with their stress diagrams.* For the sake of generality the outer forces and reactions are often taken inclined as well as vertical.*

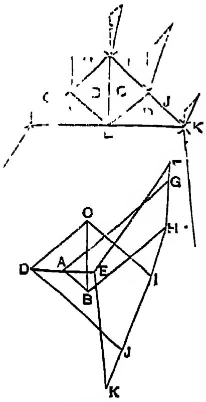


Fig. 1.

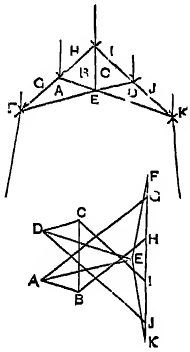


Fig. 2.

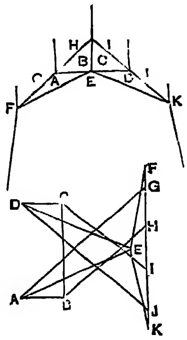


Fig. 3.

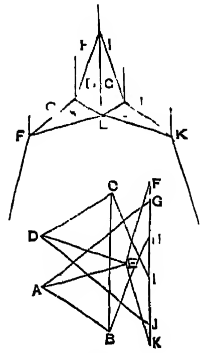


Fig. 4.

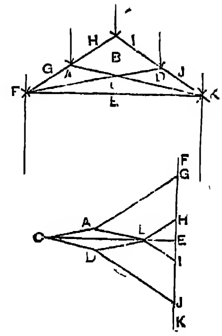


Fig. 5.

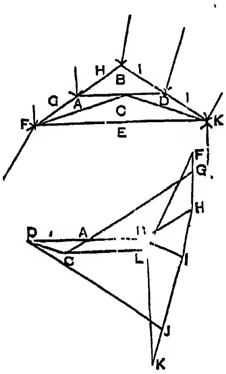


Fig. 6.

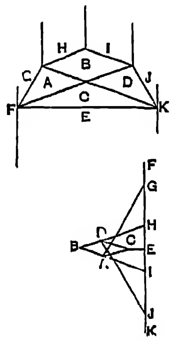


Fig. 7

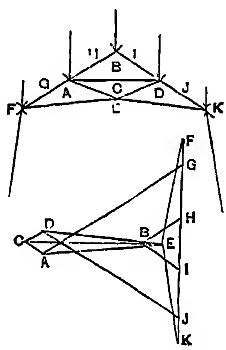


Fig. 8.

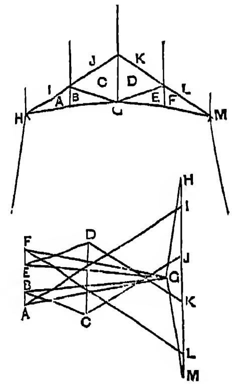


Fig. 9.

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* The student should sketch the stress diagrams for himself in each case, putting down, as he goes along, the sign (−) and (+) for compression and tension upon each member of the frame as soon as he finds it.

CHAPTER III.

GRAPHICAL STATICS. NON-CONCURRING FORCES.

Non-concurring Forces.—Let the co-planar forces F_1, F_2, F_3, F_4, etc., act at the points A_1, A_2, A_3, A_4 of any rigid body, Fig. 1.

If we lay off the forces to scale in Fig. 2, we have as before the force polygon 0 1 2 3 4, and the closing line 0 4 gives as before the resultant. If this resultant acts in the direction

FIG. 1.

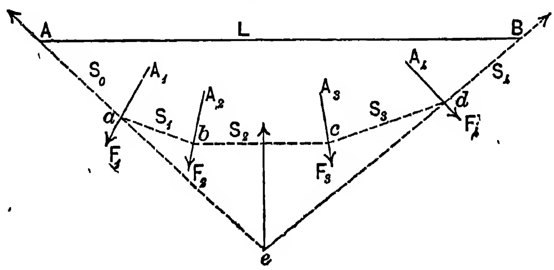
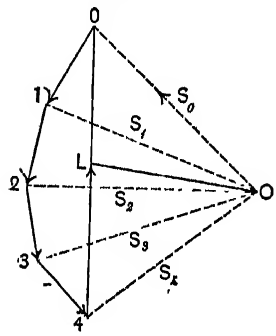


FIG. 2.



4 0 upon the rigid body, it will hold the given forces in equilibrium. If it acts in the direction 0 4, it will replace the given forces.

We thus know the magnitude and direction of the resultant. But its *position* in the plane of the forces in Fig. 1 is as yet unknown.

In order to determine this, choose any point O in Fig. 2, and draw the lines Oo and O4. This point O we call the POLE of the force polygon. Now since every line in the force polygon represents a force, by thus choosing a pole O and drawing lines Oo, O4 to the extremities of the resultant 0 4, we have *resolved the resultant into* the two forces represented by Oo and O4. This is evident from the fact that these two lines make a closed polygon with 0 4, and hence taken as acting from 4 to O and O to 0, as shown by the arrows, hold the forces F_1, F_2, F_3, F_4 in equilibrium, or replace the resultant 4 0 (page 404). As the pole O is taken anywhere we please, we can thus resolve the resultant 4 0 for equilibrium into forces in any two directions we wish.

Let us then consider the resultant 4 0 for equilibrium, replaced by the two forces 4O and Oo. Anywhere in the plane of the forces in Fig. 1 draw a line s_0 parallel to Oo and produce it till it meets F_1, produced if necessary, at a.

If then we take s_0 and F_1, Fig. 1, as acting at a, their resultant will pass through a and be parallel to s_1 in the force polygon Fig. 2, because s_1 in the force polygon is the resultant of F_1 and s_0, since it closes the polygon for those forces. Through a in Fig. 1, then, draw a line parallel to s_1 and produce it to intersection b with F_2, produced if necessary. The line s_2 in the force polygon is the resultant of s_1 and F_2. Parallel to this line then draw s_2 through b, Fig. 1, and produce to intersection c with F_3, produced if necessary. The line s_3 in the force polygon is the resultant of s_2 and F_3. Parallel to this line then draw s_3 through c, Fig. 1, and produce to intersection d with F_4, produced if necessary. Finally through d in Fig. 1 draw a line s_4 parallel to s_4 in the force polygon.

We thus find for any assumed position of s_0 in the plane of the forces in Fig. 1 the proper corresponding position of s_4. Since now s_0 and s_4 are components of the resultant in proper position and each may be considered as acting at any point in its line of direction, we have only to prolong them, and *their intersection gives a point e on the line of direction of the resultant.*

We prolong s_0 and s_4, then, in Fig. 1 to intersection e. The line of direction of the resultant passes through e. Acting in the direction from 4 to 0, it will hold the forces in equilibrium. We thus know the magnitude, direction and position of the resultant for equilibrium.

Position of Pole and of s_0 Indifferent.—The method is evidently general no matter where in the plane of the forces in Fig. 1 we take s_0 as acting, and no matter where we take the pole in Fig. 2.

Pole, Equilibrium Polygon, Rays, Closing Line.—The point O we call the POLE in the force polygon. It may be taken where we please. The polygon *abcd* in Fig. 1 we call the EQUILIBRIUM POLYGON, and *ab*, *bc*, *cd*, etc., are its *segments*. In the present case it is

FIG. 1.

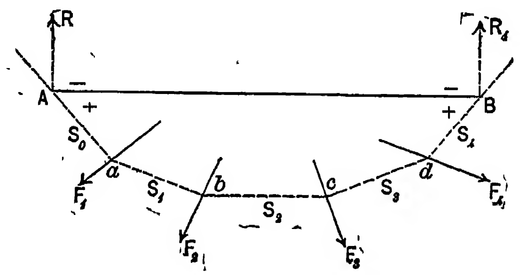
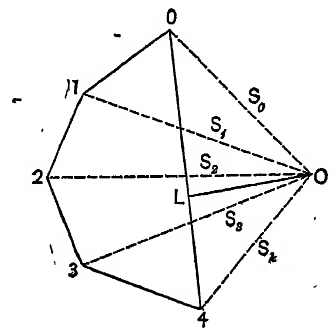


FIG. 2.



evidently the shape a string would take if suspended at any two points as A and B, in Fig. 1, on s_0 and s_4. The stresses in the segments would be tensile. These stresses are given by the lines O0, O1, O2, in the force polygon, and we call these lines RAYS. In general forces may act up as well as down, in which case some of the segments would sustain compressive stresses and our equilibrium polygon would contain struts as well as ties.

Let us take any two points, as A and B, upon the end segments s_0 and s_4, Fig. 1, and suppose them fixed. The force s_0 acting at A we shall then have to *replace* by two forces, one parallel to the resultant and one in the direction AB. So also for s_4 at B. The sum of the two components parallel to the resultant must be equal and opposite to the resultant, and the component in the direction AB must be resisted by a strut or compression member AB. This resolution we make at once by drawing through O in the force polygon a line OL parallel to AB. The line AB we call the CLOSING LINE. Thus we see from Fig. 2 that the sum of the components 4L and L0 equals the resultant.

In any case, then, we can fix any two points of the equilibrium polygon, as A, B, by drawing the closing line AB. A line OL through O parallel to AB, in the force polygon, gives the components into which s_0 and s_4 are resolved.

We can then consider the entire polygon $AabcdB$, with its closing line AB, as *a frame in equilibrium* with the given forces, and can apply to it the principles of page 406.

Thus take the apex A. Here we have the reaction $R_1 = L$0 in equilibrium with the stresses in AB and Aa. Following round in the force polygon from L to 0, 0 to O, and O to L, and transferring these directions to the apex A, we find S_0 away from A or tension, and OL towards A or compression, just as on page 406.

So also at the other apex B we have $R_2 = 4L$ in equilibrium with the stresses in AB and Bd. Following round in the force polygon from 4 to L, L to O, and O to 4, we find S_4 away from B or tension, and LO towards B or compression, as before. The components R_1 and R_2 act opposite to the resultant 0 4 which replaces the forces, and are equal to it in magnitude. The forces at A and B parallel to OL are equal and opposite. Hence the frame is in equilibrium.

Recapitulation.—Our method, then, is as follows:

1st. Draw the force polygon by laying off the forces to scale one after the other, in any order. The line which closes this polygon gives the resultant in magnitude and direction. When it is taken as acting in the direction obtained by following round the force polygon in the direction of the forces, it will cause equilibrium. In the opposite direction it replaces the forces.

2d. Choose a pole O, and draw the rays s_0, s_1, s_2, etc.

3d. Draw the equilibrium polygon.

4th. Fix any two points in the end segments of the equilibrium polygon by drawing the closing line of the equilibrium polygon between those two points.

5th. A line drawn in the force polygon parallel to the closing line of the equilibrium polygon will divide the resultant into the two reactions at the ends. We thus have a frame the stresses in which can be found as on page 406.

Graphic Construction for Centre of Parallel Co-planar Forces.—Let F_1, F_2, F_3, etc., be parallel co-planar forces acting at the points A_1, A_2, A_3, etc., of a rigid body.

We construct the force polygon Fig. 2 by laying off the forces F_1, F_2, F_3, etc. The resultant is then the algebraic sum of the forces and parallel to them.

Then choose a pole O, and draw the rays s_0, s_1, s_2, s_3, etc.

Anywhere in the plane of the forces, Fig. 1, we draw a line parallel to s_0 to intersection a with F_1; then ab parallel to s_1 to intersection b with F_2; then bc parallel to s_2 to intersection c with F_3; then s_3 through c parallel to s_3 in Fig. 2.

The intersection d of s_0 and s_3 is a point on the resultant which therefore has the direction and position dC.

Now suppose the forces F_1, F_2, F_3, etc., all turned in the same direction through a right angle.

FIG. 1.

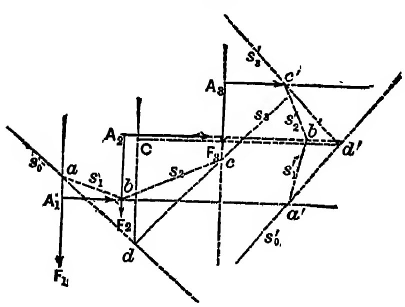
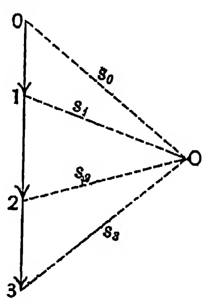


FIG. 2.



Draw the new equilibrium polygon $s_0'a'b'c's_3'$, whose sides are respectively perpendicular to those of the first.

The intersection d' of s_0' and s_3' is a point on the resultant, which therefore has the direction and position $d'C$.

The intersection C of the two resultants gives the centre of force for the system (page 189).

COR. The same construction evidently determines the centre of mass (page 21), if we divide a body into a convenient number of portions, and take the weight of each portion, F_1, F_2, F_3, etc., acting at the centre of mass of that portion.

Properties of the Equilibrium Polygon.—The equilibrium polygon has many interesting properties. We shall call attention to only two.

1st. As we have seen, the intersection of any two segments is a point in the resultant

of the forces included between those segments. Thus in the preceding Fig. 1 the intersection d of s_0 and s_3 is a point on the resultant of F_1, F_2 and F_3.

2d. Let s_0ab, Fig. 1, be a portion of the equilibrium polygon, and Fig. 2 its corresponding force polygon.

Take any line fe in Fig. 1 parallel to F_1, and draw the perpendicular $cd = x$.

Let $de = y$ be the ordinate between s_0 and s_1.

In the force polygon Fig. 2, draw the perpendicular $OH = H$ from the pole to 0 1. This is called the POLE DISTANCE of F_1.

Then by similar triangles we have

FIG. 1.

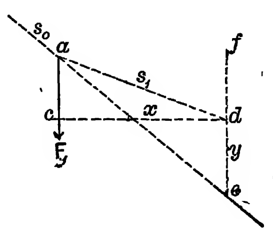
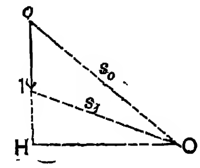


FIG. 2.



$$y : x :: F_1 : H, \quad \text{or} \quad F_1x = Hy.$$

But F_1x is the moment of F_1 with reference to any point on the line fe.

Hence *the moment of any force, as F_1, with reference to any point is equal to the ordinate through this point parallel to F_1, included between the segments of the equilibrium polygon which meet at F_1, multiplied by the pole distance of F_1 in the force polygon.*

Application to Parallel Forces.—The outer forces acting upon framed structures are generally weights and reactions of supports due to these weights. We have then in general to investigate a system of parallel forces.

Let F_1, F_2, F_3, Fig. 1, be vertical forces acting upon a rigid body or frame.

FIG. 1.

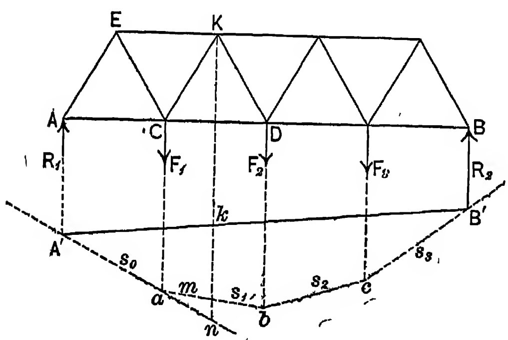
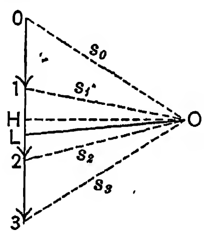


FIG. 2.



Lay off the force polygon 0 1 2 3, Fig. 2. Choose a pole O, and draw the rays s_0, s_1, s_2, s_3.

Then in the plane of the forces Fig. 1 draw s_0 to meet F_1 at a; then s_1 through a to meet F_2 at b; then s_2 through b to meet F_3 at c; and finally s_3. We thus have the equilibrium polygon s_0abcs_3. We see that the horizontal component of the stress in any segment is constant and equal to OH.

Drop verticals through A and B which meet the end segments s_0 and s_3 in A' and B'. If we fix the points A', B' by drawing the closing line $A'B'$, the reactions at A', B' will be the reactions at A and B of the frame.

Therefore in Fig. 2 draw OL parallel to $A'B'$ and we have $Lo = R_1$, and $3L = R_2$.

Draw the pole distance OH. Through the apex K of the frame drop the vertical $Kkmn$. Then, as just proved, OH (to scale of force) $\times$ kn (to scale of distance) = the moment of R_1. Again, $OH \times mn$ = the moment of F_1. The resultant moment is then given by $OH \times (kn - nm)$ or $OH \times km$.

That is, for parallel forces *the pole distance multiplied by the ordinate of the equilibrium polygon at any point, parallel to the forces included between the closing line and the polygon, gives the resultant moment of all the forces on either side of the ordinate with reference to any point in that ordinate.*

If then we make a section cutting EK, CK and CD, and take the centre of moments at

K, we have (page 401) stress in $CD \times$ lever-arm for $CD =$ algebraic sum of moments of R_1 and F_1 with reference to K. But this algebraic sum we have just seen is given by $H \times km$. Hence stress in CD is equal to $\dfrac{H \times km}{\text{lever-arm for } CD}$.

We can therefore find the moment graphically at any point by multiplying the ordinate to the equilibrium polygon at that point by the pole distance.

A few examples will make the application of the preceding principles clear.

Ex. 1. Let AB, Fig. 1, be a beam or rigid body or framed structure subjected to two unequal weights F_1 and F_2 applied at any two given points. Required the reactions at the supports A and B, also the moment at any point of all the forces right or left of that point when equilibrium exists.

FIG. 1.

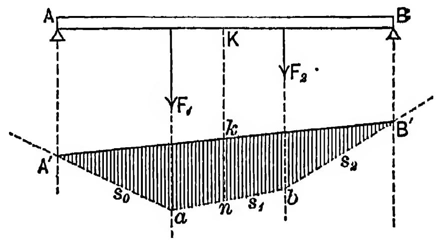
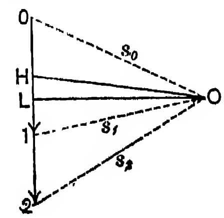


FIG. 2.



Draw the force polygon Fig. 2, choose a pole O, and draw s_0, s_1, s_2, and the pole distance H.

Construct the equilibrium polygon, Fig. 1, by drawing a parallel to s_0 to intersection a with F_1; through a a parallel to s_1 to intersection b with F_2; through b a parallel to s_2. Drop verticals from A and B, and draw the closing line $A'B'$. Parallel to $A'B'$ draw OL in Fig. 2.

Then Lo and $2L$ are the reactions at A and B; and since they act upwards, the supports must be below A and B.

The moment at any point k is equal to the ordinate kn multiplied by the pole distance H.

Ex. 2. It is well to observe that the order in which the forces are taken makes no difference in the results, although the figure obtained may be very different.

FIG. 1.

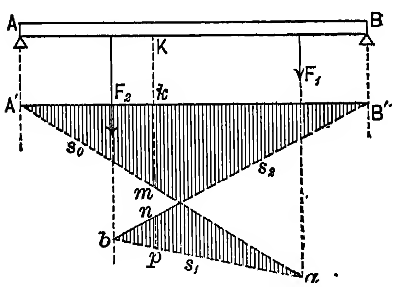
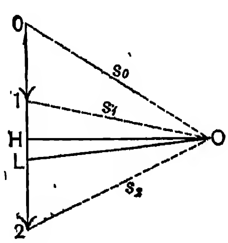


FIG. 2.



Thus take the same example as before, but number the forces in inverse order, Fig. 1.

We form the force polygon as before, choose a pole and draw s_0, s_1, s_2. Now parallel to s_0 we draw a line *till it meets* F_1 *at* a [note that s_0 must always be produced to meet F_1]; then from a a parallel to s_1 till it meets F_2 at b; then from b a parallel to s_2. Draw the closing line $A'B'$. A parallel to it in Fig. 2 gives the reactions Lo and $2L$ as before. At apex b of the equilibrium polygon we find s_2 tension, since F_2 acts downwards. At apex a we find s_0 tension, since F_1 is downward. Hence at A', s_0 acts away from A', and following round in the force polygon we obtain Lo acting upwards. At B', s_2 acts away, and hence $2L$ acts upwards also. The supports at A and B must then be below.

As to the moments, the moment of the reaction at A with reference to any point k is $H \times kn$. The moment of F_2 is $- H \times np$. The resultant moment is $H \times (km - np)$. The lower ordinates subtracted from the upper will give us the same figure as before.

Whenever, then, we obtain a double figure as in the present case, it shows that we have

taken the forces in inconvenient order. We have only to change the order to obtain the moments directly from the equilibrium polygon.

Closing Line at Right Angles to the Forces—Choice of Pole Distance.—It makes no difference what inclination the closing line may have, because, as we have seen, the ordinate in the equilibrium polygon parallel to the resultant, multiplied by the pole distance, gives the resultant moment, *with reference to any point on that ordinate*, of all the forces right or left.

We can, however, if we wish, always cause the closing line to be at right angles to the parallel forces. We have only to find first by preliminary construction the reactions or the point L. If then we take a new pole anywhere in a line through this point at right angles to the forces, the closing line will be at right angles to the forces.

As to choice of pole distance, we have only so to choose the position of the pole as to give good intersections for the polygon. The multiplication may be directly performed by properly changing the scale in the equilibrium polygon. The ordinate to this new scale will then give the moment at once. Thus if our scale of length in Fig. 1, preceding, is five feet to an inch, and the pole distance in the force polygon Fig. 2, measured to the scale of force adopted, is ten pounds, we have only to take fifty moment units to an inch as the scale for the ordinates and they will give the moments directly.

FIG. 1.

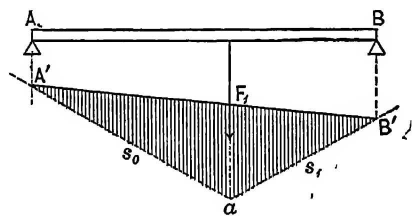
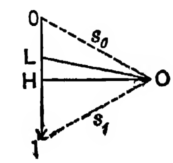


FIG. 2.



EX. 3. Let the single weight F_1 act at any point of the rigid body AB. Then the equilibrium polygon is $A'aB'$. The vertical reactions at A and B are Lo and $1L$, both acting up, and hence the supports are below A and B.

We see at once that the moment is greatest at the weight and decreases to zero at each support.

EX. 4. Let F_1 act outside of the supports A and B. Observe in constructing the equilibrium polygon that s_0 is always produced till it meets F_1; also that the closing line $A'B'$ always unites the two points vertically under the supports, upon the two end segments.

FIG. 1.

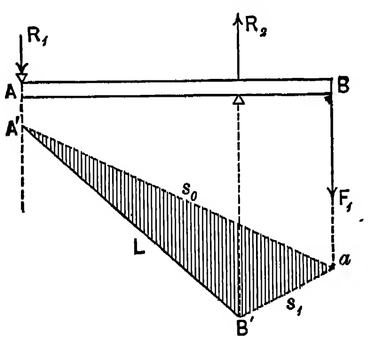
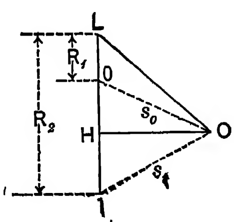


FIG. 2.



The reactions require special notice. Thus the reaction R_2 at B is the resultant of the stresses in aB' and $B'A'$, or $1L$ in the force polygon. The reaction R_1 at A is the resultant of the stresses in $A'a$ and $A'B'$, or Lo in the force polygon.

Since F_1 acts downwards at apex a, we have s_1 compression and s_0 tension. Therefore at apex A' we take s_0 acting away, and hence obtain Lo acting down, or the support is above A.

At apex B' we take s_1 acting towards B_1, and hence obtain $1L$ acting up, or the support is below B.

EX. 5. ONE DOWNWARD AND ONE UPWARD FORCE BETWEEN THE SUPPORTS.—Here we need only call special attention to the fact that as F_2 acts up and is less than F_1, s_2 in the force polygon Fig. 2 lies between s_0 and s_1.

The reaction at A is the resultant of s_0 and L or Lo. The reaction at B is the resultant of s_2 and L or $L2$. Since F_1 is down at a, we have s_0 tension, and since F_2 is up at b, we have s_2 tension. At apex A', then, s_0 acts away, and hence L is compression and Lo acts upwards and support at A is below. At apex B', s_2 acts away, and L is compression as before and $2L$ acts downwards, or support at B is above.

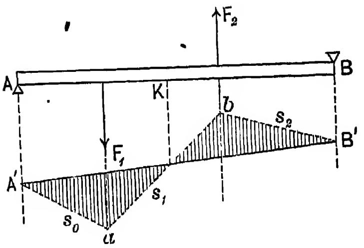


FIG. 1.

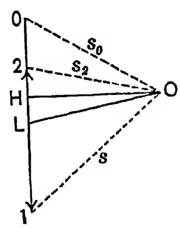


FIG. 2.

We see also that if F_2 were less, so that 2 falls below L in the force polygon, the reaction at B would be upward also, and the support would then have to be below. The student should sketch the case for F_2 greater than F_1.

At the point K we see that the moment is zero. If AB is a beam, the point K is the "point of inflection," or the point at which the curve of deflection of the beam changes from concave to convex. The beam would be concave upwards as far as K, and from there on convex upwards.

Ex. 6. In the preceding case, let the forces be equal. Laying off the force polygon Fig. 2, the first force extends from 0 to 1, and the second from 1 back to 0. Choosing a pole O, and drawing s_0, s_1, s_2, we find that s_0 and s_2 coincide.

Constructing the equilibrium polygon and drawing the closing line $A'B'$ and its parallel L in the force polygon, we see that the reaction at A or the resultant of s_0 and L is Lo, and the reaction at B or the resultant of s_2 and L is also Lo. The reactions are therefore equal. Since s_0 and s_2 are both tension, we have reaction at A upward, or support below A, and reaction at B downward, or support above B.

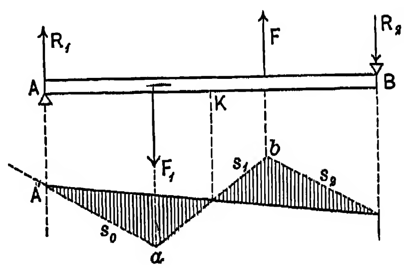


FIG. 1.

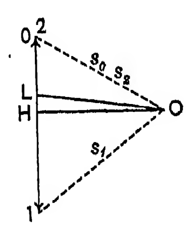


FIG. 2.

This is in accord with the principle (page 186) that a couple can only be held in equilibrium by another couple. Moreover, the resultant of s_0 and s_2 in Fig. 2 is zero, and the point of application is at the intersection of s_0 and s_2 in Fig. 1, or at an infinite distance. That is, the resultant of a couple is zero at an infinite distance (page 186).

At K the moment is zero as before, and we have a point of inflection.

Ex. 7. TWO EQUAL WEIGHTS BEYOND THE SUPPORTS.—The figure needs no explanation, except to call attention to the reactions.

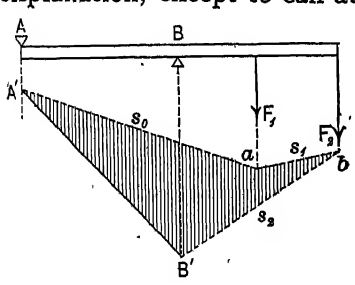


FIG. 1.

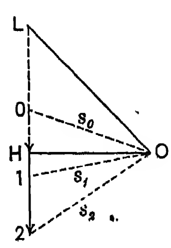


FIG. 2.

Thus the reaction at A is Lo acting down. At B it is $2L$ acting up.

The moment at any point, in all cases, is the ordinate multiplied by the pole distance H. The shaded areas then show how the moments vary.

We repeat here that the order in which the forces are taken, in all cases, as also the position of the pole, is indifferent.

The student will do well to work out cases to scale and satisfy himself that this is true.

EX. 8. TWO EQUAL AND OPPOSITE FORCES BEYOND THE SUPPORTS.—Observe that s_0 is produced till it intersects F_1 at a in Fig. 1; then s_1 from a to b; then s_2 parallel to s_2 or s_0 in Fig. 2. The closing line $A'B'$ is then drawn. A parallel to it in Fig. 2 gives L.

FIG. 1.

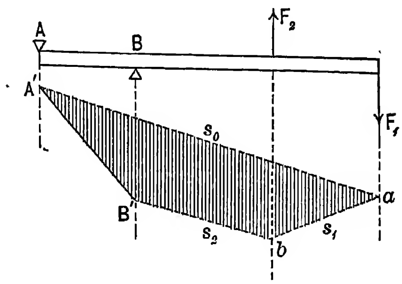
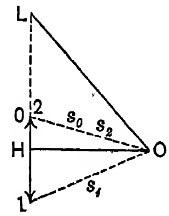


FIG. 2.



The reaction at A is Lo acting down, and at B, oL acting up.

Between B and F_2 the moment is constant. This is the graphic interpretation of the principle, page 185, that the moment of a couple is constant for any point in its plane.

EX. 9. A UNIFORMLY DISTRIBUTED LOAD.—Let the load be uniformly distributed. We might consider it as a system of equal and equidistant weights very close together.

Thus in Fig. 1 the load area, which is a rectangle of uniform density, whose height is the load per unit of length, and whose length is AB, may be divided into any number of equal parts. The weight on each of these parts acts at its centre of mass. We can then lay off the force polygon Fig. 2. Since the reactions at A and B are equal, we take the pole in a horizontal through the middle point of the force line. The closing line $A'B'$ will then be parallel to AB (page 417). We can then draw s_0, s_1, s_2, etc., and construct the equilibrium polygon. It is evident that the points a, b, c, d, etc., will enclose a curve tangent to ab, bc, cd, etc., at the points midway between, that is, where the lines of division of the load area meet the sides of the equilibrium polygon.

FIG. 1.

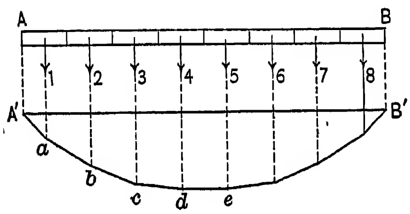
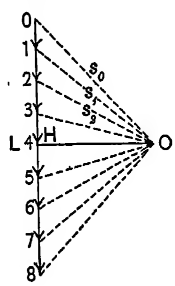


FIG. 2.



The ordinates to this curve, multiplied by the pole distance H, give the moment at any point on the ordinates.

It will be seen, however, that this method is deficient in accuracy, because the lines ab, bc; cd, etc., are so short and there are so many of them. If, however, we can find what the curve $A'abcd$, etc., is, we could draw the curve at once.

FIG. 3.

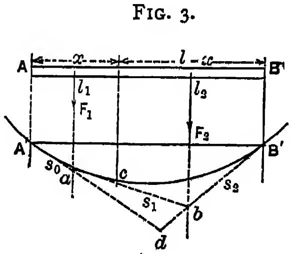
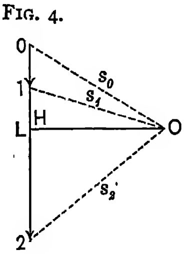


FIG. 4.



Suppose we divide the load area into only two portions of lengths x and $l - x$, where $l = AB$, Fig. 3. The entire weight over the portion x can be considered as acting at the centre e_1 of the load area. The same holds good for the portion $l - x$. We thus have two forces F_1 and F_2.

Taking the pole as before, so that the closing line $A'B'$ shall be parallel to AB, construct the equilibrium polygon $A'abB'$. The curve of moments will be tangent at A', c and B'.

Now we see that, no matter where the load area is supposed to be divided, we shall always have for the distance e_1e_2 between F_1 and F_2

$$e_1e_2 = \frac{1}{2}x + \frac{1}{2}(l - x) = \frac{1}{2}l.$$

That is, no matter where the line of division is taken, the horizontal projection of the line ab of the equilibrium polygon is constant and equal to $\frac{1}{2}l$. But ab is a tangent to the curve required. But if from any point on the line $A'd$ we draw a line ab limited by the line $B'd$, so that the horizontal projection is constant, the line ab will *envelop a parabola*.

This may easily be proved as follows: Let the load per unit of length be p. Then the entire load is pl and the reaction at each end is $\frac{pl}{2}$.

The moment at any point distant x from the left support is then

$$y = \frac{pl}{2}x - F_1\frac{x}{2}.$$

But, since F_1 is equal to px,

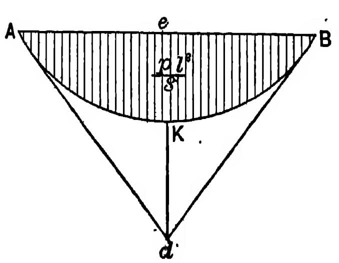
$$y = \frac{pl}{2}x - \frac{px^2}{2}.$$

This is the equation of a parabola. At the centre $x = \frac{l}{2}$, and we therefore have the centre ordinate $\frac{pl^2}{8}$.

COR. 1. We see, therefore, that when a string is suspended from two points A', B' and sustains a load uniformly distributed over the horizontal, the curve of equilibrium is a parabola.

Also, the horizontal component of the stress at any point, as is evident from the force polygon, is constant and equal to H. Also, the vertical component of the stress at any point, as c, Fig. 3, is $R_1 - F_1$, or equal to the total load between the lowest point and the point considered.

COR. 2. We have the following construction for the equilibrium curve. Lay off a perpendicular eK at the centre e and make it equal by scale to $\frac{pl^2}{8}$. Through A, K and B construct a parabola having its vertex at K. The ordinate to this parabola through any point will give the moment at that point.

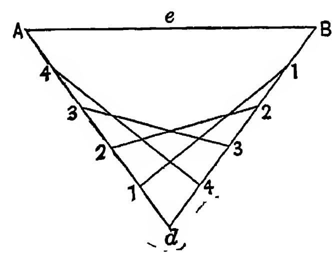


The distance Kd is also equal to $\frac{pl^2}{8}$, because the moment of the reaction with reference to e is

$$ed = \frac{pl}{2} \times \frac{l}{2} = \frac{pl^2}{4},$$

and $Kd = ed - eK = \frac{pl^2}{4} - \frac{pl^2}{8} = \frac{pl^2}{8}$.

COR. 3. *How to Draw a Parabola.*—Since we know, then, the distance $ed = \frac{pl^2}{4}$, we can always draw the lines Ad and Bd. If then we divide Ad and Bd into any number of equal parts and number these parts along one line away from d and along the other towards d, we have only to draw lines joining any two points having the same number, and these lines will all have the same horizontal projection $\frac{l}{2}$.



They will therefore enclose the parabola required. Tangent to these lines we may sketch the curve.

A better method is to plot the ordinates to the curve from its equation,

$$y = \frac{pl}{2}x - \frac{px^2}{2}.$$

Methods of Solution of Framed Structures.—In Chapter I we have given and illustrated two methods of computation for framed structures:

1st. By Resolution of Forces (page 400).

2d. By Moments or the "Method by Sections" (page 401).

In the present Chapter we have the corresponding graphic methods:

1st. By Resolution of Forces (page 404).

2d. By Moments (page 415).

Examples.—(1) *A roof-truss has a span of 50 ft. and a centre height of 12.5 ft. Each rafter is divided into four equal panels, and the lower horizontal tie is divided into six equal panels. The bracing is as shown in the figure. Find the stresses in the members, by the graphic method of moments, for a weight of 800 lbs. at each upper apex.*

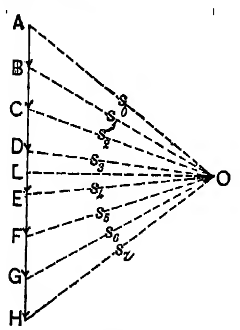


FIG 2.

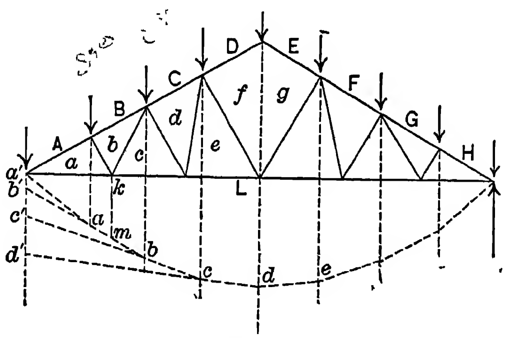


FIG. 1.

ANS. We have computed the stresses (page 402), by the two methods, resolution of forces and moments. We have also found the stresses by the graphic method of resolution of forces (page 410, ex. (1)).

We can construct the force polygon Fig. 2, and then the equilibrium polygon Fig. 1. This, however, is not advisable for reasons already given. It will be more accurate to assume the pole distance as unity, thus discarding the force polygon altogether, and construct points in a parabola from the equation

$$y = \frac{pl}{2}x - \frac{px^2}{2}.$$

In the present case the load per foot is, if we suppose half weights of 400 at the ends, $\frac{6400}{50} = 128$ lbs. $= p$.

Taking $x = \frac{1}{8}l$, $\frac{2}{8}l$, etc., we have

$$x = \frac{1}{8}l, \quad \frac{2}{8}l, \quad \frac{3}{8}l, \quad \frac{4}{8}l;$$

$$y = 17500, \quad 30000, \quad 37500, \quad 40000 \text{ lb.-ft.}$$

Laying these off to any convenient scale, we determine very accurately the points a, b, c, d of the equilibrium polygon. The other half of the polygon is precisely similar.

The ordinates to this polygon will give, to the scale adopted, the moment, for any point of the truss, of the outer forces left or right. Thus the moment with reference to k of all forces right or left is km, Fig. 1. We find by scale $km = 21666\frac{2}{3}$ lb.-ft. In the same way for the next lower apex we find the moment 35000 lb.-ft. The moment at the next lower apex or centre of the span is 40000 lb.-ft.

Now by the method of sections (page 401) we have for any member

$$\text{Stress} \times \text{lever-arm} + \Sigma \text{ moments of outer forces} = 0.$$

The second term is given by the ordinates of the equilibrium polygon to scale.

As regards the centre of moments for any member, we must observe the rule (page 401), viz.: Cut the truss entirely through by a section cutting only three members the strains in which are unknown. For any one of these take the point of moments at the intersection of the other two.

For the proper sign for the first member of the equation place an arrow on the cut member pointing away from the end belonging to the left-hand portion, and take the moment (+) or (−) according as the rotation indicated by this arrow is counter-clockwise or clockwise.

If the stress comes out positive, it indicates tension; if negative, compression.

Take, for instance, the first lower panel, La. The centre of moments must be taken at the first upper apex. The moment for this point is given by the ordinate na of the equilibrium polygon, or $-$ 17500 lb.-ft. We take the minus sign, because the rotation is clockwise. We have then

$$La \times 3.125 - 17500 = 0, \quad \text{or} \quad La = +\ 5600 \text{ lbs.},$$

where 3.125 ft. is the lever-arm of La.

In similar manner we have

$$Lc \times 6.25 - 30000 = 0, \quad \text{or} \quad Lc = +\ 4800 \text{ lbs.},$$

where 6.25 ft. is the lever-arm of Lc.

For Le we have

$$Le \times 9.375 - 37500 = 0, \quad \text{or} \quad Le = +\ 4000 \text{ lbs.},$$

where 9.375 ft. is the lever-arm of Le.

For the first upper panel, Aa, take the centre of moments at k. The moment for this point is given by the ordinate from *k to the first line of the polygon produced*. It is therefore larger than km, which gives the combined moment of the reaction and first weight. We find it by scale to be $-$ $23333\frac{1}{3}$ lb.-ft.

We have then

$$-Aa \times 3.727 - 23333\tfrac{1}{3} = 0, \quad \text{or} \quad Aa = +\ 6260 \text{ lbs.},$$

where 3.727 ft. is the lever-arm for Aa.

In like manner for Bb we have centre of moments at k, and moment $km = -\ 21666\frac{2}{3}$. Hence

$$-Bb \times 3.727 - 21666\tfrac{2}{3} = 0, \quad \text{or} \quad Bb = +\ 5813 \text{ lbs.}$$

For Cd we have

$$-Cd \times 7.454 - 35000 = 0, \quad \text{or} \quad Cd = +\ 4691 \text{ lbs.},$$

where 7.454 ft. is the arm-lever for Cd.

For Df we have

$$-Df \times 11.151 - 40000 = 0, \quad \text{or} \quad Df = +\ 3587 \text{ lbs.}$$

For all the braces the point of moments is at the left-hand end. Taking a section through Bb, ab and La, we have acting on the left-hand portion only the weight AB and the reaction. The moment of the weight relative to the left end is the ordinate $a'b'$, or by scale $-$ 5000 lb.-ft. The lever-arm for ab is 6.934 ft. Hence

$$-ab \times 6.934 - 5000 = 0, \quad \text{or} \quad ab = -\ 721 \text{ lbs.}$$

For bc we have

$$+ab \times 6.934 - 5000 = 0, \quad \text{or} \quad ab = +\ 721 \text{ lbs.}$$

For cd the moment is $a'b' + b'c'$, or $-$ 1500. We have then

$$-cd \times 13.869 - 15000 = 0, \quad \text{or} \quad cd = -\ 1081 \text{ lbs.},$$

and so on. All lever-arms can be scaled off the frame or must be computed.

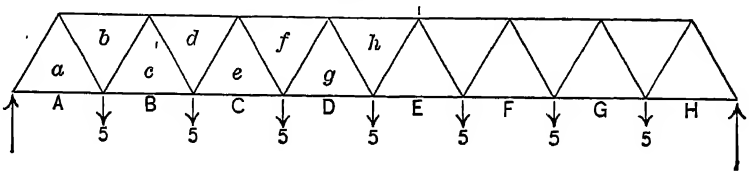
The present method is not to be recommended for the braces. In prolonging the sides *ab*, *bc*, etc., of the equilibrium polygon, a slight variation in direction will make considerable error in the ordinate at the end. Also as the sides *ab*, *bc*, etc., are short they do not give direction accurately enough.

Of all our four methods, the graphic method by resolution of forces (page 404) is the easiest of application to such cases.

The more irregular the frame the more advantageous it is.

(2) *A bridge-girder, as shown in the figure, 10 feet deep, 80 feet long, eight equal panels in the lower chord and seven equal panels in the upper chord, has a load of 5 tons at each lower apex. Find the stresses by diagram and by moments.*

FIG. 1.



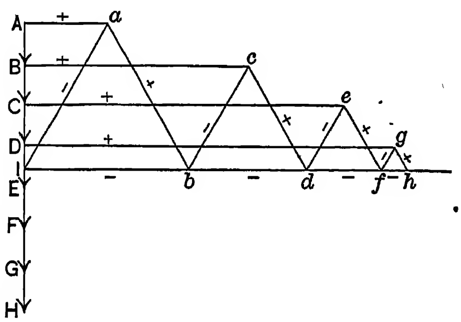


FIG. 2.

ANS. The panel length is 10 ft., $\sec\theta = 1.117$. By moments, then,

$Aa \times 10 - 17.5 \times 10 = 0,$	or $Aa = +17.5$ tons.
$Bc \times 10 - 17.5 \times 15 + 5 \times 5 = 0,$	$Bc = +23.75$ "
$Ce \times 10 - 17.5 \times 25 + 5(5 + 15) = 0,$	$Ce = +33.75$ "
$Dg \times 10 - 17.5 \times 35 + 5(5 + 15 + 25) = 0,$	$Dg = +38.75$ "
$-Ib \times 10 - 17.5 \times 10 = 0,$	$Ib = -17.5$ "
$-Id \times 10 - 17.5 \times 20 + 5 \times 10 = 0,$	$Id = -30$ "
$-If \times 10 - 17.5 \times 30 + 5(10 + 20) = 0,$	$If = -37.5$ "
$-Ih \times 10 - 17.5 \times 40 + 5(10 + 20 + 30) = 0,$	$Ih = -40$ "

$Ia = -17.5 \times 1.117 = -19.55,$	$de = -7.5 \times 1.117 = -8.38,$
$ab = +19.55,$	$ef = +8.48,$
$bc = -12.5 \times 1.117 = -13.96,$	$fg = -2.5 \times 1.117 = -2.79,$
$cd = +13.96,$	$gh = +2.79.$

CHAPTER IV.

WALLS. MASONRY DAMS.

Definitions of Parts of a Wall.—The *face* of a wall is the front surface, or outside surface, or the surface farthest from the pressure. The *back* is the rear surface, or inside surface, or the surface which sustains pressure.

The stone which forms the face is called the *facing;* that which forms the back, the *backing;* that which forms the interior, the *filling*.

A horizontal layer of stone in a wall is called a *course*. If the stones in each layer are of the same thickness, we have *regular* courses; if they are not of the same thickness, we have *irregular* or *random* courses.

The mortar layer between the stones is the *joint*. The horizontal joints are *bed-joints*.

Cut stone or squared masonry is called *ashlar*. Unsquared masonry or rough stone is called *rubble*.

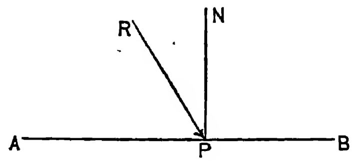
The inclination of the face or back of a wall, measured by the ratio of its horizontal to its vertical projection, is called the *batter* of the face or back. The batter is then the tangent of the angle which the face or back makes with the vertical.

Weight and Friction of Masonry.—We give here a short table of average values of the coefficient of static sliding friction μ, the density or mass per cubic foot δ, and the allowable compressive unit stress C in tons per square foot, taking 2000 lbs. to a ton, for different kinds of masonry. We also give the specific mass (page 16) $\frac{\delta}{\gamma}$, where γ is the mass of a cubic foot of water, or 62.5 lbs.

In discussing the stability of walls, the influence of the mortar is neglected, both because of its uncertain character and because such neglect is on the side of safety.

Kind of Masonry.	Coefficient of Friction μ.	Density in lbs. per Cubic Foot δ.	Specific Mass $\frac{\delta}{\gamma}$.	Allowable Compressive Unit Stress C in tons per sq ft
Limestone and granite :				
Ashlar masonry	0.6	165	2.64	25 to 30
Large mortar rubble	0.6	150	2.40	10 to 15
Small dry rubble	0.6	125	2.00	6 to 10
Concrete	0.6	150	2.40	12 to 17
Sandstone :				
Ashlar masonry	0.6	150	2.40	20 to 25
Large mortar rubble	0.6	130	2.08	10 to 15
Small dry rubble	0.6	110	1.76	6 to 10
Brick work	0.6	100	1.60	6 to 10

Stability of a Masonry Joint.—Let AB represent a masonry joint, and R the resultant of all the external forces acting upon it at the point P.



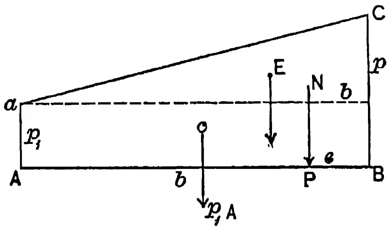
Then we must have the following conditions for stability:

1. The resultant R of all the external forces must intersect the joint at some point P within the joint; otherwise we have no rotation.

2. The resultant R must make an angle RPN with the normal to the joint whose tangent is less than the coefficient of friction; otherwise we have sliding.

3. The greatest unit pressure at any point of the joint must not exceed the allowable compressive unit stress C for the material; otherwise the joint is overloaded.

Determination of this Greatest Unit Pressure.—Let N be the normal component of the resultant R of all the external forces acting at the point P, and let e be the distance of P from the *nearest edge* B.



Then the least unit pressure p_1 will act along the farthest edge A, and the greatest unit pressure p will act along the nearest edge B. If we lay off $Aa = p_1$ and $Bc = p$, the unit pressure at any point will be given by the co-ordinate to the straight line ac. Let A be the area of joint, and b the breadth of joint AB.

We have then the mean unit pressure $\frac{p_1 + p}{2}$, and hence the total normal pressure

$$N = \frac{(p_1 + p)A}{2}, \quad \text{or} \quad p_1 = \frac{2N}{A} - p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The entire load area is made up of the rectangular area $aABb$ and the triangular area abc. The load represented by the rectangular area is $p_1 A$, and its centre of action is at c at a distance $\frac{b}{2}$ from the edge B. The load represented by the triangular area is $\frac{(p - p_1)A}{2}$ acting at E at a distance $\frac{b}{3}$ from the edge B.

We have then, taking moments about the edge B,

$$p_1 A \times \frac{b}{2} + \frac{(p - p_1)A}{2} \times \frac{b}{3} = Ne. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

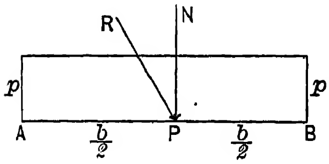
Substituting the value of p_1 from (1), we have for the greatest unit pressure

$$p = \frac{2N}{A}\left(2 - \frac{3e}{b}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where N is the normal pressure on the joint of area A and breadth b, and e is the nearest edge distance from N.

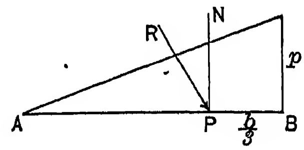
From (3) we can find in any case the greatest unit pressure, and this must not exceed the allowable compressive unit stress C, otherwise the joint is overloaded.

"Middle Third" Rule.—From (3) we see that when $e = \frac{b}{2}$ we have $p = \frac{N}{A}$. That is,

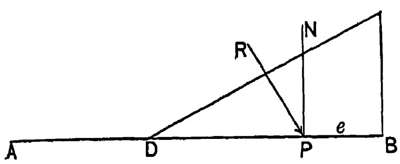


when the resultant R of all the external forces acts at the centre of mass of the joint the load N is uniformly distributed and the unit pressure at every point is $p = \frac{N}{A}$, so that $p_1 = p$.

As the point of application P of R approaches B, p increases and p_1 decreases, and when $e = \frac{b}{3}$ we have $p = \frac{2N}{A}$ and $p_1 = 0$. That is, when the resultant R acts at $\frac{1}{3}b$ from the nearest edge, the unit pressure at the farthest edge is zero, and the greatest unit pressure at the nearest edge is twice as great as if the load were uniformly distributed.



If then e is less than $\frac{b}{3}$, the *entire joint is not brought into action.* The effective breadth of joint is then $3e = DB$ and the portion AD affords no resistance, if we disregard the tensile strength of the mortar.



If l is the length of joint, the area in action is $3el$ and the greatest unit pressure $p = \frac{2N}{3el}$.

We see, then, that in order to have the entire joint in action the resultant R of all the external forces must intersect the joint *inside the middle third.*

This is called the "middle third rule," and in a well-proportioned masonry structure it should be complied with.

If then the point P lies within the middle third, we have the greatest unit pressure

$$p = \frac{2N}{bl}\left(2 - \frac{3e}{b}\right).$$

If the point P is just at the limit of the middle third, we have

$$p = \frac{2N}{bl}.$$

If the point P is outside the middle third, we have

$$p = \frac{2N}{3el}.$$

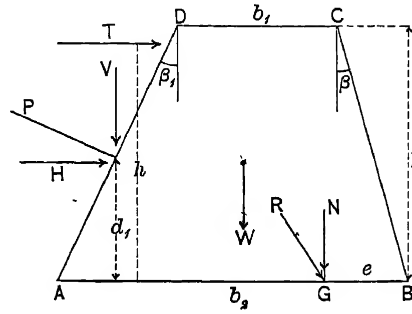
In any case the value of p must not exceed the allowable compressive unit stress C, otherwise the joint is overloaded.

Stability of a Wall in General.—We can now investigate the stability of a wall in general

NOTATION.—Let h_1 be the height of the wall, b_1 the top base, b_2 the bottom base, β_1 the batter angle of the back, β the batter angle of the face, δ the density of the wall.

Let there be a horizontal thrust T per unit of length of wall acting at the distance h from

the bottom, and let there be a pressure P per foot of length of the wall acting in any direction at a distance d_1 from the bottom.* Let H and V be the horizontal and vertical components per ft. of length of the wall of this pressure P. Let W be the weight per ft. of length of the wall acting at the centre of mass O, and let R be the resultant of H, V, W and T, intersecting the base at a point G distant e from the front edge. Let N be the total normal pressure upon the base per foot of length of the wall.



The weight W per foot of length of the wall is given then by

$$W = \frac{\delta(b_1 + b_2)h_1}{2}, \qquad (1)$$

and the total pressure per foot of length of the wall upon the base is

$$N = W + V. \qquad (2)$$

STABILITY FOR SLIDING.—If μ is the coefficient of friction as given in the table page 424, we have the friction per foot of length of the wall

$$\mu N = \mu(W + V). \qquad (3)$$

If $T + H$ is greater than this and the joint AB extends through without break, we have sliding on the base; if $T + H$ is less than this, there is no sliding. For safety let us take the friction equal to n times $T + H$, so that n is the factor of safety for sliding. We have then

$$\mu N = n(T + H), \quad \text{or} \quad n = \frac{\mu(W + V)}{T + H}. \qquad (\text{I})$$

If then, in any case, n is less than unity, we have sliding for a through joint AB. If $n = 1$, the wall is on the point of sliding. If n is greater than unity, there is no sliding and the greater n is the greater the security.

In practice we should have n at least 2 or even more if shocks are to be apprehended. If the joint AB does not extend through, or if the masonry courses are irregular, so that no joint extends through, there is no possibility of sliding and equation (I) need not be applied.

STABILITY FOR ROTATION.—The distance $\bar{x}$ of W from the front edge B is, from page 25,

$$\bar{x} = \frac{b_1^2 + b_1 b_2 + b_2^2 + h_1(2b_1 + b_2)\tan\beta}{3(b_1 + b_2)}, \qquad (4)$$

or, putting $h_1 \tan\beta = b_2 - b_1 - h_1 \tan\beta_1$,

$$\bar{x} = \frac{2b_2^2 + 2b_1 b_2 - b_1^2 - h_1(2b_1 + b_2)\tan\beta_1}{3(b_1 + b_2)}. \qquad (5)$$

The distance of W from G is then $\bar{x} - e$. The distance of V from G is $b_2 - e - d_1 \tan\beta_1$. The distance of H from G is d_1. The distance of T from G is h. If then we take moments about G, we have, if the resultant R passes through G,

$$-Th - Hd_1 + V(b_2 - e - d_1 \tan\beta_1) + W(\bar{x} - e) = 0.$$

* For both water-pressure and earth-pressure h is the distance to the water or earth surface, and $d_1 = \frac{1}{3}h$ (pages 431, 454).

Hence the distance e from the front edge B of the point G where the resultant R cuts the base is given by

$$e = \frac{W\bar{x} + V(b_2 - d_1 \tan \beta_1) - Hd_1 - Th}{W + V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

If in any case e as given by (II) is negative, the point G falls outside of the base AB and we have rotation. If e is positive, the point G is within the base and there is no rotation.

STABILITY FOR PRESSURE.—We have still to test as to whether the base is overloaded or not. We have $N = W + V$. From page 426 we have for the greatest unit pressure on the base, *provided we take e as the distance from G to the nearest edge*,

$$\left.\begin{aligned} &\textit{when e is less than } \frac{1}{3}b_2 \ldots\ldots\ldots p = \frac{2(W+V)}{3e}, \\ &\textit{when e is equal to } \frac{1}{3}b_2 \ldots\ldots\ldots p = \frac{2(W+V)}{b_2}, \\ &\textit{when e is greater than } \frac{1}{3}b_2 \ldots\ldots\ldots p = \frac{2(W+V)}{b_2}\left(2 - \frac{3e}{b_2}\right). \end{aligned}\right\} \quad . \quad . \quad \text{(III)}$$

In these equations e is always the distance of *G from the nearest edge*. In any case the value of p should be less than the allowable unit stress C given in the table page 424, or else the base is overloaded.

It is then a very simple matter to test by equations (I), (II) and (III) the stability of a wall whose dimensions are given in any case where T, H and V are given.

High and Low Wall.—The distance e from the front edge B of the point G where the resultant cuts the base (figure, page 427) is given by equation (II). As we have seen (page 425), if e is less than $\frac{1}{3}b_2$, the entire base is not brought into action. In such case the joints on the inside tend to open, and in a well-designed wall this should not occur. Also, the greatest unit pressure p should never exceed the allowable unit stress C as given by the table page 424.

In a properly designed wall, then, e must be equal to or greater than $\frac{1}{3}b_2$, and p must be less than, or at most equal to, C.

If, in any case when $e = \frac{1}{3}b_2$, p is less than, or at most equal to, C, the wall is said to be low. If, when $p = C$, we have e greater than $\frac{1}{3}b_2$, the wall is said to be high.

If then in the second of equations (III), we make $p = C$, we have for $e = \frac{1}{3}b_2$

$$b_2 = \frac{2(W+V)}{C}.$$

For trapezoid section

$$W = \frac{\delta(b_1 + b_2)h_1}{2}.$$

Hence we have for the limit of low wall

$$limit\ h_1 = \frac{b_2C - 2V}{\delta(b_1 + b_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

Equation (I) gives the limit of h_1 for low wall of trapezoid section, for given top base b_1 and bottom base b_2. For h_1 greater than this limit the wall is high. For h_1 less than this the wall is low.

For rectangular section we have $b_2 = b_1$, and in this case we have

$$limit\ h_1 = \frac{b_1C - 2v}{2\delta b_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) gives the limit of h_1 for low wall of rectangular section for given base b_1. For h_1 greater than this limit the wall is high. For h_1 less than this the wall is low.

Design of Low Wall—Trapezoid Section.—For both water and earth pressure h (figure, page 427) is the distance from base to water or earth surface, and d_1, as we shall see hereafter (page 431), is $\frac{1}{3}h$.

BOTTOM BASE.—If then we make $e = \frac{1}{3}b_2$ in equation (II), page 428, put $d_1 = \frac{1}{3}h$, and substitute the values of W and x as given by equations (1) and (5), page 427, and solve for b_2, we have

$$b_2 = -B_1 + \sqrt{B_1^2 + E_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II)$$

where the quantities B_1 and E_1 are given by

$$B_1 = \frac{1}{2}\left(b_1 + \frac{4V}{\delta h_1} - h_1 \tan\beta_1\right), \qquad E_1 = b_1(b_1 + 2h_1 \tan\beta_1) + \frac{2h}{\delta h_1}(V \tan\beta_1 + H + 3T).$$

Equation (II) gives the bottom base b_2 for a properly designed low wall of trapezoid section. If the value of b_2, as given by (II), substituted in (I), gives the limit h_1 greater than the actual height, the wall is low. If less, the wall is high and equation (II) does not apply.

TOP BASE.—For economy the top base b_1 should be assumed as small as possible consistent with practical and local considerations. We should not in any case assume b_1 greater than b_2. If then we make $b_2 = b_1$ in (II), we have for the greatest allowable value of b_1

$$max.\ b_1 = -B + \sqrt{B + E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the quantities B and E are given by

$$B = \frac{1}{2}\left(\frac{4V}{\delta h_1} - 3h_1 \tan\beta_1\right), \qquad E = \frac{2h}{\delta h_1}(V \tan\beta_1 + H + 3T).$$

Equation (2) gives then the largest allowable value of b_1 for low wall, that is, when h_1 is less than or equal to limit h_1, as given by (1).

Design of High Wall—Trapezoid Section.—As we have seen, if the value of b_2 as given by (II), substituted in (I), gives the limit h_1 greater than the actual height, the wall is low. If less, the wall is high. In this case e is greater than $\frac{1}{3}b_2$, and $p = C$.

LOWER BASE.—From the third of equations (III), page 428, we have

$$p = \frac{2(W+V)}{b_2}\left(2 - \frac{3e}{b_2}\right).$$

For $p = C$ we have

$$e = \frac{2}{3}b_2 - \frac{Cb_2^2}{b(W+V)}.$$

If we equate this to the value of e given by equation (II), page 428, make $d_1 = \frac{1}{3}h$, substitute the values of W and x from equations (1) and (5), page 427, and solve for b_2, we have for the value of b_2

$$b_2 = -B_2 + \sqrt{B_2^2 + E_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

where the quantities B_2 and E_2 are given by

$$B_2 = \frac{\delta h_1}{2C}\left(\frac{2V}{\delta h_1} - h_1 \tan \beta_1\right),$$

$$E_2 = \frac{\delta h_1 b_1}{C}(b_1 + 2h_1 \tan \beta_1) + \frac{2h}{C}(V \tan \beta_1 + H + 3T).$$

TOP BASE.—For economy the top base b_1 should be assumed as small as possible consistent with practical and local considerations. We should not in any case have b_1 greater than b_2. If then in (III) we make $b_2 = b_1$, we have

$$max.\ b_1 = -B + \sqrt{B^2 + E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the quantities B and E are given by

$$B = \frac{\delta h_1}{2(C - \delta h_1)}\left(\frac{4V}{\delta h_1} - 3h_1 \tan \beta_1\right),$$

$$E = \frac{2h}{C - \delta h_1}(V \tan \beta_1 + H + 3T).$$

Equation (3) gives then the largest allowable value of b_1 for high wall, that is, when h_1 is greater than limit h_1 as given by (1).

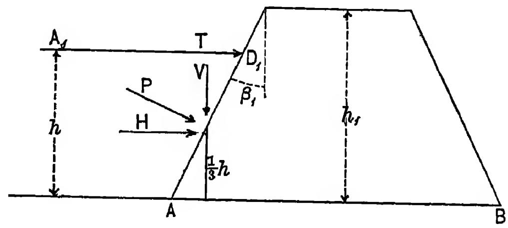
Design of Wall in General—Trapezoid Section.—In general, then, in order to design a wall of trapezoid section for given b_1 and h_1 when V, H and T are known, we first find from (II), page 429, the value of b_2 for low wall. If this value of b_2 substituted in (I), page 429, gives the limit h_1 greater than the actual height, the wall is low, and the value of b_2 given by (I) is the value required. If limit h_1 is less than the actual height, the wall is high, and we must find b_2 from (III), page 430.

In the case of low wall the value of b_1 assumed should not be greater than max. b_1 given by equation (2), page 429. In the case of high wall the value of b_1 assumed should not be greater than max. b_1 given by equation (3), page 430. In any case, for economy we should take b_1 as small as local or practical considerations permit.

Water-pressure.—It is a well-known principle of Physics that the direction of water-pressure upon a submerged surface is always perpendicular to that surface. Also, upon a plane surface the pressure is equal to the weight of a prism of water whose base is the submerged area and whose height is the distance from the water-level to the centre of mass of

the submerged area. This pressure acts at a point at a distance from the water-level equal to $\frac{2}{3}$ of the depth of the water, or at a distance d_1 from the bottom equal to $d_1 = \frac{1}{3}h$, where h is the depth of water.

Thus, in the figure, let A_1D_1 be the water-level and h the depth of water above the base AB. Then the pressure P per foot of length of the wall upon the submerged area $\overline{AD_1} \times 1$ ft. is perpendicular to that area. This pressure P acts at a point at a distance from the bottom equal to $\frac{1}{3}h$. The centre of mass of the submerged area is at a distance of $\frac{1}{2}h$ below the water-surface. The pressure P is then equal to the weight of a prism of water whose base is $\overline{AD_1} \times 1$ ft., and whose height is $\frac{1}{2}h$. If γ is the density of water, the pressure P per foot of length of the wall is then



$$P = \gamma \times \overline{AD_1} \times \frac{h}{2}.$$

Let β_1 be the batter angle of the back. Then $\overline{AD_1} \cdot \cos\beta_1 = h$, or $\overline{AD_1} = \frac{h}{\cos\beta_1}$. Hence the pressure P per foot of length of the wall is

$$P = \frac{\gamma h^2}{2\cos\beta_1}. \quad \ldots \ldots \quad (1)$$

The vertical component V of P per foot of length of the wall is then

$$V = P\sin\beta_1 = \frac{\gamma h^2}{2}\tan\beta_1, \quad \ldots \ldots \quad (2)$$

and the horizontal component H of P per foot of length of the wall is

$$H = P\cos\beta_1 = \frac{\gamma h^2}{2}. \quad \ldots \ldots \quad (3)$$

Both these components act at a point at a distance from the bottom equal to $\frac{1}{3}h$.

Example.—*A dam 20 feet high whose back has a batter of 1 to 2 sustains water-pressure. The water-level is 2 feet below the top. Find the horizontal and vertical pressure per foot of length.*

ANS. We have $h_1 = 20$ ft., $h = 18$ ft., $l = 1$ ft., $\tan\beta_1 = \frac{1}{2}$. For water $\gamma = 62.5$ lbs. per cubic ft.

Hence the horizontal pressure is

$$H = \frac{\gamma h^2}{2} = \frac{62.5 \times 18 \times 18}{2} = 10125 \text{ pounds,}$$

and the vertical pressure is

$$V = \frac{\gamma h^2}{2}\tan\beta_1 = \frac{62.5 \times 18 \times 18}{2} \cdot \frac{1}{2} = 5062.5 \text{ pounds.}$$

These pressures act at $\frac{1}{3}h = 6$ ft. from the bottom.

Ice and Wave Pressure.—A dam has to sustain, in addition to the water-pressure, a horizontal pressure per foot of length at the water-level due to waves or to the pressure of ice. We denote this horizontal thrust per foot of length by T. For waves we may take, on the basis of experiments made by Stevenson, $T = 24000$ pounds per foot of length, and for ice, on the basis of the Report of the Aqueduct Commission on the Quaker Bridge Dam, 1889, we may take $T = 40000$ pounds per foot of length. Since both these do not act together, we have only to consider T for ice in cold climates and T for waves in warm climates, if the body of water back of the dam is sufficiently extensive for wave-action to occur.

Stability of a Dam—Trapezoid Section.—We have then only to use this value of T and the values of V and H given by equations (2) and (3), page 431, in the general equations (I), (II) and (III), page 428, and make $d_1 = \frac{1}{3}h$, in order to investigate the stability of any given dam of trapezoidal section.

Examples.—(1) *Investigate the stability of a dam of granite ashlar 20 feet high, with vertical back and face. The water-level is 2 feet below the top of the dam. The top base is 2 feet. Section trapezoid.*

ANS. We have $b_1 = 2$ ft., $b_2 = 2$ ft., $\beta_1 = 0$, $\beta = 0$, $h_1 = 20$ ft., $h = 18$ ft., $\gamma = 62.5 \frac{\text{lbs.}}{\text{cub. ft.}}$ Also, from the table page 424, $\delta = 165 \frac{\text{lbs.}}{\text{cub. ft.}}$, $C = 30 \times 2000 = 60000 \frac{\text{pounds}}{\text{sq. ft.}}$, $\mu = 0.6$.

Hence we obtain $V = 0$, $H = \frac{62.5 \times 18 \times 18}{2} = 10125$ pounds per ft., $W = 165 \times 2 \times 20 = 6600$ pounds per ft. From (I), page 427, then, disregarding ice and wave thrust, we have

$$n = \frac{\mu W}{H} = \frac{6 \times 6600}{10 \times 10125} = 0.39.$$

The dam is therefore not secure against sliding even if we disregard ice and wave thrust.

We have from (4), page 427, $\bar{x} = 1$ ft., and from (II), page 428, disregarding ice and wave thrust,

$$e = \frac{W - \frac{Hh}{3}}{W} = \frac{6600 - 60750}{6600} = -8.2.$$

Since e is negative, the resultant R falls outside the base, and we have rotation even when ice and wave thrust are neglected. The dam is therefore unstable both for sliding and rotation.

(2) *Let us change the bottom base to 12 feet, the rest being the same as before.*

ANS. We have $b_1 = 2$ ft., $b_2 = 12$ ft., $\beta_1 = 0$, $\tan \beta = \frac{1}{2}$, $h_1 = 20$ ft., $h = 18$ ft., $\gamma = 62.5 \frac{\text{lbs.}}{\text{cub. ft.}}$, $\delta = 165 \frac{\text{lbs.}}{\text{cub. ft.}}$, $C = 60000 \frac{\text{pounds}}{\text{sq. ft}}$, $\mu = 0.6$.

Hence we have $W = 165 \times 7 \times 20 = 23100$ pounds per ft., $V = 0$, and $H = 10125$ pounds per ft., as before.

From (I), page 427, then, disregarding ice and wave thrust,

$$n = \frac{\mu W}{H} = \frac{6 \times 23100}{10 \times 10125} \doteq 1.3.$$

We ought to have $n = 2$ at least.

The dam, then, is not secure against sliding even when we disregard ice and wave thrust. Let us suppose, however, that the courses are irregular, so that sliding cannot take place.

We have from (4), page 427,

$$\bar{x} = \frac{4 + .24 + 144 + 10(4 + 12)}{3 \times 16} = 6.916 \text{ ft.}$$

From (II), page 428, disregarding ice and wave thrust, we have

$$e = \frac{23100 \times 6.9 - 60750}{23100} = 4.27.$$

Since e is positive, the resultant R falls within the base and there is no rotation if we disregard ice and wave thrust. Since e is greater than $\frac{1}{3}b_2 = 4$ and less than $\frac{2}{3}b_2 = 8$, we have, from the third of equations (III), page 428, for the greatest unit pressure

$$p = \frac{2 \times 23100}{12}\left(2 - \frac{3 \times 4.27}{12}\right) = 3619 \text{ pounds per sq. ft.}$$

This is much less than the allowable compressive unit stress $C = 60000$ pounds per square foot.

If we take into account wave-pressure, we have $T = 24000$ and

$$e = \frac{23100 \times 6.9 - 60750 - 432000}{23100} = -14.4.$$

Since e is negative, the resultant R now falls outside the base and we have rotation.

The dam, then, if constructed so that sliding is not possible, is stable when we neglect wave-pressure, but is unstable if we take wave-pressure into account.

(3) *The San Mateo dam, California, is built of concrete having a density $\delta = 150$ lbs. per cubic foot. The height $h_1 = 170$ feet, top base $b_1 = 20$ feet, bottom base $b_2 = 176$ feet. The back batter is 1 to 4, or $\tan \beta_1 = \frac{1}{4}$. Investigate the stability for depth of water $h = 165$ feet.* (See example (6), page 438.)

ANS. We have $b_2 = b_1 + h_1 \tan \beta_1 + h_1 \tan \beta$; hence $\tan \beta = \frac{227}{340}$.

Also, from table page 424,

$$C = 17 \times 2000 = 34000 \frac{\text{pounds}}{\text{sq. ft.}}, \qquad \mu = 0.6.$$

For a section one foot in length we have

$$V = \frac{\gamma h^2}{2} \tan \beta_1 = \frac{62.5 \times 165 \times 165}{2 \times 4} = 212700 \text{ pounds per ft.,}$$

$$H = \frac{\gamma h^2}{2} = \frac{62.5 \times 165 \times 165}{2} = 850780 \text{ pounds per ft.,}$$

$$W = \frac{\delta(b_1 + b_2)h_1}{2} = 150 \times 98 \times 170 = 2\,499\,000 \text{ pounds per ft.}$$

For the climate of San Mateo we have no ice. For wave-thrust, taking $T = 24000$, we have, from (I), page 427, for security against sliding

$$n = \frac{\mu(W + V)}{T + H} = \frac{0.6 \times 2711700}{874780} = 1.8.$$

There are no through joints, and sliding is impossible by construction. But even if it were not, the dam would be safe even if wave-thrust is taken into account, although the coefficient of safety is not as high as 2.

From (4), page 427, we have $\bar{x} = 101$ feet. From (II), page 428, we have then, taking $T = 24000$, for wave-pressure $e = 87$ feet. This is greater than $\frac{1}{3}b_2 = 58\frac{2}{3}$ ft. and less than $\frac{2}{3}b_2 = 117\frac{1}{3}$ ft. The resultant R of the weight, pressure and wave-pressure cuts the base, then, within the middle third.

From the third of equations (III), page 428, then, we have for the greatest unit pressure

$$p = 15930 \text{ pounds per sq. ft.}$$

This is much less than the allowable unit stress $C = 34000$ pounds per sq. ft.

If the dam is empty, we have $H = 0$, $V = 0$, $T = 0$, and $e = \bar{x} = 101$ ft. The weight then cuts the base within the middle third, and we have

$$p = 8579 \text{ pounds per sq. ft.}$$

The dam is then stable, never overloaded, and safe even for through joints and wave-pressure.

Design of Dam—Trapezoid Section.—If we insert the value of T as given (page 432), and the values of V and H given by equations (2) and (3), page 431, in the general equations (I), (II) and (III), pages 429 to 430, we can design a dam of trapezoid section.

LIMIT b_2.—From (I), page 429, we have then

$$limit\ h_1 = \frac{b_2 C - \gamma h^2 \tan \beta_1}{\delta(b_1 + b_2)}. \qquad \text{(I)}$$

Equation (I) gives the limit of h_1 for low dam of trapezoid section for given top base b_1 and bottom base b_2. For h_1 greater than this limit the dam is high.

For rectangular section we have $b_2 = b_1$, and hence

$$limit\ h_1 = \frac{b_1 C - \gamma h^2 \tan \beta_1}{2\delta b_1}. \qquad (1)$$

Equation (1) gives the limit of h_1 for low dam of rectangular section. If h_1 is greater than this, the dam is high.

LOW DAM—BOTTOM BASE.—From (II), page 429, we have for low dam

$$b_2 = -B_1 + \sqrt{B_1^2 + E_1}, \qquad \text{(II)}$$

where the quantities B_1 and E_1 are given by

$$B_1 = \frac{1}{2}b_1 + \frac{\gamma h_1 \tan \beta_1}{2\delta}\left(\frac{2h^2}{h_1^2} - \frac{\delta}{\gamma}\right),$$

$$E_1 = b_1(b_1 + 2h_1 \tan \beta_1) + \frac{\gamma h^3(1 + \tan^2 \beta_1) + 6Th}{\delta h_1}.$$

Equation (II) gives the bottom base b_2 for a properly designed low dam of trapezoid section. If the value of b_2 as given by (II), substituted in (I), gives the limit h_1 less than the actual height, the dam is high and (II) does not apply.

LOW DAM—TOP BASE.—For economy the top base should be assumed as small as possible consistent with practical and local considerations. We should not in any case assume b_1 greater than b_2. If then we make $b_2 = b_1$ in (II), we have rectangular section, and for the greatest allowable value of b_1

$$max.\ b_1 = -B + \sqrt{B^2 + E}, \qquad (2)$$

where the quantities B and E are given by

$$B = \frac{\gamma h_1 \tan \beta_1}{2\delta}\left(\frac{2h^2}{h_1^2} - \frac{3\delta}{\gamma}\right), \qquad E = \frac{\gamma h^3(1 + \tan^2 \beta_1) + 6Th}{\delta h_1}.$$

Equation (2) gives the largest allowable value of b_1 for low dam, trapezoid section; that is, when h_1 is less than or equal to limit h_1 as given by (1).

LOW DAM—BEST VALUE OF β_1.—We see from the table page 424 that $\frac{\delta}{\gamma}$ is greater than 2 for all materials except brickwork and small dry rubble. The ratio $\frac{h^2}{h_1^2}$ can never exceed unity. For all materials, then, except brickwork and small dry rubble the quantity

$\left(\frac{2h^2}{h_1^2} - \frac{\delta}{\gamma}\right)$ is always negative, and even for the last two materials it is also negative if h is not greater than $\frac{9}{10}h_1$.

In general, then, as β_1 decreases the value of B_1 increases and the value of E decreases. Also, B_1 increases more rapidly than $\sqrt{B_1^2 + E}$. Hence b_2 in equation (II) has its least value when $\beta_1 = 0$, or *the most economic section for low dam, trapezoid section, is that for which the back is vertical.*

LOW DAM—ECONOMIC TRAPEZOID SECTION.—If then we make $\beta_1 = 0$ in (II), we have for low dam, economic trapezoid section,

$$b_2 = -\frac{b_1}{2} + \sqrt{\frac{5b_1^2}{4} + \frac{\gamma h^3 + 6Th}{\delta h_1}}, \quad \ldots \ldots \quad \text{(II}')$$

and for the greatest allowable value of b_1, making $b_2 = b_1$, we have for rectangular section

$$max.\ b_1 = \sqrt{\frac{\gamma h^3 + 6Th}{\delta h_1}} \quad \ldots \ldots \quad (2')$$

Equation (2′) gives the value of b_1 for low dam, which makes the economic section rectangular.

HIGH DAM—BOTTOM BASE.—Inserting the values of V and H in (III), page 43, we have for high dam

$$b_2 = -B_2 + \sqrt{B_2^2 + E_2}, \quad \ldots \ldots \quad \text{(III)}$$

where the quantities B_2 and E_2 are given by

$$B_2 = \frac{\gamma h_1^2 \tan \beta_1}{2C}\left(\frac{h^2}{h_1^2} - \frac{\delta}{\gamma}\right), \qquad E_2 = \frac{\delta h_1 b_1}{C}(b_1 + 2h_1 \tan \beta_1) + \frac{\gamma h^3}{C}(1 + \tan^2 \beta_1) + \frac{6Th}{C}.$$

Equation (III) gives the bottom base b_2 for a properly designed high dam of trapezoid section.

HIGH DAM—TOP BASE.—For economy the top base should be assumed as small as possible consistent with practical and local considerations. We should not in any case assume b_1 greater than b_2. If then we make $b_2 = b_1$ in (III), we have for the greatest allowable value of b_1

$$max.\ b_1 = -B + \sqrt{B^2 + E}, \quad \ldots \ldots \quad (3)$$

where the quantities B and E are given by

$$B = \frac{\gamma h_1^2 \tan \beta_1}{2(C - \delta h_1)}\left(\frac{h^2}{h_1^2} - \frac{\delta}{\gamma}\right), \qquad E = \frac{\gamma h^3}{C - \delta h_1}(1 + \tan^2 \beta_1) + \frac{6Th}{C - \delta h_1}.$$

Equation (3) gives the largest allowable value for b_1 for high dam, trapezoid section.

HIGH DAM—BEST VALUE OF β_1.—We see from the table page 424 that $\frac{\delta}{\gamma}$ is greater than unity for all materials. The ratio $\frac{h^2}{h_1^2}$ can never exceed unity. For all materials, then, the quantity $\left(\frac{h^2}{h_1^2} - \frac{\delta}{\gamma}\right)$ is always negative, and $-B_2$ is therefore always positive and decreases

as β_1 decreases. So also does E_2. Hence b_2 in (III) has its least value when $\beta_1 = 0$, or *the most economic section for high dam of trapezoid section is that for which the back is vertical.*

HIGH DAM—ECONOMIC TRAPEZOID SECTION.—If then we make $\beta_1 = 0$ in (III), we have for high dam, economic trapezoid section,

$$b_2 = \sqrt{\frac{\delta h_1 b_1^2}{C} + \frac{\gamma h^3 + 6Th}{C}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{III}')$$

and for the greatest allowable value of b_1

$$max.\ b_1 = \sqrt{\frac{\gamma h^3 + 6Th}{C - \delta h_1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3')$$

CONDITIONS FOR RECTANGULAR SECTION.—If we make $b_2 = b_1$ in (I), and $\beta_1 = 0$, we have for limit of h_1 for low dam of rectangular section

$$h_1 = \frac{C}{2\delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and from (2′) for the value of b_1

$$b_1 = \sqrt{\frac{\gamma h^3 + 6Th}{\delta h_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From this, if we make $h = h_1$ and solve for h_1; we have for water level with top of dam

$$h_1 = \sqrt{\frac{\delta b_1^2}{\gamma} - \frac{6T}{\gamma}}.$$

We see from this that if b_1 is equal to $\sqrt{\frac{6T}{\delta}}$, we have $h_1 = 0$, and if b_1 is less than this, h_1 is imaginary. We have then

$$min.\ b_1 = \sqrt{\frac{6T}{\delta}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and from (5), making $h = h_1$, and inserting the value of h_1 from (4),

$$max.\ b_1 = \sqrt{\frac{\gamma C^2}{4\delta^3} + \frac{6T}{\delta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Local and practical considerations must control the choice of top base b_1, and we should take it as small as such considerations will allow. We should not take it greater than given by (7). If we take it greater than given by (6), the section can be rectangular.

If we take it less than given by (6), the section cannot be rectangular.

Examples.—(1) *Design a rectangular dam of granite ashlar 20 feet high, the water-level to be 2 feet below the top, with and without ice-pressure.*

ANS. We have $h_1 = 20$, $h = 18$, $b_2 = b_1$, $\beta_1 = 0$, $P = 40000$, and, from table page 424, $C = 60000$ $\delta = 165$.

From equation (I), page 434, making $\beta_1 = 0$ and $b_2 = b_1$, we have for the limit h_1 for low dam

$$limit\ h_1 = \frac{C}{2\delta} = \frac{60000}{330} = 181.8 \text{ ft}.$$

The dam is therefore low; that is, when $e = \frac{1}{3}b_1$, p is less than C.

From equation (II′), page 435, making $b_2 = b_1$, we have for the value of b_1 in order that e shall be just equal to $\frac{1}{3}b_1$, without ice-pressure,

$$b_1 = \sqrt{\frac{\gamma h^3}{\delta h_1}} = \sqrt{\frac{62.5 \times 5832}{165 \times 20}} = 10.5 \text{ ft.}, \qquad A = 210 \text{ sq. ft.}$$

With ice-pressure

$$b_1 = \sqrt{\frac{\gamma h^3 + 6Th}{\delta h_1}} = \sqrt{\frac{62.5 \times 5832 + 6 \times 40000 \times 18}{165 \times 20}} = 37.67 \text{ ft.}, \qquad A = 753.4 \text{ sq. ft.}$$

We see that the breadth must be quite large in order to bring the entire base into action.

(2) *Design a trapezoidal dam of granite ashlar 20 feet high with vertical face, the tangent of the back batter angle being $\frac{1}{10}$ and the water-level 2 feet below the top, with and without ice-pressure.*

ANS. We have $h_1 = 20$, $h = 18$, $\beta = 0$, $\tan \beta_1 = \frac{1}{10}$, $b_2 = b_1 + 2$, $T = 40000$, and, from table page 424, $\delta = 165$, $C = 60000$.

From equation (II), page 434, making $b_2 = b_1 + 2$ and solving for b_1, we have for low dam

without ice-pressure $\qquad b_1 = -0.61 + \sqrt{109.48} = 9.85$ ft., hence $b_2 = 11.85$ ft.;

with ice-pressure $\qquad b_1 = -0.61 + \sqrt{1418.55} = 37.05$ ft., hence $b_2 = 39.05$ ft.

If now in (I) page 434, we insert these values of b_1 and b_2, we find that the limit of h_1 for low dam in both cases is much greater than the height $h_1 = 20$. The dam is therefore low in both cases, and the values of b_1 and b_2 just found hold good.

We have then for the area of cross-section

without ice-pressure $\qquad A = 217$ sq. ft.,

with ice-pressure $\qquad A = 761$ sq. ft.

We see by reference to the preceding example that we have saved no material by having a back batter. We could, however, save material by having a back and a face batter, as we shall see from the next example.

(3) *Design a trapezoidal dam of granite ashlar 20 feet high, the tangent of the front and back batter angles being $\frac{1}{10}$ and the water-level 2 feet below the top, with and without ice-pressure.*

ANS. We have $h_1 = 20$, $h = 18$, $\tan \beta_1 = \tan \beta = \frac{1}{10}$, $b_2 = b_1 + 4$, $T = 40000$, $\delta = 165$, $C = 60000$ (table page 424).

From equation (II), page 434, making $b_2 = b_1 + 4$ and solving for b_1, we have for low dam without ice-pressure

$$b_1 = -3.52 + \sqrt{126.65} = 7.73 \text{ ft., and hence } b_2 = 11.73;$$

and with ice-pressure

$$b_1 = -3.52 + \sqrt{1435.65} = 34.36 \text{ ft., and hence } b_2 = 38.36.$$

From (I), page 434, inserting these values of b_1 and b_2, we have for the limit of h_1 for low dam

without ice-pressure $\qquad$ *limit* $h_1 = 218.5$ ft.,

with ice-pressure $\qquad$ *limit* $h_1 = 191.6$ ft.

Both these limits are greater than 20 ft. The dam is then low in both cases, and the values of b_1 and b_2 just found hold good.

We have then for the area of cross-section

without ice-pressure $\qquad A = 194.6$ sq. ft.,

with ice-pressure $\qquad A = 727.2$ sq. ft.

We see by reference to the preceding examples that we have saved material by giving the face and back a batter.

We might, however, have saved more still by making the back vertical, as we shall see from the next example.

(4) *Design a trapezoidal dam of granite ashlar 20 feet high with vertical back, the tangent of the front batter angle being $\frac{1}{10}$ and the water-level 2 feet below the top, with and without ice-pressure.*

ANS. We have $h_1 = 20$, $h = 18$, $\beta_1 = 0$, $\tan\beta = \frac{1}{10}$, $b_2 = b_1 + 2$, $T = 40000$, and, from table page 424, $\delta = 165$, $C = 60000$.

From equation (II), page 434, making $b_2 = b_1 + 2$ and solving for b_1, we have for low dam without ice-pressure

$$b_1 = -3 + \sqrt{115.45} = 7.74 \text{ ft., and hence } b_2 = 9.74 \text{ ft.;}$$

and with ice-pressure

$$b_1 = -3 + \sqrt{1424.54} = 34.74 \text{ ft., and hence } b_2 = 36.74 \text{ ft.}$$

From (I), page 434, inserting these values of b_1 and b_2, we have for the limit of h_1 for low dam

without ice-pressure $\quad$ *limit* $h_1 = 268.5$ ft.,

with ice-pressure $\quad$ *limit* $h_1 = 188.5$ ft.

Both these limits are greater than 20. The dam is then low in both cases, and the values of b_1 and b_2 just found hold good.

We have then for the area of cross-section

without ice-pressure $\quad A = 174.8$ sq. ft.,

with ice-pressure $\quad A = 714.8$ sq. ft.

This is the most economic section for the given proportions.

We see, then, that there is a saving of material over the preceding examples by making the back vertical. We could save still more, however, by increasing the front batter angle or decreasing the top base, as in the next example.

(5) *Design a trapezoidal dam of granite ashlar 20 feet high, top base 2 feet, water-level 2 feet below top, for economic section.*

ANS. We have $h_1 = 20$, $h = 18$, $b_1 = 2$, $T = 40000$, $\delta = 165$, $C = 60000$, and for economic section $\beta_1 = 0$.

From (II'), page 435, we have for low dam

without ice-pressure $\quad b_2 = -1 + \sqrt{5 + \frac{62.5 \times 5832}{165 \times 20}} = 10.16$ ft.,

with ice-pressure $\quad b_2 = -1 + \sqrt{5 + \frac{62.5 \times 5832}{165 \times 20} + \frac{6 \times 40000 \times 18}{165 \times 20}} = 36.74$ ft.

From (I), page 434, inserting these values of b_2, we have for the limit of h_1 for low dam

without ice-pressure $\quad$ *limit* $h_1 = \frac{b_2 C}{\delta(b_1 + b_2)} = \frac{10.16 \times 60000}{165 \times 12.16} = 303.8$ ft.,

with ice-pressure $\quad$ *limit* $h_1 = \frac{36.74 \times 60000}{165 \times 38.74} = 344.8$ ft.

Both these limits are greater than 20 ft. The dam is then low in both cases and the values of b_2 just found hold good.

We have then for the area of cross-section

without ice-pressure $\quad A = 121.6$ sq. ft.,

with ice-pressure $\quad A = 387.4$ sq. ft.

We see that there is a large saving of material over the preceding cases. This is the most economic section for the given dimensions. We could save more only by reducing the top base still more.

(6) *Design the San Mateo dam* (page 433). *taking the top base at 20 feet and the back batter 1 to 4 as actually built, and taking into account wave-pressure.*

ANS. We have $h_1 = 170$, $h = 165$, $b_1 = 20$, $\tan\beta_1 = \frac{1}{4}$, $\gamma = 62.5$, $\delta = 150$, $C = 34000$, $T = 24000$.

From equation (II), page 434, we have for low dam

$$b_2 = -5\,432 + \sqrt{29.51 + 15461.38} = 119.03 \text{ ft.}$$

From equation I), page 434, we have for the limit of h_1 for low dam

$$limit\ h_1 = \frac{119.03 \times 34000}{150 \times 139.03} - \frac{4 \times 150 \times 139.03}{62.5 \times 165^2} = 194 \text{ ft.}$$

This is greater than the height 170 ft. The dam is therefore low and the value of b_2 just found holds good.

We have then the area of cross-section

$$A = 11818 \text{ sq. ft.}$$

The weight per foot of length is

$$W = \delta A = 150 \times 11818 = 1\,772\,700 \text{ pounds per ft.,}$$

the vertical pressure V (page 431) is

$$V = \frac{\gamma l h^2}{2} \tan \beta_1 = \frac{62.5 \times 170^2}{8} = 22578 \text{ pounds per ft.,}$$

and from the second of equations (III), page 428, the greatest unit pressure is

$$p = \frac{2(W + V)}{b_2 l} = 33580 \text{ pounds per sq. ft.,}$$

or less than the allowable stress $C = 34000$ pounds per sq. ft., the value of e being $\frac{1}{3}b_2$.

The dam as actually built (see page 433) has a base $b_2 = 176$ ft., an area $A = 16660$ sq. ft., the value of e is greater than $\frac{1}{3}b_2$, and the greatest unit pressure $p = 15930$ pounds per sq. ft.

We see, then, that by properly designing we have for $e = \frac{1}{3}b_2$, so that the entire base is still in action, a saving of over 29 per cent, and the dam is still safe against rotation and crushing.

For sliding we have, as on page 427, for the coefficient of safety

$$n = \frac{\mu(W + V)}{H + T} = \frac{0.6 \times 1998480}{874780} = 1.37.$$

Since n is greater than unity, the resistance to sliding for through joint at base is greater than the thrust $H + T$; but if friction only is relied upon, the coefficient is not large enough. By breaking joints, however, sliding is impossible.

By making the back vertical we could make a still greater saving, as we see from the next example.

(7) *Design the San Mateo dam for economic section, taking into account wave-pressure and the top base 20 ft. as actually built.*

ANS. We have $b_1 = 20$, $h_1 = 170$, $h = 165$, $\gamma = 62.5$, $\delta = 150$, $c = 34000$, $T = 24000$, and for economic section $\beta_1 = 0$.

From equation (II′), page 435, we have for low dam

$$b_2 = -10 + \sqrt{500 + \frac{62.5 \times (165)^3}{150 \times 170} + \frac{6 \times 24000 \times 165}{150 \times 170}} = 101.54 \text{ ft.}$$

From equation (I), page 434, we have for the limit of h_1 for low dam

$$limit\ h_1 = \frac{101.5 \times 34000}{150 \times 121.54} = 189 \text{ ft.}$$

This is greater than the height 170 ft. The dam is therefore low and the value of b_2 just found holds good

We have then the area of cross-section

$$A = 10331 \text{ sq. ft.,}$$

the weight per foot of length

$$W = \delta A = 150A = 1\,549\,650 \text{ pounds per ft.},$$

and from the second of equations (III), page 428, the greatest unit pressure

$$p = \frac{2W}{b_2} = 30523 \text{ pounds per sq. ft.},$$

or less than the allowable unit stress $C = 34000$ pounds per sq. ft.

The dam as actually built (see page 433) has a base $b_2 = 176$ ft., a back batter of 1 to 4, and an area $A = 16660$ sq. ft. The value of e is greater than $\frac{1}{3}b_2$, and the greatest unit pressure is only $p = 15930$ pounds per sq. ft.

By making the back vertical and $e = \frac{1}{3}b_2$, so that the entire base is still in action, we save over 38 per cent on the section, and the dam is safe against rotation and crushing.

For sliding we have, as on page 427, for the coefficient of safety

$$n = \frac{\mu W}{H + T} = \frac{0.6 \times 1549650}{874780} = 1.06.$$

Since n is greater than unity, the resistance to sliding for through joint at base is greater than the thrust $H + T$, but the coefficient is not large enough if friction only is relied upon. By breaking joints, however, sliding is impossible.

(8) *Suppose the height in the last example to be taken at 200 ft.; the water level 5 ft. from the top.*

ANS. We have $b_1 = 20$, $h_1 = 200$, $h = 195$, $\gamma = 62.5$, $\delta = 150$, $C = 34000$, $T = 24000$, and for economic section $\beta_1 = 0$.

From equation (II′), page 435, we have for low dam

$$b_2 = -10 + \sqrt{\frac{500 + 62.5 \times (195)^3 + 6 \times 24000 \times 195}{150 \times 200}} = 120 \text{ ft.}$$

From equation (I), page 434, we have for the limit of h_1 for low dam

$$limit\ h_1 = \frac{120 \times 34000}{150 \times 140} = 197 \text{ ft.}$$

This is less than the height 200 ft. The dam is therefore high and the value of b_2 just found does not hold good. We should then use equation (III′), page 436. We obtain, then,

$$b_2 = \sqrt{\frac{150 \times 200 \times (20)^2 + 62.5 \times (195)^3 + 6 \times 24000 \times 195}{34000}} = 120.38 \text{ ft.}$$

The area of cross-section is then

$$A = 14038 \text{ sq. ft.}$$

Design of Dam—Economic Section.—We have seen, pages 435 and 436, that for low and high dam and trapezoid section the greatest economy is for back vertical. We have also seen, page 436, that *for rectangular section* b_1 must not be greater than

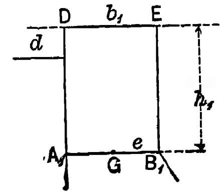
$$max.\ b_1 = \sqrt{\frac{\gamma C^2}{4\delta^3} + \frac{6T}{\delta}}$$

nor less than

$$min.\ b_1 = \sqrt{\frac{6T}{\delta}}. \qquad (1)$$

Local and practical considerations must control the choice of top base b_1, and we should take it as small as such considerations will allow.

FIRST SUB-SECTION.—For given top base b_1, then, less than *max.* b_1 and greater than *min.* b_1 as given by (1), the section should be a rectangle, A_1B_1DE. We can run this rectangular section down to a distance h_1 below the top, until e shall be just equal to $\frac{1}{3}b_1$, so that the entire joint A_1B_1 acts, provided this joint is not overloaded.



We have from equation (2′), page 435, for rectangular section and $e = \frac{1}{3}b_1$

$$\delta b_1^2 h_1 = \gamma h^3 + 6Th.$$

Let d be the distance of the water-level below the top. Then $h = h_1 - d$. Substituting this value of h and solving for h_1, we have

$$\gamma h_1^3 - 3\gamma d h_1^2 - (\delta b_1^2 - 3\gamma d^2 - 6T)h_1 = \gamma d^3 + 6Td. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

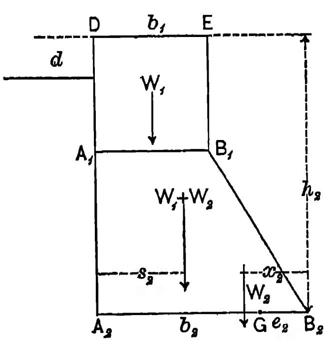
Equation (2) will give the value of h_1 for the first sub-section.
From equation (I), page 434, we have for the limit of h_1 for rectangular section

$$limit\ h_1 = \frac{C}{2\delta} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We can then run the first rectangular sub-section A_1B_1ED down for the distance h_1 given by equation (2) without overloading, provided that h_1 as given by (2) is less than the limit of h_1 given by (3).

If b_1 is taken less than $\sqrt{\frac{6T}{\delta}}$, there can be no rectangular sub-section, but the first sub-section should be a trapezoid, as given in the next article.

SECOND SUB-SECTION.—If the height of dam is greater than the value of h_1 as given by equation (2), we still continue the back vertical, but the breadth must increase so that e_2 shall be equal to $\frac{1}{3}b_2$. We have then a second sub-section, $A_1B_1B_2A_3$, with front batter and vertical back. Since the vertical pressure remains unchanged for dam empty, we can run this second sub-section down to a distance h_2 below the top, until the back edge distance s_2 for dam empty is also equal to $\frac{1}{3}b_2$, provided the greatest unit stress p is less than or equal to C. Or we can run it down until $p = C$, provided s_2 is greater than or equal to $\frac{1}{3}b_2$.



We have then two cases, $s_2 = e_2 = \frac{1}{3}3b_2$, and p less than or equal to C, and $e_2 = \frac{1}{3}b_2$, $p = C$, and s_2 greater than or equal to e_2.

The height of the second sub-section is then $(h_2 - h_1)$. It is required to find h_2 and b_2.

Since the resultant for full dam is to cut the base $b_2 = A_2B_2$ in both cases at a distance $e_2 = \frac{1}{3}b_2$ from B_2, we have in both cases

$$e_2 = \frac{W_1\left(b_2 - \frac{b_1}{2}\right) + W_2\bar{x}_2 - \frac{H(h_2 - d)}{3} - T(h_2 - d)}{W_1 + W_2}.$$

From equation (5), page 427,

$$\bar{x}_2 = \frac{2b_2^2 + 2b_1b_2 - b_1^2}{3(b_1 + b_2)}.$$

We have also

$$W_1 = \delta b_1 h_1, \qquad W_2 = \frac{\delta(b_1 + b_2)(h_2 - h_1)}{2}, \qquad H = \frac{\gamma(h_2 - d)^2}{2}.$$

Inserting these values, making $e_2 = \frac{1}{3}b_2$, and solving for b_2, we have for the base b_2 of the second sub-section in both cases

$$b_2 = -B_2 + \sqrt{B_2^2 + E_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the quantities B_2 and E_2 are given by

$$B_2 = \frac{(h_2 + 3h_1)b_1}{2(h_2 - h_1)}, \qquad E_2 = b_1^2 + \frac{3b_1^2h_1}{h_2 - h_1} + \frac{\gamma(h_2 - d)^3 + 6T(h_2 - d)}{\delta(h_2 - h_1)}.$$

If we have b_1 less than $\sqrt{\frac{6T}{\delta}}$, there is no rectangular section (page 436), and we have $h_1 = 0$ in (4).

The weight W_1 per foot of length of the first sub-section acts at a distance $\frac{1}{2}b_1$ from A_2 and W_2 at the distance $(b_2 - \bar{x}_2)$ from A_2. The resultant $W_1 + W_2$ acts then at the distance s_2 from A_2 given by

$$s_2 = \frac{\frac{W_1b_1}{2} + W_2(b_2 - \bar{x}_2)}{W_1 + W_2}.$$

Inserting the values of W_1, W_2 and $\bar{x}_2$, we have

$$s_2 = \frac{b_1^2(h_2 + 2h_1) + b_1b_2(h_2 - h_1) + b_2^2(h_2 - h_1)}{3b_1(h_2 + h_1) + 3b_2(h_2 - h_1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When $s_2 = \frac{1}{3}b_2$ we have, solving for h_2 for the limit of h_2,

$$\left.\begin{array}{l}\text{when } e_2 = s_2 = \frac{1}{3}b_2 \text{ and} \\ p \text{ less than or equal to } C,\end{array}\right\} \textit{limit } h_2 = \frac{2h_1(b_2 - b_1)}{b_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and when $e_2 = \frac{1}{3}b_2$ and $p = C$, we have from the second of equations (III), page 428,

$$b_2 = \frac{2(W_1 + W_2)}{C},$$

or, inserting the values of W_1 and W_2 and solving for h_2, we have for the limit of h_2

when $s_2 > \frac{1}{3}b_2 = e_2$ and p equal to C,

$$limit\ h_2 = \frac{Cb_2 + \delta h_1(b_2 - b_1)}{\delta(b_1 + b_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

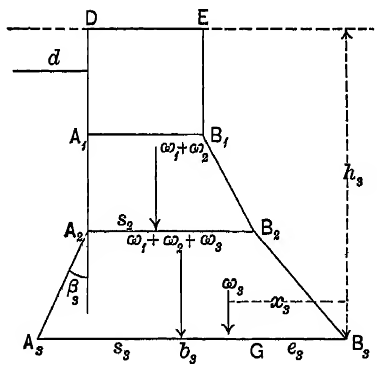
If we substitute the value of b_2 given by (4) in equations (6) and (7), the *least of the two values for limit h_2 thus obtained* will be the limiting value of h_2 for the second sub-section. We can then find b_2 from (4) for any value of h_2 up to this limiting value.

If we have b_1 less than $\sqrt{\frac{6T}{d}}$, there is no rectangular section, and we make $h_1 = 0$ in (6) and (7). The limit h_2 in this case is

$$limit\ h_2 = \frac{Cb_2}{\delta(b_1 + b_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

THIRD SUB-SECTION.—If the height of the dam is greater than the limiting value of h_2 just found, we have a third sub-section. The limit h_2 for the second sub-section may be given by equation (6) or by equation (7) as we have seen, and we thus have two cases to consider.

1st Case.—If the limit h_2 is given by equation (6), that is, if $e_2 = s_2 = \frac{1}{3}b_2$ and p less than or equal to C, we must batter both face and back, so that e_3 shall be greater than $\frac{1}{3}b_3$ and $p = C$ for dam full. We have then the sub-section $A_2B_2B_3A_3$ with front and back batter. The back batter angle we denote by β_3.



We can run this sub-section *down to the base of the dam*, so that h_3 is the given height of dam; and since the vertical pressure is unchanged for dam empty, we should have the edge distance s_3 from A_3 for dam empty equal to the edge distance e_3 from B_3 for dam full. It is required to find b_3 and β_3.

The resultant $W_1 + W_2$ is at the distance $s_2 = \frac{1}{3}b_2$ from A_2 for dam empty. The weight W_3 is at a distance $\bar{x}_3$ from B_3 given from equation (5), page 427, by

$$\bar{x}_3 = \frac{2b_3^2 + 2b_2b_3 - b_2^2 - (h_3 - h_2)(2b_2 + b_3)\tan\beta_3}{3(b_2 + b_3)}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

There is a vertical water-pressure V on the inclined back A_2A_3. For the sake of simplicity we neglect V. This omission, which is on the side of safety, involves no practical error.

Neglecting V, then, we have the edge distance s_3 from A_3 given by

$$s_3 = \frac{(W_1 + W_2)\left[\frac{b_2}{3} + (h_3 - h_2)\tan\beta_3\right] + W_3(b_3 - \bar{x}_3)}{W_1 + W_2 + W_3}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For dam full, neglecting V, we have the edge distance e_3 from B_3,

$$e_3 = \frac{(W_1+W_2)\left[b_3 - \frac{b_2}{3} - (h_3 - h_2)\tan\beta_3\right] + W_3\bar{x}_3 - \frac{H(h_3-d)}{3} - T(h_3-d)}{W_1+W_2+W_3}, \qquad (11)$$

where W_1 and W_2 are already known and

$$W_3 = \frac{\delta(b_2+b_3)(h_3-h_2)}{2}, \qquad H = \frac{\gamma(h_3-d)^2}{2}.$$

From the third of equations (III), page 428, we have when e_3 is greater than $\frac{1}{3}b_3$ and $p = C$, neglecting V,

$$e_3 = \frac{2}{3}b_3 - \frac{Cb_3^2}{6(W_1+W_2+W_3)}. \qquad (12)$$

If we equate (11) and (12), insert the values of W_3, H and $\bar{x}_3$, and solve for b_3, we have for the base b_3

$$b_3 = -B_3 + \sqrt{B_3^2 + E_3}, \qquad (13)$$

where the quantities B_3 and E_3 are given by

$$B_3 = \frac{2(W_1+W_2) - \delta(h_3-h_2)^2\tan\beta_3}{2C},$$

$$E_3 = \frac{6(W_1+W_2)}{C}\left[\frac{b_2}{3} + (h_3-h_2)\tan\beta_3\right] + \frac{\delta(h_3-h_2)b_2}{C}\left[b_2 + 2(h_3-h_2)\tan\beta_3\right] + \frac{\gamma(h_3-d)^3 + 6T(h_3-d)}{C}.$$

If we have b_1 less than $\sqrt{\frac{6T}{\delta}}$, there is no rectangular section (page 436), and we make $W_1 = 0$, $h_1 = 0$.

If we equate (10) and (12), insert the value of W_3, and solve for B_3, we have for B_3

$$\tan\beta_3 = \frac{2(W_1+W_2)(2b_3-b_2) + \delta(h_3-h_2)(b_3^2 + b_2b_3 - b_2^2) - Cb_3^2}{(h_3-h_2)[\delta(h_3-h_2)(b_3+2b_2) + 6(W_1+W_2)]}. \qquad (14)$$

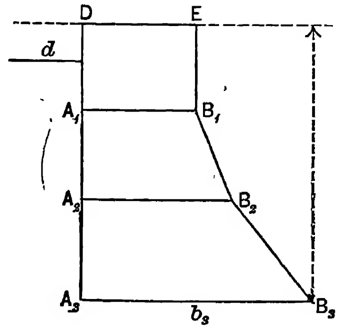
If we have b_1 less than $\sqrt{\frac{6T}{\delta}}$, there is no rectangular section (page 436), and we make $W_1 = 0$, $h_1 = 0$.

From (13) and (14) we can find b_3 and β_3.

2d Case.—If the limit h_2 is given by equation (7), that is, if s_2 is greater than $e_2 = \frac{1}{3}b_2$ and p equal to C, and if the height of the dam is greater than this limit, we have also a third sub-section.

In this case we still continue the back vertical, but the breadth must increase so that $e_3 = s_3$ shall be greater than $\frac{1}{3}b_3$ and $p = C$. We have then the third sub-section $A_2B_2B_3A_3$ with vertical back.

Since the vertical pressure remains unchanged for dam empty, we can run this sub-section down with vertical back until the edge distance s_3 from the back for dam empty is equal to the edge distance e_3 from B_3 for dam full. The height of this sub-section is $h_3 - h_2$. It is required to find h_3 and b_3.



Evidently we have only to make $\beta_3 = 0$ and $\frac{b_2}{3} = s_2$ in equations (13) and (14) and we have

$$b_3 = -B + \sqrt{B_3^2 + E_3}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where the quantities B_3 and E_3 are given by

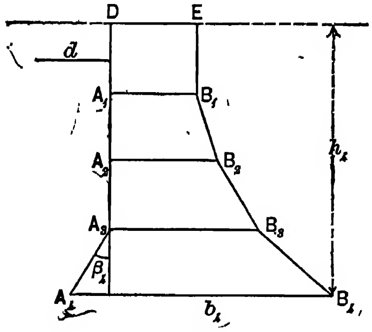
$$B_3 = \frac{W_1 + W_2}{C}, \qquad E_3 = \frac{6(W_1 + W_2)s_2}{C} + \frac{\delta(h_3 - h_2)b_2^2}{C} + \frac{\gamma(h_3 - d)^3 + 6T(h_3 - d)}{C},$$

$$limit\ h_3 = h_2 + \frac{Cb_3^2 - 2(W_1 + W_2)(2b_3 - 3s_2)}{\delta(b_3^2 + b_2 b_3 - b_2^2)}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If we substitute the value of b_3 given by (15) in (16) and solve for h_3, we shall have the limiting value of h_3 for the third sub-section in this case. We can then find b_3 from (15) for any value of h_3 up to this limiting value.

If we have b_1 less than $\sqrt{\frac{6T}{\delta}}$, there is no rectangular section (page 436), and we make $W_1 = 0$ and $h_1 = 0$.

FOURTH SUB-SECTION.—If the height of the dam is greater than the limiting value of h_3, when we have the third sub-section with vertical back, we must batter both face and back so that e_4 shall be greater than $\frac{1}{3}b_4$ and $p = C$ for dam full.



We have then a fourth sub-section, $A_3B_3B_4A_4$, with front and back batter. We denote the back batter angle by β_4.

We can run this sub-section *down to the base of the dam*, so that h_4 is the given height of dam; and since the vertical pressure is unchanged for dam empty, we should have the edge distance s_4 from A_4 for dam empty equal to the edge distance e_4 from B_4 for dam full. It is required to find b_4 and β_4.

In this case we have from (9) and (10), by making $\beta_3 = 0$,

$$\bar{x}_3 = \frac{2b_3^2 + 2b_2b_3 - b_2^2}{3(b_2 + b_3)},$$

$$s_3 = \frac{(W_1 + W_2)s_2 + W_3(b_3 - \bar{x}_3)}{W_1 + W_2 + W_3},$$

and from (13) and (14)

$$b_4 = -B_4 + \sqrt{B_4^2 + E_4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where the quantities B_4 and E_4 are given by

$$B_4 = \frac{2(W_1 + W_2 + W_3) - \delta(h_4 - h_3)^2 \tan \beta_4}{2C}$$

$$E_4 = \frac{6(W_1+W_2+W_3)}{C}[s_3+(h_4-h_3)\tan\beta_4] + \frac{\delta(h_4-h_3)b_3}{C}[b_3+2(h_4-h_3)\tan\beta_4] + \frac{\gamma(h_4-d)^3+6T(h_4-d)}{C},$$

$$\tan \beta_4 = \frac{2(W_1+W_2+W_3)(2b_4-3s_3)+\delta(h_4-h_3)(b_4^2+b_3b_4-b_3^2)-Cb_4^2}{(h_4-h_3)[\delta(h_4-h_3)(b_4+2b_3)+6(W_1+W_2+W_3)]}. \quad . \quad (18)$$

From (17) and (18) we can find b_4 and β_4.

If we have b_1 less than $\sqrt{\frac{6T}{\delta}}$, there is no rectangular section (page 436) and we make $W_1 = 0$, $h_1 = 0$.

For dam full and empty $p = C$, and in both cases $e_4 = s_4$, or, from (12),

$$e_4 = s_4 = \frac{2}{3}b_4 - \frac{Cb_4^2}{6(W_1+W_2+W_3+W_4)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Examples.—(1) *The height of the proposed Quaker Bridge Dam, New York, is 170 ft., top thickness 20 ft., density of masonry 150 pounds per sq. ft., depth of water 163 ft., allowable compressive stress 20000 pounds per sq. ft. Find the economic section without ice-pressure or wave-thrust.*

ANS. We have $d = 7$, $b_1 = 20$, $C = 20000$, $\delta = 150$, $\gamma = 62.5$, and without ice-pressure or wave-thrust $T = 0$.

The top base $b_1 = 20$ is less than maximum b_1 and greater than minimum h_1 as given by equations (1) page 440, viz.,

$$max.\ b_1 = \sqrt{\frac{\gamma C^2}{4\delta^3}} = 43 \text{ ft.} \quad \text{and} \quad min.\ b_1 = \sqrt{\frac{6T}{\delta}} = 0.$$

The first sub-section is then rectangular.

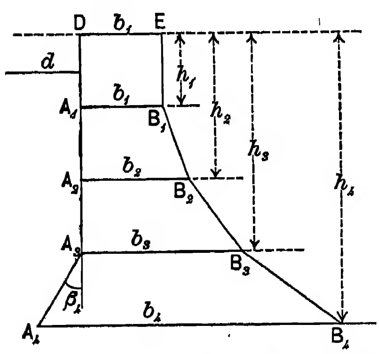
First Sub-section.—For the height of this rectangular sub-section we have, from equation (2), page 441,

$$h_1^3 - 21h_1^2 - 813h_1 = 343, \quad \text{or} \quad h_1 = 41.02 \text{ ft.}$$

This value of h_1 is less than the limit h_1 given by equation (3), page 441, viz.,

$$limit\ h_1 = \frac{C}{2\delta} = \frac{20000}{300} = 66\tfrac{2}{3} \text{ ft.}$$

The base is then not overloaded. The unit stress is then, from the second of equations (III), page 428,



$$p = \frac{2W_1}{b_1}.$$

We have for the weight W_1 and cross-section A_1

$$W_1 = \delta b_1 h_1 = 123060 \text{ pounds per ft.} \qquad A_1 = 820 \text{ sq. ft.},$$

and hence for the greatest unit stress at front edge B_1

$$p = 12306 \text{ pounds per sq. ft.},$$

and for the greatest unit stress at back edge A_1, for dam empty,

$$p = \frac{W_1}{b_1} = 61530 \text{ pounds per sq. ft.}$$

Second Sub-section.—If in equation (4), page 442, we assume for a first approximation $h_2 = 78$, we have

$$B_2 = 54.54, \quad E_2 = 5760.32 \quad \text{and}$$

$$b_2 = -54.54 + \sqrt{8734.92} = 38.91.$$

Substituting this value of b_2 in equations (6) and (7), page 443, we find the least value of limit h_2 given by (6), and from (6) we have limit $h_2 = 77.5$, or less than we assumed it.

If we assume again $h_2 = 78.5$ in equation (4), we have $b_2 = 39.33$, and from (6), $h_2 = 79.2$, or greater than we assumed it.

The value of h_2 is then between 78 and 78.5. If we try again for $h_2 = 78.1$, we have, from (4), $b_2 = 39.07$. and, from (6), $h_2 = 78.187$, or slightly larger than we assumed it. The value of h_2 is then between 78 and 78.1, It will then be quite accurate if we take

$$h_2 = 78.1 \text{ ft.} \quad \text{and} \quad b_2 = 39.07 \text{ ft.}$$

The weight per ft. of the second sub-section and the area of cross-section are then

$$W_2 = \frac{\delta(b_1 + b_2)(h_2 - h_1)}{2} = 164262 \text{ pounds per ft.,} \quad A_2 = 1095 \text{ sq. ft.}$$

Since the least value of limit h_2 is given by equation (6), page 442, we have $s_2 = b_2 = \frac{1}{3}b_2$, and hence from the second of equations (III), page 428, we have for the greatest unit stress for both dam full and empty

$$p = \frac{2(W_1 + W_2)}{b_2} = 14713 \text{ pounds per sq. ft.}$$

The tangent of the front batter angle is given by $\frac{b_2 - b_1}{h_2 - h_1} = 0.51$.

Third Sub-section.—Since the least value of limit h_2 is given by equation (6), page 442, we have the first case, given on page 443. The third sub-section then extends to the base of the dam and has a front and back batter. We have the base b_3 and back batter β_3 from equations (13) and (14), page 444.

If in (13) we assume $\tan \beta_3 = 0.2$, we have $B_3 = 8.02$, $E_3 = 18288.38$ and

$$b_3 = -8.02 + \sqrt{18352.7} = 127.45.$$

If we substitute this value of b_3 in (14), we have $\tan \beta_3 = 0.16$, or less than we assumed it.

If we try again for $\tan \beta_3 = 0.15$, we have $B_3 = 9.61$, $E_3 = 17643.69$ and

$$b_3 = -9.61 + \sqrt{17736.04} = 123.56.$$

With this value of b_3 in (14) we have $\tan \beta_3 = 0.17$, or greater than we assumed it.

The value of $\tan \beta_3$ is then between 0.2 and 0.15. If we assume $\tan \beta_3 = 0.17$, we have $B_3 = 8.97$ $E = 17901.57$ and

$$b_3 = -8.97 + \sqrt{17982.03} = 125.12.$$

With this value of b_3 in (14) we have $\tan \beta_3 = 0.169$, or almost exactly what we assumed it.

We have then for the third section

$$h_3 - h_2 = 91.9, \qquad b_3 = 125.12, \qquad \tan \beta_3 = 0.17.$$

The weight per foot of the third sub-section and the cross-section are then

$$W_3 = \frac{\delta(b_2 + b_3)(h_3 - h_2)}{2} = 1\,131\,670 \text{ pounds per ft.,} \qquad A_3 = 7544 \text{ sq. ft.}$$

The tangent of the front batter angle is given by

$$\frac{b_3 - b_2 - (h_3 - h_2) \times 0.17}{h_3 - h_2} = 0.76.$$

The unit pressure back and front is 20000 pounds per sq. ft.

The front edge distance is, from (12), $e_3 = 46.64$, and the back edge distance the same.

We have then the following table:

h	b	A	$\tan \beta$ back.	$\tan \beta$ front.	e	s	p back.	p front.
41.02	20	820	0	0	6.6	10	61530	123060
78.1	39.07	1095	0	0.51	13.02	13.02	14713	14713
170	125.12	7544	0.17	0.76	46.64	46.64	20000	20000
		9459						

In this table the first column contains the distance in feet from the top of the dam to the bottom of each sub-section, the second the base in feet of each sub-section, the third the area in square feet of each sub-section, the fourth and fifth the tangent of the back and front batter angles, the sixth and seventh the back and front edge distances in feet, the last two the greatest unit stress back and front in pounds per square foot.

The total area is 9459 sq. ft. We see, from example (3), page 433, that in the case of the San Mateo dam, which has the same height and density and an even greater allowable unit stress C, the area is 15810 sq. ft. There is a saving in this case of 40 per cent, due to economic section.

(2) *Design the proposed Quaker Bridge Dam, New York, taking ice-pressure into account, dimensions and data as given in the preceding example.*

ANS. We have $T = 40000$ and, as before, $d = 7$, $b_1 = 20$, $C = 20000$, $\delta = 150$, $\gamma = 62.5$.

The top base $b_1 = 20$ ft. is less than $\sqrt{\frac{6T}{\delta}} = 40$ ft., and therefore (page 436) there is no rectangular sub-section. We have then to make $W_1 = 0$ and $h_1 = 0$ wherever they occur in our equations.

First Sub-section.—The first sub-section, then, is given by equations (4) and (7), page 443, remembering to make $W_1 = 0$, $h_1 = 0$.

If we assume $h_2 = 102$ ft. in (4), we have $B_2 = 10$, $E_2 = 5239.48$ and

$$b_2 = -10 + \sqrt{5339.98} = 63.07 \text{ ft.}$$

Substituting this in (7), we have $h_2 = 99.6$ ft., or less than we assumed.

If we assume $h_2 = 101$ ft. in (4), we have $B_2 = 10$, $E_2 = 5315.61$ and

$$b_2 = -10 + \sqrt{5415.61} = 63.59 \text{ ft.}$$

Substituting this in (7), we have $h_2 = 101.43$ ft., or greater than we assumed. The value of h_2 is then between 101 and 102 ft.

Assuming $h_2 = 101.5$ ft. in (4), we have $B_2 = 10$, $E_2 = 5353.97$ and

$$b_2 = -10 + \sqrt{5453.97} = 63.7 \text{ ft.}$$

Substituting this in (7), we have $h_2 = 101.47$ ft., or almost exactly what we assumed.

We have then for the first sub-section a trapezoid with vertical back and height and lower base given by

$$h_2 = 101.5 \text{ ft.}, \qquad b_2 = 63.7 \text{ ft.}$$

We have then the weight W_2 per foot of length and the area of cross-section A_2,

$$W_2 = \frac{\delta(b_1 + b_2)h_2}{2} = 633166 \text{ pounds per ft.}, \qquad A_2 = 4214 \text{ sq. ft.}$$

We have also for the dam full $e_2 = \frac{1}{3}b_2 = 21.33$ ft., and the greatest unit pressure $p = C = 20000$ pounds per sq. ft.

For the dam empty we have from (5), page 428, making $h_1 = 0$, $s_2 = 22.82$ ft., and from the third of equations (III), page 442, the greatest unit pressure $p = 18388$ pounds per sq. ft.

The tangent of the front batter angle is given by $\frac{b_2 - b_1}{h_2} = 0.43$.

Second Sub-section.—Since the limit of h_2 is given by equation (7), page 443, we have the second case, given on page 445 by equations (15) and (16), page 445, after making $h_1 = 0$, $W_1 = 0$.

If we assume $h_3 = 107$ ft. in (15), we have $B_3 = 31.66$, $E_3 = 8827.015$, and

$$b_3 = -31.66 + \sqrt{9829.365} = 67.48.$$

Substituting this in (16), we have $h_3 = 111.04$, or greater than we assumed.

If we assume $h_3 = 110.5$ ft. in (15), we have $B_3 = 31.66$, $E_3 = 9315.26$ and

$$b_3 = -31.66 + \sqrt{10317.61} = 69.91.$$

Substituting this in (16), we have $h_3 = 110.81$, or greater than we assumed.

If we assume $h_3 = 110.8$ in (15), we have $B_3 = 31.66$, $E_3 = 9358.2$ and

$$b_3 = -31.66 + \sqrt{10360.55} = 70.12.$$

Substituting this in (16), we have $h_3 = 110.81$, or almost exactly what we assumed.

We have then for the second sub-section a trapezoid with vertical back and height and lower base given by

$$h_3 - h = 9.3 \text{ ft.}, \qquad b_3 = 70.12 \text{ ft.}$$

The weight W_3 per ft. of length and the area of cross-section A_3 are then

$$W_3 = \frac{\delta(b_2 + b_3)(h_3 - h_2)}{2} = 93340 \text{ pounds per ft.}, \qquad A_3 = 622 \text{ sq. ft.}$$

We have for the dam full and empty $p = C = 20000$ pounds per square foot, and in both cases $e_3 = s_3$, or from equation (12), page 444, making $W_1 = 0$,

$$e_3 = s_3 = 24.18 \text{ ft.}$$

The tangent of the front batter angle is $\dfrac{b_3 - b_2}{h_3 - h_2} = 0.69$.

Third Sub-section.—We have the last sub-section given by equations (17) and (18), page 446, after making $W_1 = 0$ and $h_1 = 0$.

If we assume $\tan \beta_4 = 0.1$ in (17), we have $B_4 = 35.06$, $E_4 = 23439.6$ and

$$b_4 = -35.06 + \sqrt{24668.8} = 122.$$

Substituting this in (18), we have $\tan \beta_4 = 0.29$, or greater than we assumed.

If we assume $\tan \beta_4 = 0.3$, we have $B_4 = 32.47$, $E_4 = 24434.8$ and

$$b_4 = -32.47 + \sqrt{25489.1} = 127.18.$$

Substituting this in (18), we have $\tan \beta_4 = 0.29$, or less than we assumed.

If we assume $\tan \beta_4 = 0.2$, we have $B_4 = 32.56$, $E_4 = 24385$ and

$$b_4 = -32.56 + \sqrt{25445.15} = 126.94.$$

Substituting this in (18), we have $\tan d_4 = 0.29$, or what we assumed. We have then for the last sub-section a trapezoid whose back batter is given by

$$\tan d_4 = 0.29,$$

whose height $h_4 - h_3 = 59.2$ ft. and whose base $b_4 = 126.94$ ft. The tangent of the front batter angle then is $\dfrac{b_4 - b_3 - (h_4 - h_3)\tan\beta_4}{h_4 - h_3} = 0.67$. The weight per foot of length and the area of cross-section are

$$W_4 = \frac{\delta(b_4 + b_3)(h_4 - h_3)}{2} = 8749476 \text{ pounds per ft.}, \quad A_4 = 5833 \text{ sq. ft.}$$

We have for dam full and empty $p = C = 20000$ pounds per sq. ft., and in both cases $e_4 = s_4$, or from equation (19), page 446, making $W_1 = 0$,

$$e_4 = s_4 = 51.08.$$

We have then the following table:

h	b	A	tan β back.	tan β front.	e	s	p back.	p front.
101.5	63.7	4214	0	0.43	21.33	22.82	18388	20000
110.8	70.12	622	0	0.69	24.18	24.18	20000	20000
170	126.94	5833	0.29	0.67	51.08	51.08	20000	20000
		10669						

In this table the first column gives the distance in feet from the top of the dam to the bottom of each sub-section, the second the base in feet of each sub-section, the third the cross-section in square feet of each sub-section, the fourth and fifth the tangent of the back and front batter angles, the sixth and seventh the back and front edge distances in feet, the last two the greatest unit stress back and front in pounds per square foot.

The total area is 10669 sq. ft. We see from example (3), page 443, that in the case of the San Mateo dam, which has the same height and density and an even greater allowable C, the area is 15810 sq. ft. There is then a saving in the present case, due to economic section, of over 32 per cent.

Arch Dam.—When the dam is made in the form of an arch, so that it supports the water-pressure back of it wholly by reason of its action as an arch, it is called an arch dam.

The water-pressure upon the back must always be normal to the surface, and the pressure upon a unit area always the same at the same depth.

Let *aaa*, Fig. 1, be the centre line of a horizontal cross-section of the dam, one foot in height. Let P_1 and P_2 be the equal normal pressures upon the equal portions $a'a'$, $a'a'$, and H the horizontal pressure at the crown.

FIG. 1.

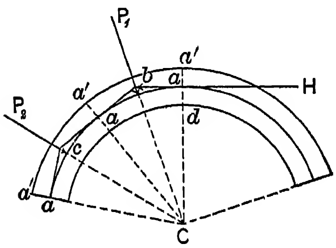
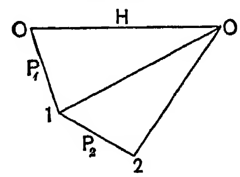


FIG. 2.



In Fig. 2, lay off H from O to o horizontally, and let Oo represent the magnitude of H. Then lay off o 1 and 1 2 parallel and equal in magnitude to P_1 and P_2, and draw the rays $O1$, $O2$.

In Fig. 1, let H act at a, and prolong its direction till it meets P_1 at b. From b draw bc parallel to $O1$ till it meets P_2 at c. From c draw ca parallel to $O2$.

Then (page 413) *abca*, Fig. 1, is the equilibrium polygon. We have by similar triangles

$$P_1 : H :: cb : bC \text{ or } cC; \qquad \therefore \frac{P_1}{cb} = \frac{H}{cC}.$$

The same holds true no matter how many equal portions $a'a'$ we take. But as we increase the number of portions, the polygon approaches a curve. For an indefinitely great number of portions we have for the curve of equilibrium $cb = ds$, also $\frac{P_1}{ds^2} = p =$ the unit pressure and $cC = r =$ the radius of curvature. Hence

$$p \times ds = \frac{H}{r},$$

or

$$H = rp \times ds.$$

But p must be constant, and we see from the construction that H is constant. Therefore r is constant and *the equilibrium curve is a circle.*

If, then, we make the dam circular in cross-section, as shown in Fig. 1, the curve of equilibrium will coincide with the centre line, and the horizontal pressure H at the crown acts at the centre line and is given by

$$H = rp \times ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, since in Fig. 2 the force polygon o 1 2 becomes a circle of radius H when the segments of the arch are indefinitely great in number, and since any ray, as o 1 in Fig. 2, gives the stress in the corresponding segment, cb, Fig. 1, of the equilibrium polygon (page 413), it is evident that *the pressure at every point of the centre line is tangent to the centre line at that point and equal to H.*

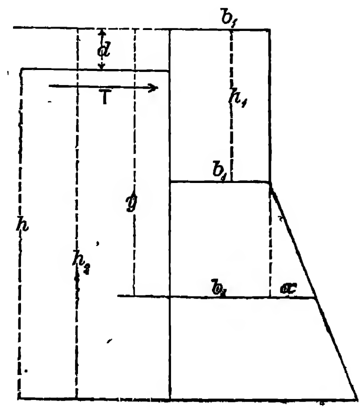
If, then, C is the allowable stress per square foot, we have for the area A of cross-section of the arch one foot in depth at any distance below the surface

$$A = \frac{H}{C} = \frac{rp \times ds}{C},$$

where p is the unit pressure at any distance below the surface of the water, and C is given by our table page 424.

FIRST SUB-SECTION.—Let the water-level be at a distance d below the top (Fig. 3). Let the top base be b_1, and let the dam be rectangular for a distance h_1 below the top.

FIG. 3.



At this depth, then, we have

$$b_1 = \frac{A}{ds} = \frac{rp}{C}.$$

But the unit pressure of the water at the depth $h_1 - d$ is $\gamma(h_1 - d)$, where γ is the mass of a cubic foot of water, or 62.5 lbs.

If T is the ice-thrust at the surface per unit of length, $\frac{T}{h}$ is the unit pressure due to ice-thrust if h is the depth of water. The total unit pressure is then

$$p = \gamma(h_1 - d) + \frac{T}{h}.$$

We have then

$$b_1 = \frac{r\gamma}{C}(h_1 - d) + \frac{rT}{hC},$$

or the height h_1 of the first rectangular sub-section in order that the allowable unit stress C may be just attained is given by

$$h_1 = d + \frac{Cb_1}{\gamma r} - \frac{T}{\gamma h}. \quad \ldots \ldots \ldots \quad (2)$$

TOP THICKNESS.—The choice of top thickness b_1 must in general be determined by local and practical considerations. We can scarcely take $b_1 = 0$, or have no rectangular portion, as it would not be allowable to bring the top to an edge. Some breadth must be assumed.

We have, from (2),

$$b_1 = \frac{\gamma r}{C}(h_1 - d) + \frac{rT}{hC}. \quad \ldots \ldots \ldots \quad (3)$$

Equation (3) will give b_1 for any value of h_1 and d we choose. Thus if we take $b_1 = \frac{rT}{hC}$, we shall have $h_1 = d$. We ought not to take b_1 *less* than this. If we take

$$b_1 = \frac{\gamma r}{C}(h_2 - d) + \frac{rT}{hC},$$

we shall have h_1 equal to the entire height h_2 of the dam, and the entire cross-section will be rectangular. We cannot take b_1 *greater* than this.

Between these two values of

$$b_1 = \frac{rT}{hC} \quad \text{and} \quad b_1 = \frac{\gamma r}{C}(h_2 - d) + \frac{rT}{hC} \quad \ldots \ldots \ldots \quad (4)$$

we can assume b_1 what we wish, and will have then h_1 greater than d and less than h_2. There will then be a second sub-section not rectangular.

SECOND SUB-SECTION.—For any distance y below the top greater than h_1 and less than h_2 (Fig. 3) we have the thickness

$$t = b_1 + x = \frac{A}{ds} = \frac{rp}{C},$$

and the total unit pressure

$$p = \gamma(y - d) + \frac{T}{h}.$$

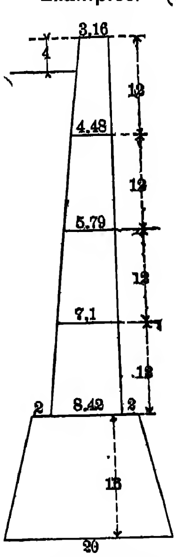
Hence we have for the thickness at any distance y from the top when $y > h_1$

$$t = b_1 + x = \frac{\gamma r}{C}(y - d) + \frac{rT}{hC}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The arch dam requires far less masonry than the gravity dam, but the pressure on the arch stones increases with the depth and span, and so does the thickness. When the thickness becomes great we cannot be sure that each arch stone will take its own share of the pressure. The distribution of the pressure over the cross-section is then uncertain. For such reasons the arch dam is best restricted to short and low dams. It is also evidently unwise to make the stability of a dam depend wholly upon its action as an arch, except under the most favorable conditions as to rigid side hills for abutments, and the most unfavorable conditions as to cost of masonry.

Although it is not, then, generally wise to make the stability of a dam depend wholly upon its action as an arch, it is well to make a gravity dam curved up-stream, so that the arch action may give additional security.

There are but two examples of the pure arch type: the Zola dam in the city of Aix in France, and the Bear Valley dam in the San Bernardino Mountains, Southern California. The first is of rubble masonry, height 120 ft., radius 158 ft., thickness at top 19 ft., at base 42 ft. The Bear Valley dam is of granite, height 64 ft., radius 300 ft., thickness at top 3.16 ft., at base 20 ft.



Examples.—(1) *The Bear Valley dam in the San Bernardino Mountains, Southern California, is an arch dam of granite ashlar, radius $r = 300$ ft., top base $b_1 = 3.17$ ft., bottom base $b_2 = 20$ ft., depth of water $h = 60$ ft., height $h_2 = 64$ ft., and therefore $d = 4$ ft., face vertical and other dimensions as shown in the following figure. Examine its stability.*

ANS. We have from the given dimensions and from equation (5), neglecting the ice-thrust T, for

$y =$	12	24	36	48	64 ft.
$t =$	4.48	5.79	7.1	8.42	20 "
$C = \frac{\gamma r}{t}(y - d) =$	16.72	32.22	42.25	49.	28.12 tons per square foot.

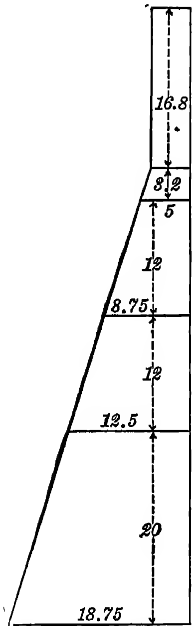
From page 424, the allowable unit stress C ought not to exceed 30 tons per square foot. The dam as built then has, except at top and bottom, a much higher unit stress than ordinary practice would consider allowable.

(2) *Design an arch dam of the same height and radius as the Bear Valley dam, for same depth of water and for an allowable compressive stress of 30 tons per square foot.*

ANS. We have $h = 60$, $h_2 = 64$, $d = 4$, $r = 300$, $\gamma = 62.5$, $C = 60000$. Taking the ice-thrust $T = 40000$ pounds per ft., we have, from equations (4), page 451, $b_1 = 3\frac{1}{8}$ ft. and $b_1 = 22.08$ ft. We should not take b_1 less than the first value, in which case h_1 will equal d, nor greater than the second, in which case h_1 will equal h_2. Let us then take $b_1 = 4$ ft.

1st. *Without Ice-thrust.*—For the first sub-section we have, from equation (2), page 451,

$$h_1 = 4 + \frac{60000 \times 4}{62.5 \times 300} = 16.8 \text{ ft.}$$



The cross-section is then rectangular for a distance $h_1 = 16.8$ ft. from the top. Below this point the thickness should increase. We have then, from equation (5), page 452, for any value of y greater than 16.8 the corresponding thickness

$$t = \frac{62.5 \times 300}{60000}(y - 4).$$

If we take

$$y = 16.8 \qquad 20 \qquad 32 \qquad 44 \qquad 64 \text{ ft.},$$

we have

$$t = 4 \qquad 5 \qquad 8.75 \qquad 12.5 \qquad 18.75 \text{ ft.}$$

If we make the face vertical and batter the back, we have then the cross-section shown in the left-hand figure, 4 ft. thick for the first 16.8 ft

2d. *With Ice-thrust.*—For the first sub-section we have, from equation (2), page 451,

$$h_1 = 4 + \frac{60000 \times 4}{62.5 \times 300} - \frac{40000}{62.5 \times 60} = 6.14 \text{ ft.}$$

The cross-section is then rectangular for a distance $h_1 = 6.14$ ft. from the top. Below this point we have, from equation (5), page 452, for any value of y greater than 6.14 the corresponding thickness

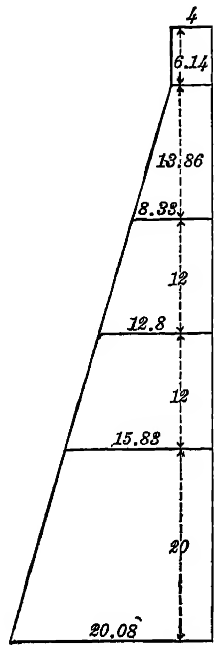
$$t = \frac{62.5 \times 300}{60000}(y - 4) + \frac{300 \times 40000}{60 \times 60000}.$$

If we take

$$y = 6.14 \qquad 20 \qquad 32 \qquad 44 \qquad 64 \text{ ft.},$$

we have

$$t = 4 \qquad 8.33 \qquad 12.08 \qquad 15.83 \qquad 22.08 \text{ ft.}$$



CHAPTER V.

RETAINING WALLS. EARTH PRESSURE. EQUILIBRIUM OF EARTH.

Retaining Wall.—A wall designed to resist the pressure of earth back of it is called a retaining wall.

In the case of a dam we know that the water-pressure is normal to the submerged surface, and its magnitude is as given on page 431. This pressure acts at a distance from the bottom of $d_1 = \frac{1}{3}h$, where h is the depth of water. Its vertical and horizontal components V and H are then known in magnitude and point of application, as given on page 431, and in the preceding chapter we have seen how to investigate the stability and design a dam.

In the case of a retaining wall the earth-pressure is not in general normal to the surface, and the rule for magnitude of earth-pressure no longer holds for earth. A special investigation is therefore necessary in order to find V and H in the case of a retaining wall. When once these are known in magnitude and point of application the general principles and equations already given in the preceding chapter, apply at once and we can investigate the stability or design a retaining wall.

It then only remains necessary to find V and H and the point of application for earth-pressure.

Point of Application of Earth-pressure.—In treating retaining walls it is allowable to neglect the cohesion of the earth. We can therefore consider the earth-pressure as zero at the earth-surface, and increasing for any point of the back of the wall directly as the depth of that point below the earth-surface. The point of application of the pressure is then just the same as for water, viz., at a distance of $d_1 = \frac{1}{3}h$ from the bottom.

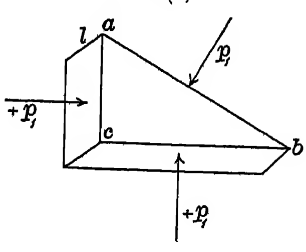
We have already used this value of d_1 in finding the general equations of pages 429 to 432. These general equations apply, then, directly as they stand to retaining walls, if we make the ice- or wave-thrust T zero wherever it occurs.

It only remains to determine the magnitude and direction of the pressure P in the case of earth-pressure.

Magnitude and Direction of Earth-pressure—Graphic Determination.—We shall first discuss some cases of equilibrium of a prism in general.

CASE I.—Let abc, Fig. 1 (a), be any small prism of thickness l, and let $+p_1$ be the normal pressure per unit of area upon the faces ac and bc at right angles, the $(+)$ sign indicating direction up and to the right.

FIG. 1 (a).

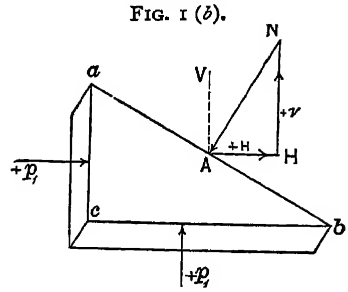


We shall prove that for equilibrium *the pressure per unit of area upon the third face ab is also normal and equal to* p_1.

For if we multiply the area $\overline{ac} \times l$ of the face ac by $+p_1$, we have the total horizontal force $+H = +p_1 \times \overline{ac} \times l$; and if we multiply the area $\overline{bc} \times l$ of the face bc by p_1, we have the total vertical force $+V = +p_1 \times \overline{bc} \times l$.

If we lay these forces off in Fig. 1 (b) from A to H and H to N, so that $AH = +H$ and $HN = +V$, the resultant for equilibrium is NA, making with the vertical an angle ANH whose tangent is

FIG. 1 (b).



$$\frac{H}{V} = \frac{p_1 \times \overline{ac} \times l}{p_1 \times \overline{bc} \times l} = \frac{\overline{ac}}{\overline{bc}}.$$

The angle of NA with the vertical AV is then the same as the angle of ab with the horizontal bc. Since AV is perpendicular to qc, *NA must be perpendicular to ab.*

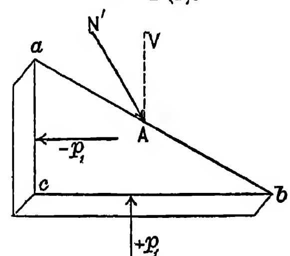
We have also

$$NA = \sqrt{p_1^2 \times \overline{ac}^2 \times l^2 + p_1^2 \times \overline{bc}^2 \times l^2} = p_1 l \sqrt{\overline{ac}^2 + \overline{bc}^2} = p_1 l \times \overline{ab},$$

or for equilibrium *the pressure per unit of area upon the face ab is normal to the face and equal to p_1.*

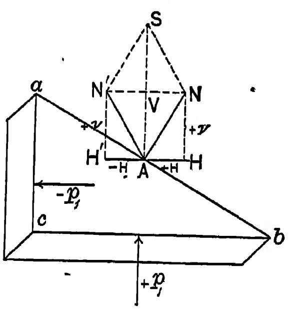
CASE 2.—Suppose now the normal unit pressure on the face ac to be *reversed* in direction, so that it is $-p_1$, Fig. 2 (a).

FIG. 2 (a).



We shall prove that for equilibrium *the pressure per unit of area upon the third face ab is also p_1 as before, but its direction is no longer normal to ab but makes the angle $N'AV$ on the other side of AV equal to the angle NAV in the first case of Fig. 1 (b).*

FIG. 2 (b).



For we have the total pressure on the face bc equal to $+p_1 \times \overline{bc} \times l = +V$ the same as before, and the total pressure on the face ac is now $-p_1 \times \overline{ac} \times l = -H$, or just the same as before in magnitude but opposite in direction.

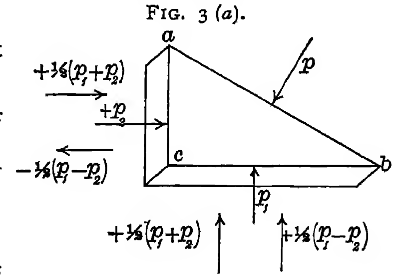
If we lay these forces off, in Fig. 2 (b), from A to H', and H' to N', so that $AH' = -H$, and $HN' = +V$, the resultant for equilibrium is $N'A$. It is at once evident that the magnitude of $N'A$ is the same as NA before, but its direction makes the angle $N'AV$ on the other side of AV equal to the angle NAV in the first case of Fig. 1 (a).

In this second case, then, for equilibrium *the pressure per unit of area upon the face ab is still p_1, but makes an angle $N'AV$ on the other side of AV equal to the angle NAV in the first case.*

If then we lay off in Fig. 2 (b) $AN = p_1$ normal to ab, and with N as a centre describe an arc of a circle intersecting the vertical AV at S, then the line SN will give the direction and magnitude for equilibrium of the unit pressure on the face ab in the second case of Fig. 2 (b).

CASE 3.—Suppose now that the normal pressures per unit of area on the two faces ac and bc, Fig. 5, are *unequal* and given by $+p_2$ and $+p_1$. Required to find the magnitude and direction of the unit pressure p on ab for equilibrium.

FIG. 3 (a).

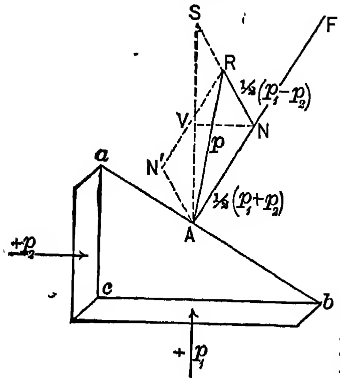


We can divide the normal pressure $+p_1$ on the face bc into two parts, one equal to $+\frac{1}{2}(p_1+p_2)$ and the other equal to $+\frac{1}{2}(p_1-p_2)$, as indicated in Fig. 3 (a).

We can also divide the normal pressure $+p_2$ on the face ac into two parts, one equal to $+\frac{1}{2}(p_1+p_2)$ and the other equal to $-\frac{1}{2}(p_1-p_2)$.

Now, by Case 1, the unit pressure normal to the face *ab* which balances $+\frac{1}{2}(p_1+p_2)$ on the face *bc* and $+\frac{1}{2}(p_1+p_2)$ on the face *ac* is just the same, or $NA=\frac{1}{2}(p_1+p_2)$, Fig. 3 (*b*), laid off normal to *ab*.

FIG. 3 (*b*).



Also, by Case 2, the unit pressure on the face *ab* which balances $+\frac{1}{2}(p_1-p_2)$ on the face *bc* and $-\frac{1}{2}(p_1-p_2)$ on the face *ac* is just the same, or $\frac{1}{2}(p_1-p_2)$, but it makes an angle $N'AV$ on the other side of AV equal to NAV.

If then we lay off, in Fig. 3 (*b*), NA normal to *ab* and equal to $\frac{1}{2}(p_1+p_2)$, and with N as a centre and NA as a radius describe the arc of a circle intersecting the vertical AV at the point S, then SN will give the direction of $\frac{1}{2}(p_1-p_2)$ acting on the face *ab*.

If then we lay off along this line SN the distance $NR=\frac{1}{2}(p_1-p_2)$ and draw RA, *the line RA will give the magnitude and direction of the resultant unit pressure p on the face ab when the nmoral unit pressures p_1 and p_2 on the faces bc and ac are unequal.*

It is also evident that the angle RNF is equal to twice the angle VAN, or, if we bisect the angle RNF, we shall have the direction of AV or p_1.

Suppose, now, the faces *ac* and *bc*, Fig. 3 (*b*), to remain invariable in direction and the normal unit pressures p_1 and p_2 on these faces to remain unchanged, but let the third face, *ab*, vary its inclination. Then the magnitudes of $NA=\frac{1}{2}(p_1+p_2)$ and of $RN=\frac{1}{2}(p_1-p_2)$ in Fig. 3 (*b*) will be unchanged, but their directions will change according as the face *ab* changes its inclination. It is evident that the greatest possible value of the angle RAN which the resultant pressure $p=RA$ on the face *ab* makes with the normal NA to that face will be when RN is perpendicular to RA, or when the angle $ARN=90°$.

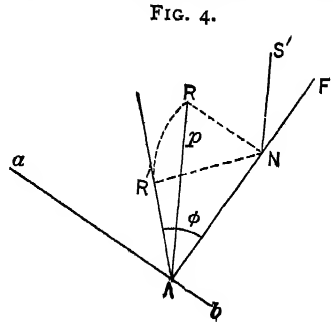
Now for earth-pressure the greatest possible value of the angle RAN *is the angle of friction or of repose $\phi°$ for earth on earth.*

The angle RNF is then equal to $90°+\phi°$, and, as we have just seen, if we bisect this we have the direction of p_1. Hence the angle VAN of p_1 with the normal NA is $\frac{1}{2}(90°+\phi°)=45°+\frac{\phi°}{2}$.

CASE 4.—In Fig. 4, let *ab* be any earth-surface, and let the resultant unit pressure $RA=p$ on this surface be given in direction and magnitude. It is required to find $\frac{1}{2}(p_1+p_2)$, $\frac{1}{2}(p_1-p_2)$ and the direction of p_1.

Draw AF normal to the surface *ab*, and AR' making the angle of friction ϕ with this normal. Then find by trial a point N in this normal AF such that if we take N as a centre and NR as a radius, the arc ARR' will be just tangent to AR'. When this point N is thus found by trial, then, by Case 3, the distance AN will be $\frac{1}{2}(p_1+p_2)$, and $R'N=RN$ will be $\frac{1}{2}(p_1-p_2)$. Also, if we bisect the angle RNF by the line NS', we have the direction of p_1.

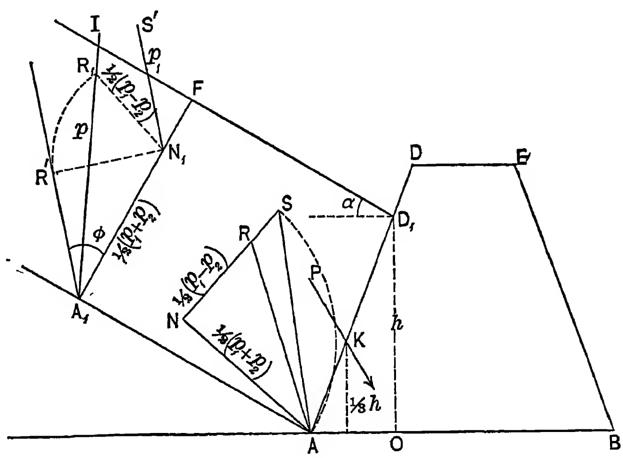
FIG. 4.



APPLICATION TO THE RETAINING WALL.—Let AD, Fig. 5, be the back of the wall, and D_1FI the earth-surface

making the angle α with the horizontal. Pass a plane AA_1 through the foot of the wall A parallel to the earth-surface, and draw A_1I vertical and A_1F normal to the earth-surface. The pressure upon every square foot of this plane AA_1 is vertical and equal to the weight of a column of earth of vertical height A_1I and cross-section 1 sq. ft. $\times$ cos α.

FIG. 5.



If γ_1 is the mass of a cubic foot of earth, then we have

$$\gamma_1 \times \overline{A_1I} \times 1 \text{ sq. ft.} \times \cos \alpha$$

for the mass of this column. But $\overline{A_1I} \cos \alpha = A_1F$, hence the mass of this column is

$$\gamma_1 \times \overline{A_1F} \times 1 \text{ sq. ft.}$$

If then we draw A_1F perpendicular to the earth-surface and revolve A_1F about A_1 as centre to the vertical A_1R_1, we have for the vertical unit pressure p

$$p = \gamma_1 \times \overline{A_1R_1} \text{ pounds per sq. ft.,}$$

where γ_1 is the mass in lbs. of a cubic foot of earth, and A_1R_1 is measured in feet. We have then, as in Case 4, p given in magnitude and direction, and we can find, as in Case 4, $\frac{1}{2}(p_1 + p_2)$, $\frac{1}{2}(p_1 - p_2)$, and the direction of p_1.

Thus, as in Fig. 4, draw A_1R' (Fig. 5) making with the normal A_1F the angle $R'A_1F$, equal to the angle of friction or repose ϕ for earth on earth. Find by trial a point N_1 on the normal A_1F such that the arc of a circle with N_1 as a centre passes through R_1 and is tangent to A_1R'. Then, as in Case 4,

$$\frac{1}{2}(p_1+p_2) = \gamma_1 \cdot \overline{N_1A_1},$$

$$\frac{1}{2}(p_1-p_2) = \gamma_1 \cdot \overline{N_1R_1}.$$

Bisect the angle R_1N_1F by the line N_1S'. Then the line N_1S' gives the direction of p_1, as in Fig. 4.

Now lay off at the foot of the wall A (which may be considered as identical with A_1) the distance $NA = N_1A_1$ in a direction perpendicular to the back of the wall at A. Draw the line AS parallel to N_1S', the direction of p_1 already found. Then, as in Fig. 3 (*b*), with N as a centre and NA as a radius describe an arc of a circle intersecting AS at S, and lay off along NS the distance $NR = N_1R_1$. Then, as in Case 3, RA represents the magnitude and direction of the pressure per square foot at the foot of the wall. Thus, if γ_1 is the density of earth and we measure RA in feet, the pressure per square foot at the foot A of the wall is given in magnitude by

$$\gamma_1 \times \overline{RA},$$

and its direction is the direction of RA.

Since the pressure is zero at the top D_1 and greatest at the foot A, and varies for any point directly as the distance of that point below D_1, the average pressure per square foot is

$$\frac{1}{2}\gamma_1 \times \overline{RA}.$$

The total pressure P is then for a wall of one foot in length

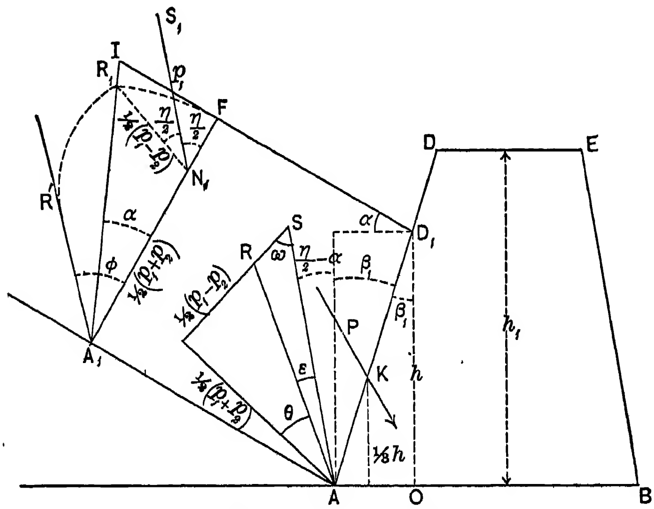
$$P = \frac{1}{2}\gamma_1 \times \overline{RA} \times D_1A,$$

where γ_1 is the mass of a cubic foot of earth and RA, D_1A are taken in feet, and P is the pressure per foot of length.

This pressure P acts (page 454) at a point K at a distance above the base of the wall equal to $\frac{1}{3}h$, where h is the distance D_1O of the earth-surface above the base of the wall, and is parallel to the direction of RA already found.

We can thus find by a simple graphic construction, in any given case, the magnitude, direction and point of application of the earth-pressure P on the back of the wall. The vertical and horizontal components V and H of P are then easily found, and then the general principles and equations already given in the preceding chapter, apply at once, and we can investigate the stability of, or design, a retaining wall.

Magnitude and Direction of Earth-pressure—Analytic Determination.—From the graphic construction just given we can easily derive the corresponding general formulas for the magnitude and direction of the earth-pressure P.



NOTATION.—Let $h = D_1O$ be the height of the earth-surface at D_1 above the base AB of the wall; the angle of the earth-surface with the horizontal is α; the batter angle of the back of the wall with the vertical is β_1; the earth-pressure P makes the angle θ with the normal to the back of the wall; the angle $R'A_1N_1 = \phi$ is the angle of repose for earth on earth; the angle $R_1N_1F = \eta$, and the angles $R_1N_1S_1$ and S_1N_1F are each $\frac{\eta}{2}$; the angle $RAS = \epsilon$: the angle $RSA = \omega$; γ_1 is the mass of a cubic foot of earth.

SOLUTION.—Now by the graphic construction we have

$$\frac{1}{2}(p_1 + p_2) \sin \phi = \frac{1}{2}(p_1 - p_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have also

$$AD_1 = \frac{h}{\cos \beta_1}, \qquad A_1F = AD_1 \cos(\beta_1 - \alpha) = \frac{h}{\cos \beta_1}\cos(\beta_1 - \alpha),$$

and since $A_1R_1 = A_1F$, we have by construction

$$\frac{1}{2}(p_1 - p_2) \sin \eta = \frac{\gamma_1 h}{\cos \beta_1} \cos (\beta_1 - \alpha) \sin \alpha. \qquad (2)$$

We have also by construction

$$\left[\frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_1 - p_2) \cos \eta\right]^2 + \left[\frac{1}{2}(p_1 - p_2) \sin \eta\right]^2 = \left[\frac{\gamma_1 h}{\cos \beta} \cos (\beta_1 - \alpha)\right]^2, \qquad (3)$$

and also

$$\frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_1 - p_2) \cos \eta = \frac{\gamma_1 h}{\cos \beta} \cos (\beta_1 - \alpha) \cos \alpha. \qquad (4)$$

From (1), (2) and (3), eliminating $\frac{1}{2}(p_1 + p_2)$ and $\frac{1}{2}(p_1 - p_2)$, we obtain

$$\cos \eta = -\frac{\sin^2 \alpha}{\sin \phi} + \sqrt{(1 - \sin^2 \alpha)\left(1 - \frac{\sin^2 \alpha}{\sin^2 \phi}\right)}. \qquad (I)$$

We have also directly from the figure $\omega =$ angle NSA = angle NAS, or

$$\omega = 90 - \beta_1 - \frac{\eta}{2} + \alpha. \qquad (II)$$

From (2) and (1) we have

$$p_1 = \frac{\gamma_1 h \cos (\alpha - \beta_1) \sin \alpha(1 + \sin \phi)}{\cos \beta_1 \sin \phi \sin \eta} \qquad (5)$$

$$p_2 = \frac{\gamma_1 h \cos (\alpha - \beta_1) \sin \alpha(1 - \sin \phi)}{\cos \beta_1 \sin \phi \sin \eta}. \qquad (6)$$

We have also from the figure

$$\tan \epsilon = \frac{\overline{RS} \sin \omega}{\overline{AS} - \overline{RS} \cos \omega}.$$

But $\gamma_1 . \overline{RS} = p_2$, and $\gamma_1 . \overline{AS} = (p_1 + p_2) \cos \omega$. Therefore

$$\tan \epsilon = \frac{p_2 \sin \omega}{p_1 \cos \omega}.$$

Substituting the values of p_1 and p_2 from (5) and (6),

$$\tan \epsilon = \frac{1 - \sin \phi}{1 + \sin \phi} \tan \omega = \tan^2 \left(45 - \frac{\phi}{2}\right) \tan \omega. \qquad (III)$$

We have also directly from the figure

$$\theta = \omega - \epsilon. \qquad (IV)$$

Also,

$$\gamma_1 . \overline{RA} = \sqrt{p_2^2 \sin^2 \omega + (\gamma_1 . \overline{AS} - p_2 \cos \omega)^2} = \sqrt{p_2^2 \sin^2 \omega + p_1^2 \cos^2 \omega},$$

or, substituting the values of p_1 and p_2 from (5) and (6), we have for the earth-pressure P

$$P = \frac{1}{2}\gamma_1 . \overline{RA} . \overline{AD_1} = \frac{h}{\cos \beta_1} . \frac{1}{2}\gamma_1 . \overline{RA},$$

or

$$P = \frac{\gamma_1 h^2 \cos (\beta_1 - \alpha) \sin \alpha}{2 \cos^2 \beta_1 \sin \phi \sin \eta} \sqrt{(1 + \sin \phi)^2 - 4 \sin \phi \sin^2 \omega}. \quad . \quad . \quad . \quad . \quad (V)$$

From (1) and (4) we obtain

$$p_1 = \frac{\gamma_1 h \cos (\beta_1 - \alpha) \cos \alpha (1 + \sin \phi)}{\cos \beta_1 (1 + \sin \phi \cos \eta)}.$$

Comparing this with (5), we have

$$\frac{\sin \alpha}{\sin \phi \sin \eta} = \frac{\cos \alpha}{1 + \sin \phi \cos \eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Making this substitution in (V), we obtain an equivalent expression for P which can be used when $\alpha = 0$, viz.,

$$P = \frac{\gamma_1 h^2 \cos (\beta_1 - \alpha) \cos \alpha}{2 \cos^2 \beta_1 (1 + \sin \phi \cos \eta)} \sqrt{(1 + \sin \phi)^2 - 4 \sin \phi \sin^2 \omega}. \quad . \quad . \quad . \quad (VI)$$

We see from the figure page 458 that the earth-pressure P makes the angle $\theta + \beta_1$ with the horizontal. We have then for the vertical and horizontal components of P

$$\left.\begin{aligned} V &= P \sin (\theta + \beta_1), \\ H &= P \cos (\theta + \beta_1). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (VII)$$

The values of V and H being thus known, the principles and general equations already given in the preceding chapter, apply at once.

The case of water-pressure is a special case of the equations just deduced. Thus for water we have γ in place of γ_1, since the surface is level we have $\alpha = 0$, and since there is no friction $\phi_1 = 0$. We have then, from (I), $\eta = 0$; from (II), $\omega = 90 - \beta_1$; from (III), $\epsilon = \omega$; from (IV), $\theta = 0$. The pressure of water is therefore normal to the surface. From (VI) we have

$$P = \frac{\gamma h^2}{2 \cos \beta_1},$$

and from (VII)

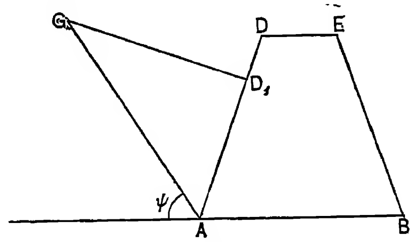
$$V = \frac{\gamma h^2}{2} \tan \beta_1,$$

$$H = \frac{\gamma h^2}{2}.$$

These are precisely the values of V and H given on page 431 for water-pressure. We see, then, that water-pressure is but a special case.

Surface of Rupture.—If there were no wall and we were to disregard cohesion, a prism of earth AD_1G would tend to slide off along a plane AG which would make with the horizontal the angle of repose ϕ for earth on earth.

But on account of the wall this plane AG makes with the horizontal an angle ψ greater than ϕ.



This angle ψ we call the *angle of rupture*, the plane AG is the *plane of rupture*, and the prism AD_1G which thus tends to separate along AG and force the wall is the *prism of rupture*.

If, in the figure page 458, p_1 remains unchanged in direction and magnitude, while AA_1 is revolved about A until the pressure upon AA_1 makes with the normal A_1N_1 the angle ϕ, then this new position of AA_1 is the plane of rupture. But for this new position p_1 makes the angle $45° + \frac{\phi}{2}$ with the normal. The normal A_1N_1, and hence the plane AA_1, have then been revolved through the angle $45 + \frac{\phi}{2} - \frac{\eta}{2}$. The angle which the plane of rupture AG makes with the horizontal is then

$$\psi = 45° + \frac{\phi}{2} - \frac{\eta}{2} + \alpha. \quad \ldots\ldots\ldots \quad \text{(VIII)}$$

In the case of water, $\alpha = 0$, $\phi = 0$ and, from (I), $\eta = 0$. We have then for water $\psi = 45°$.

General Method.—We have in any case the following general method:

1st. Find η from (I).
2d. Find ω from (II).
3d. Find ϵ from (III).
4th. Find θ from (IV).

The angle θ gives the inclination of the earth-pressure with the normal to the back of the wall.

5th. Find P from (V) or (VI).
6th. Find V and H from (VII).

If desired we can find the angle of rupture from (VIII).

Knowing now V and H, we can use the general equations pages 426 to 430, making in them $T = 0$.

Special Cases.—The formulas (I) to (VIII) just given are general and admit of simplification for special cases.

CASE I. EARTH-SURFACE HORIZONTAL.—If the earth-surface is horizontal, we have $\alpha = 0$, hence, from (I), $\eta = 0$ and, from (II), $\omega = 90° - \beta_1$. We have then, from (III),

$$\tan \epsilon = \tan^2\left(45 - \frac{\phi}{2}\right) \text{cotan } \beta_1, \quad \ldots\ldots\ldots \quad (8)$$

and then, from (IV),

$$\theta = 90° - \beta_1 - \epsilon; \quad \ldots\ldots\ldots \quad (9)$$

from (VI),

$$P = \frac{\gamma_1 h^2}{2}\sqrt{\frac{1}{\cos^2\beta_1} - \frac{4\sin\phi}{(1+\sin\phi)^2}}; \quad \ldots\ldots\ldots \quad (10)$$

and from (VII),

$$V = P\sin(90° - \epsilon), \qquad H = P\cos(90° - \epsilon). \quad \ldots\ldots \quad (11)$$

From (VIII) the surface of rupture makes with the horizontal the angle

$$\psi = 45^\circ + \frac{\phi}{2}. \qquad (12)$$

CASE 2. EARTH-SURFACE HORIZONTAL—BACK VERTICAL.—If in the preceding case we make $\beta_1 = 0$, we have $\epsilon = 90^\circ$, $\theta = 0$ and therefore the earth-pressure *is normal to the back of the wall, or horizontal.*

From (10), then,

$$P = H = \frac{\gamma_1 h^2}{2} \cdot \frac{1 - \sin\phi}{1 + \sin\phi} = \frac{\gamma_1 h_2}{2} \tan^2\left(45^\circ - \frac{\phi}{2}\right), \qquad (13)$$

and from (11) $V = 0$.

The surface of rupture makes as before the angle ψ with the horizontal given by

$$\psi = 45^\circ + \frac{\phi}{2}.$$

CASE 3. EARTH-SURFACE HORIZONTAL—BACK BATTER ANGLE EQUAL TO $90^\circ - \psi$.—In this case $\alpha = 0$, $\beta_1 = 90^\circ - \psi$. We have, from (I), $\eta = 0$ and, from (VIII), $\psi = 45^\circ + \frac{\phi}{2}$. Hence $\beta_1 = 45^\circ - \frac{\phi}{2}$, and, from (II), $\omega = \psi = 45^\circ + \frac{\phi}{2}$; from (III), $\epsilon = 45^\circ - \frac{1}{2}\phi$; from (IV),

$$\theta = \phi,$$

or the pressure *makes the angle of friction ϕ with the normal to the back.*

From (VI) we have

$$P = \frac{\gamma_1 h^2}{2\cos\left(45^\circ - \frac{\phi}{2}\right)(1 + \sin\phi)} \sqrt{(1 + \sin\phi)^2 - 4\sin\phi \sin^2\left(45^\circ - \frac{\phi}{2}\right)}.$$

But by Trigonometry

$$\frac{1 + \sin\phi}{1 - \sin\phi} = \frac{\sin^2\left(45^\circ + \frac{\phi}{2}\right)}{\cos^2\left(45^\circ + \frac{\phi}{2}\right)}.$$

Hence

$$\sin\phi = 1 - 2\cos^2\left(45^\circ + \frac{\phi}{2}\right), \qquad 1 + \sin\phi = 2 - 2\cos^2\left(45^\circ + \frac{\phi}{2}\right),$$

$$\cos^2\phi = 4\cos^2\left(45^\circ + \frac{\phi}{2}\right) - 4\cos^4\left(45^\circ + \frac{\phi}{2}\right).$$

Inserting these values and reducing, we have

$$P = \frac{\gamma_1 h^2 \cos^2\left(45^\circ + \frac{\phi}{2}\right)}{\cos\phi \cos\left(45^\circ - \frac{\phi}{2}\right)}. \qquad (14)$$

From (VII),

$$V = P\sin\left(45^\circ + \frac{\phi}{2}\right), \qquad H = P\cos\left(45^\circ + \frac{\phi}{2}\right). \qquad (15)$$

CASE 4. EARTH-SURFACE INCLINED AT THE ANGLE OF REPOSE.—In this case $\alpha = \phi$. We have, from (I), $\cos \eta = -\sin \phi$, or $\eta = 90 + \phi$; from (II), $\omega = 45° - \beta_1 + \frac{\phi}{2}$; from (III),

$$\tan \epsilon = \tan^2 \left(45° - \frac{\phi}{2}\right) \tan \left(45° - \beta_1 + \frac{\phi}{2}\right). \quad . \quad . \quad . \quad . \quad (16)$$

From (IV),

$$\theta = 45° - \beta_1 + \frac{\phi}{2} - \epsilon. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From (V),

$$P = \frac{\gamma_1 h^2 \cos (\beta_1 - \phi)}{2 \cos^2 \beta_1 \cos \phi} \sqrt{(1 + \sin \phi)^2 - 4 \sin \phi \sin^2 \left(45° - \beta_1 + \frac{\phi}{2}\right)}. \quad . \quad . \quad . \quad (18)$$

From (VII),

$$V = P \sin \left(45° + \frac{\phi}{2} - \epsilon\right), \qquad H = P \cos \left(45° + \frac{\phi}{2} - \epsilon\right). \quad . \quad . \quad . \quad . \quad (19)$$

From (VIII),

$$\psi = \phi,$$

or the surface of rupture is parallel to the earth-surface.

CASE 5. EARTH-SURFACE INCLINED AT THE ANGLE OF REPOSE—BACK VERTICAL.—In this case we have only to make $\beta_1 = 0$ in the preceding case, and we then have $\eta = 90 + \phi$, $\omega = 45 + \frac{\phi}{2}$, and, from (16),

$$\tan \epsilon = \tan^2 \left(45° - \frac{\phi}{2}\right) \tan \left(45° + \frac{\phi}{2}\right),$$

or, since, by Trigonometry,

$$\frac{1 + \sin \phi}{1 - \sin \phi} = \tan^2 \left(45° + \frac{\phi}{2}\right), \qquad \frac{1 - \sin \phi}{1 + \sin \phi} = \tan^2 \left(45° - \frac{\phi}{2}\right),$$

$$\tan \epsilon = \frac{1 - \sin \phi}{1 + \sin \phi} \sqrt{\frac{1 + \sin \phi}{1 - \sin \phi}} = \sqrt{\frac{1 - \sin \phi}{1 + \sin \phi}} = \tan \left(45° - \frac{\phi}{2}\right).$$

Hence $\epsilon = 45° - \frac{\phi}{2}$, and, from (17),

$$\theta = \phi,$$

or *the pressure makes the angle of friction* ϕ *with the normal to the back.*

From (18),

$$P = \frac{\gamma_1 h^2}{2} \sqrt{(1 + \sin \phi)^2 - 4 \sin \phi \sin^2 \left(45° + \frac{\phi}{2}\right)}.$$

or, reducing as in Case 3,

$$P = \frac{\gamma_1 h^2 \cos \phi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

From (19),

$$V = \frac{\gamma_1 h^2 \sin 2\phi}{4}, \qquad H = \frac{\gamma_1 h^2 \cos^2 \phi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The surface of rupture makes, as before, the angle ϕ with the horizontal.

Values of ϕ, μ and γ_1.—We give in the following table the value of the angle of friction ϕ, of the coefficient of friction $\mu = \tan \phi$, and of the density γ_1 for earth, sand and gravel.

Material.	Angle of Repose ϕ.	Coefficient of Friction μ.	Density in lbs. per cu. ft. γ_1.
Gravel, round	30°	0.58	100
" sharp	40	0.84	110
Sand, dry	35	0.70	100
" moist	40	0.84	110
" wet	30	0.58	125
Earth, dry	40	0.84	90
" moist	45	1.00	95
" wet	32	0.62	115

Examples.—(1) *At Northfield, Vt., on the line of the Central Vermont R. R., is a retaining wall 15 ft. high, top base 2 ft., bottom base 6 ft. The wall is composed of large blocks of limestone without cement. The density of the masonry is about 170 lbs. per cubic foot. The face of the wall has a batter of 1 inch horizontal for every foot of height. The wall is over 30 years old and in as good condition as when laid. Investigate the stability, taking the angle of repose 38°, the density of the earth 90 lbs. per cubic foot, and the coefficient of friction for the masonry* $\mu = 0.66$ (page 424).

ANS. We have $h = h_1 = 15$, $b_1 = 2$, $b_2 = 6$, $\delta = 170$, $\tan \beta_1 = \frac{2.75}{15}$, or $\beta_1 = 10° \ 23'$, $\phi = 38$, $\gamma_1 = 90$, $\mu = 0.66$.

Take a section of the wall 1 foot in length. Then the weight of this section is

$$W = \frac{\delta(b_1 + b_2)h}{2} = 10200 \text{ pounds per ft.}$$

From Case 1, page 461, we have

$$P = \frac{90 \times 225}{2} \sqrt{\frac{1}{0.967} - \frac{2.463}{2.6}} = 2983 \text{ pounds per ft.},$$

$$\tan \epsilon = \tan^2 26° \cot 10° \ 23', \quad \text{or} \quad \epsilon = 52° \ 26',$$

$$V = P \sin (90 - \epsilon) = 1814 \text{ pounds per ft.}, \qquad H = P \cos (90 - \epsilon) = 2364 \text{ pounds per ft.}$$

Now that we know V and H, we can proceed as on pages 426 to 430, making $T = 0$ in all equations in which it occurs.

Thus for stability for sliding we have from (I), page 427, for the factor of safety for sliding

$$n = \frac{0.66(10200 + 1814)}{2364} = 3.6.$$

There is therefore ample security against sliding even for through joints. If there are no through joints, there is in any case no possibility of sliding.

For stability for rotation we have, from equation (5), page 427,

$$\bar{x} = \frac{72 + 24 - 4 - 15(4 + 6)\frac{2.75}{15}}{3 \times 8} = 2.646 \text{ ft.},$$

and from (II), page 428, since $d_1 = \frac{1}{3}h = 5$,

$$e = \frac{10200 \times 2.646 + 1814\left(6 - \frac{5 \times 2.75}{15}\right) - 2364 \times 5}{10200 + 1814} = 2.03.$$

Since e is positive, the resultant pressure on the base falls within the base and there is no rotation.

For stability for pressure, since e is greater than $\frac{1}{3}b_2 = 2$, we have, from the third of equations (III), page 428, for the greatest unit pressure

$$p = \frac{2(10200 + 1814)}{6}\left(2 - \frac{3 \times 2.03}{6}\right) = 3944.6 \text{ pounds per sq. ft.}$$

From our table page 424 we see that the allowable unit stress is $C =$ from 50000 to 60000 pounds per sq. ft. The wall is then abundantly secure against sliding, rotation and crushing. We also see that the bottom base might have been considerably less, thus saving much material, without being insecure.

Check by Graphic Construction, page 457.

(2) *Design the preceding wall properly for the same top base, height and back batter, for an allowable stress of $C = 40000$ pounds per sq. ft.*

ANS. We have $h = h_1 = 15$, $b_1 = 2$, $\delta = 170$, $\tan \beta_1 = \frac{2.75}{15}$, $\mu = 0.66$, and of course the same values for V and H as before, viz.,

$$V = 1814, \qquad H = 2364.$$

From equation (II), page 429, making $T = 0$, we have for the value of b_2 for low wall

$$b_2 = -1.065 + \sqrt{1.134 + 50.95} = 6.16 \text{ ft.}$$

Substituting this value of b_2 in equation (I), page 429, we have for the limit of h_1 for low wall

$$\textit{limit } h_1 = \frac{6.16 \times 40000 - 3628}{170 \times 8.16} = 175 \text{ ft.}$$

Since this is greater than the actual height, the wall is low and the value of b_2 just found is the value required. We have for this base $e = \frac{1}{3}b_2$ and p less than C.

We have then the weight per ft.

$$W = \frac{\delta(b_1 + b_2)h}{2} = \frac{170 \times 8.16 \times 15}{2} = 10404 \text{ pounds per ft.}$$

From (I), page 427, we have the coefficient of safety for sliding

$$n = \frac{0.66(10494 + 1814)}{2364} = 3.4.$$

There is no danger of sliding even for through joints.

Check by Graphic Construction, page 457.

(3) *Design the wall of the preceding example for the same top base and height and vertical back, for an allowable stress $C = 40000$ pounds per square foot, and show that there is a saving of over 17 per cent due to vertical back.*

ANS. We have $h = h_1 = 15$, $b_1 = 2$, $\delta = 170$, $\beta_1 = 0$, $\phi = 38°$, $\gamma_1 = 90$, $\mu = 0.66$.

From Case 2, page 462, we have

$$V = 0, \qquad H = \frac{90 \times 225}{2}\tan^2 26° = 2408 \text{ pounds per ft.}$$

From equation (II), page 429, making $T = 0$, we have

$$b_2 = -1 + \sqrt{33.33} = 4.77 \text{ ft.}$$

Substituting this value of b_2 in equation (I), page 429, we find the limit of h_1 for low wall greater than the given height of dam. The wall is then low and the value of b_2 just found is the value required. For this base we have $e = \frac{1}{3}b_2$ and p less than C.

We have then the weight per ft.

$$W = \frac{\delta(b_1 + b_2)h}{2} = \frac{170 \times 6.77 \times 15}{2} = 8632 \text{ pounds per ft.,}$$

as against 10404 pounds per ft. in the preceding example, or a saving of 17.03 per cent.

From (I), page 427, we have the coefficient of safety for sliding

$$n = \frac{0.66 \times 8632}{2408} = 2.3.$$

There is then no danger of sliding even for through joints.

Check by Graphic Construction, page 457.

(4) *Find the bottom base of a trapezoidal retaining wall of granite ashlar with vertical back, 20 feet high, earth surface horizontal and level with the top,* $\phi = 33° \, 40'$, $\gamma = 100$ *pounds per cubic foot.*

ANS. In this case $\beta_1 = 0$, $V = 0$, $h = h_1$. From Case (2), page 462, we have the pressure P normal to the back and given by

$$P = H = \frac{100 \times 20^2}{2} \tan^2 28° \, 15' = 5774 \text{ pounds per ft.}$$

From equation (II), page 429, we have

$$b_2 = -\frac{1}{2}b_1 + \sqrt{\frac{5b_1^2}{4} + \frac{2H}{\delta}}.$$

If we take the top base $b_1 = 2$ ft. and $\delta = 165$ lbs. per cubic ft. (page 424), we have $b_2 = 7.66$ ft.

Check by Graphic Construction, page 457.

(5) *Same as preceding example, with back batter* $\beta_1 = 8°$.

ANS. $P = 6420$ pounds per ft., $\theta = 18° \, 9'$, $H = 5758$ pounds per ft., $V = 2825$ pounds per ft., $b_2 = 7.9$ ft.

(6) *Design a retaining wall 20 ft. high, back batter* $\beta_1 = 8°$, $\delta = 170$ *lbs. per cubic foot, earth-surface inclined to horizontal at angle of repose* $\phi = 33° \, 40'$, $\gamma_1 = 100$ *lbs. per cubic ft., earth-surface at top of wall.*

ANS. From Case 4, page 463, we have $\epsilon = 21° \, 22'$, $\theta = 32° \, 28'$, $P = 21740$ pounds per ft., $H = 16522$ pounds per ft., $V = 13230$ lbs. per ft.

If we take the top base $b_1 = 2$ ft., we have, from equation (II), page 429, $b_2 = 9.6$ ft.

Check by Graphic Construction, page 457.

(7) *A wall 15 ft. high retains an earth filling which supports a double-track railway. The top base is 3.5 ft. Find the bottom base for* $\gamma_1 = 100$, $\phi = 33° \, 40'$, $\beta_1 = 8°$, $\delta = 170$.

ANS. If we take the train-load at 6000 pounds per linear ft. and top of fill 15 ft., the pressure per square ft. on top is 400 pounds, which is equivalent to a column of earth 4 ft. high. We have then $h_1 = 15$, $h = 19$, and from Case 1, page 461,

$$P = 5795 \text{ pounds per ft.,} \quad \theta = 18°, \quad H = 5200 \text{ pounds per ft.,} \quad V = 2540 \text{ pounds per ft.}$$

From equation (II), page 429, $b_2 = 7$ ft.

Check by Graphic Construction, page 457.

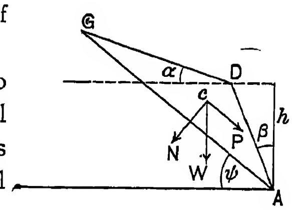
COHESION OF EARTH.—Cohesion is that resistance to motion which occurs when two surfaces of the same kind are in contact. It is found by experiment that cohesion is directly proportional to the area of contact, varies with the nature of the surfaces in contact, and is independent of the pressure.

For any given surfaces, then, the cohesion is

$$cA,$$

where A is the area of contact and c is the *coefficient of cohesion.*

EQUILIBRIUM OF A MASS OF EARTH.—Let ADG be a mass of earth, the batter angle of the face being β.



If there were no cohesion, a prism of earth ADG would tend to slide off along a plane AG which would make with the horizontal the angle of repose ϕ. But if there is cohesion, this plane, which is called the plane of rupture, will make an angle ψ with the horizontal greater than the angle of repose ϕ. We call ψ the angle of rupture.

Let the angle of the earth-surface DG with the horizontal be α, and the weight of the prism ADG per unit of length be W.

The weight W acting at the centre of mass C can be resolved into a force N normal to the surface of rupture AG and a force P parallel to the surface.

We have then

$$P = W \sin \psi, \qquad N = W \cos \psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The force P tends to cause sliding. This force is resisted by the friction and the cohesion. The friction is μN, where $\mu = \tan \phi$ is the coefficient of static sliding friction, and the cohesion is $c \,.\, AG$, where c is the coefficient of cohesion.

We have then for equilibrium

$$P - \mu N - c \,.\, AG = 0, \quad \text{or} \quad P - \mu N = c \,.\, AG,$$

or

$$c = \frac{P - \mu N}{AG}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, for any plane which makes an angle with the horizontal greater or less than ψ there will be no sliding, and for that plane $P - \mu N$ will be less than $c \,.\, AG$. For the plane of rupture, then, we must have

$$\frac{P - \mu N}{AG} = \text{a maximum.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the vertical height of the prism be h. Then $AD = \dfrac{h}{\cos \beta}$, the angle DAG is $90 - (\beta + \psi)$, the area of the prism ADG is then

$$\frac{AG}{2} \,.\, AD \cos (\beta + \psi) = \frac{AG \,.\, h \cos (\beta + \psi)}{2 \cos \beta},$$

and the weight W per foot of length is, if γ_1 is the density of the earth,

$$W = \frac{\gamma h \,.\, AG \,.\, \cos (\beta + \psi)}{2 \cos \beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Inserting this value of W in (1) and the corresponding values of P and N in (3), we have, since $\mu = \tan\phi$,

$$\frac{\gamma h \cos(\beta+\psi)}{2\cos\beta\cos\phi} \cdot \sin(\psi-\phi) = c = \text{a maximum.} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

ANGLE OF RUPTURE.—Equation (5) is a maximum when

$$\cos(\beta+\psi) = \sin(\psi-\phi) = \cos[90^\circ - (\psi-\phi)],$$

or when

$$\beta+\psi = 90^\circ - (\psi-\phi), \quad \text{or when} \quad \psi = 45^\circ + \frac{(\phi-\beta)}{2}. \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) gives, then, the angle of rupture, or the angle which the plane of rupture AG makes with the horizontal.

COEFFICIENT OF COHESION.—If we insert this value of ψ in (5), we have

$$\gamma h \cos[45 + \tfrac{1}{2}(\phi+\beta)] \sin[45 - \tfrac{1}{2}(\phi+\beta)] = 2c\cos\beta\cos\phi,$$

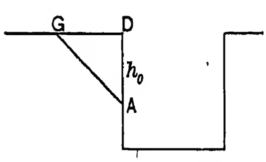
or

$$\gamma h[1 - \sin(\phi+\beta)] = 4c\cos\beta\cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let $\beta = 0$, and let h in this case be h_0. Then we have, from (7),

$$c = \frac{\gamma h_0(1-\sin\phi)}{4\cos\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (8) we can determine c by experiment. Thus if a trench with vertical sides, of considerable length as compared to its width, is dug in the earth, with a transverse trench at each end, so that lateral cohesion may not prevent rupture, it will be observed after a few days to have caved in along some plane, as AG. Let the observed depth AD be h_0. Then from (8) we can compute the coefficient of cohesion c.



HEIGHT OF SLOPE.—If we substitute the value of c from (8) in (7), we obtain

$$h = \frac{h_0(1-\sin\phi)\cos\beta}{1-\sin(\phi+\beta)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (6) and (9) we see that the angle of rupture ψ and the height of slope h are independent of the inclination α of the earth-surface.

Equation (9) gives the limiting height h of slope when sliding is about to take place. Let n be a factor of safety, so that if n is 2 or 3, the height can be taken two or three times the safe height before sliding begins. Then we have for the safe height

$$h = \frac{h_0(1-\sin\phi)\cos\beta}{n[1-\sin(\phi+\beta)]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (10) gives the safe height of slope for any given batter angle β.

ANGLE OF SLOPE.—If h is given and the corresponding batter angle β is required, we have, from (10),

$$\frac{1 - \sin(\phi + \beta)}{\cos \beta} = \frac{h_0(1 - \sin \phi)}{nh} = a,$$

where the second member, being a known quantity, is denoted by a.

If we develop the numerator in the first member and substitute for $\sin \beta$ and $\cos \beta$ their values in terms of $\tan \frac{1}{2}\beta$, viz.,

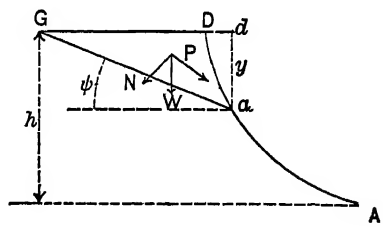
$$\sin \beta = \frac{2 \tan \frac{1}{2}\beta}{1 + \tan^2 \frac{1}{2}\beta}, \qquad \cos \beta = \frac{1 - \tan^2 \frac{1}{2}\beta}{1 + \tan^2 \frac{1}{2}\beta},$$

we obtain a quadratic whose solution gives

$$\tan \frac{1}{2}\beta = \frac{1}{1 + a + \sin \phi}[\cos \phi - \sqrt{a(a + 2 \sin \phi)}]. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Equation (11) gives the batter angle for a factor of safety n when the height is given.

CURVE OF SLOPE.—Let a be any point of the slope DaA, whose vertical distance below D is $da = y$, and let aG be the plane of rupture at the point a, making the angle ψ with the horizontal.



Then the prism DGa of weight W per foot of length tends to slide along aG and is prevented by friction and cohesion. Let N and P be the components of W normal and parallel to aG. Then if n is the factor of safety and μ is the coefficient of static sliding friction, we have

$$n(P - \mu N) - c \,.\, aG = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let A be the area daD between the curve of the slope and any ordinate $da = y$. Then $\gamma_1 A$ is the weight of the prism daD per foot of length, if γ_1 is the mass of a unit of volume of earth. The area daG is $\frac{\gamma_1 y^2 \cot \psi}{2}$ per foot of length. Hence the weight W of the prism DaG is

$$W = \gamma_1\left(\frac{y^2 \cot . \psi}{2} - A\right).$$

If we insert this value of W in the equations (1) for P and N, and then substitute in (12), we obtain, since $aG = \frac{y}{\sin \psi}$,

$$n\gamma_1\left(\frac{y^2 \cot \psi}{2} - A\right)(\sin \psi - \mu \cos \psi) - c\frac{y}{\sin \psi} = 0,$$

or, dividing by sin ψ,

$$n\gamma_1\left(\frac{y^2 \cot \psi}{2} - A\right)(1 - \mu \cot \psi) - cy(1 + \cot^2 \psi) = 0. \quad . \quad . \quad . \quad . \quad (13)$$

If aG makes an angle with the horizontal greater or less than ψ, we have, from (12), $n(P - \mu N)$ less than $c \cdot aG$, or the left side of equation (13) less than zero. The value of ψ must then make (13) a maximum.

If then we differentiate (13) with reference to cot ψ and put the first derivative equal to zero, we obtain

$$\frac{n\gamma_1 y^2}{2}(1 - \mu \cot \psi) - n\gamma_1\mu\left(\frac{y^2 \cot \psi}{2} - A\right) - 2cy \cot \psi = 0. \quad . \quad . \quad . \quad (14)$$

Eliminating cot ψ from (13) and (14), we obtain

$$A = \frac{y}{2n\mu^2\gamma_1}[n\mu\gamma_1 y + 4c - 2\sqrt{2c(n\mu\gamma_1 y + 2c)(1 + \mu^2)}]. \quad . \quad . \quad . \quad (15)$$

Equation (15) gives the area A between the curve of the slope and any ordinate $da = y$.

Examples.—(1) *A bank of earth without cohesion stands 30 ft. high with a slope of 50 ft. Find the coefficient of friction and the angle of repose.*

ANS. The horizontal projection of the slope is 40 ft. Hence $\mu = \tan \phi = \frac{30}{40} = 0.75$, and ϕ is about 35.

(2) *A bank of earth with vertical face is found to cave for a distance of 3 ft. below the surface. The same earth loose and without cohesion takes a slope of 1.25 to 1 horizontal. Find the slope after rupture. Also, if the mass of a cubic foot is 100 lbs., find the coefficient of cohesion.*

ANS. We have $\beta = 0$ and, from equation (6), page 468, $\psi = 45° + \frac{\phi}{2}$. The tangent of the angle of repose is $\mu = \tan \phi = 0.75$. Hence ϕ is about 35° and ψ is about 62°.

From equation (8), page 468, since $h_0 = 3$ ft., $\gamma_1 = 100$ lbs. per cubic ft., $\phi = 35°$,

$$c = \frac{100 \times 3(1 - \sin 35°)}{4 \cos 35°} = \frac{128}{3.28} = 39 \text{ pounds per sq. ft.}$$

(3) *A bank of earth the same as in the preceding example has a height of 30 ft. and a batter of 45°. Find the limiting height for the same slope, also the factor of safety.*

ANS. From equation (9), page 468, since $h_0 = 3$, $\beta = 45°$, $\phi = 35°$, the limiting height is

$$h = \frac{3(1 - \sin 35°) \cos 45°}{1 - \sin 80°} = \text{about 60 ft.,}$$

or the factor of safety is 2.

(4) *A bank of earth the same as in example (2) is required to have a height of 30 ft. and a factor of safety of 2. Find the batter of the face.*

ANS. From equation (11), page 469, $\beta = 45°$.

(5) *A bank of earth with vertical face caves for a distance of 5 ft. below the surface. The same earth loose and without cohesion takes a slope of 1.25 to 1 horizontal. The mass of a cubic foot is 100 lbs. Find the angle of rupture and the coefficient of cohesion. If the batter of the face is made 45° and the height 30 ft., find the factor of safety.*

ANS. The angle of repose ϕ is about 35°. The angle of rupture ψ is about 62°. The coefficient of cohesion is $c = 65$ pounds per sq. ft. From equation (10), page 468,

$$n = \frac{5(1 - \sin 35°) \cos 45°}{30(1 - \sin 80°)} = 3\tfrac{1}{2}.$$

(6) *Find the batter angle of the slope in the preceding example for a height of 30 ft. and a factor of safety of $3\frac{1}{8}$*

ANS. From equation (11), page 469, $\beta = 45°$.

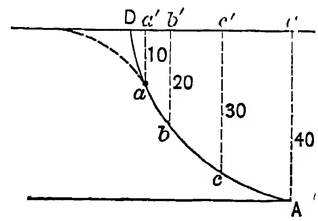
(7) *Find the curve of the slope in example (5) for a factor of safety of 3 and a height of 40 feet.*

ANS. We have $\mu = 0.75$, $c = 65$, $n = 3$, $\gamma_1 = 100$, and equation (15), page 470, becomes

$$A = \frac{y}{337.5}\left[225y + 260 - 2\sqrt{\frac{1625}{8}(225y + 130)}\right].$$

If we take	$y = 10$	20	30	40 ft.,
we have	$A = 33$	167	413	777 sq. ft.

Considering the area between the slope and any ordinate as made up of trapezoids as shown in the figure, we have



$$\frac{1}{2} \cdot 10 \times Da' = 33, \quad \text{or} \quad Da' = 6.6 \text{ ft.};$$

$$33 + \frac{10+20}{2} \times a'b' = 167, \quad \text{or} \quad a'b' = 9 \text{ ft.};$$

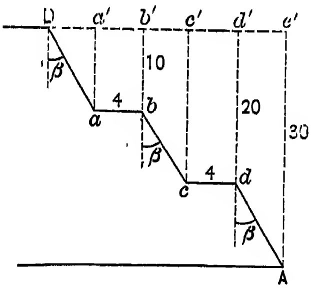
$$167 + \frac{20+30}{2} \times b'c' = 413, \quad \text{or} \quad b'c' = 9.8 \text{ ft.};$$

$$413 + \frac{30+40}{2} \times c'd' = 777, \quad \text{or} \quad c'd' = 10.4 \text{ ft.}$$

We see from equation (15), page 470, that for small values of y, A is negative. The equation should not be used for y less than h_0, and the upper part of the slope should be rounded off as shown in the figure.

(8) *It is desired to cut a bank 30 ft. high into three terraces as shown in the figure, with a factor of safety of 1.5. The height of each terrace is to be 10 ft., and there are to be two steps, ab and cd, each 4 ft. wide. The mass per cubic foot is $\gamma_1 = 100$ lbs., and ϕ and h_0 as found by experiment are $\phi = 31°$, $h_0 = 5$ ft. Find the batter for each terrace.*

ANS. We have $\mu = \tan\phi = 0.6$, and from equation (8), page 468, $c = 71$. Equation (15), page 470, becomes



$$A = \frac{y}{108}\left[284 + 90y - 2\sqrt{189(90y + 142)}\right].$$

For	$y = 10$	20	30
we have	$A = 27$	159	421

We have then

$$\frac{10}{2} \times Da' = 27, \quad \text{or} \quad Da' = 5.4 \text{ ft.};$$

$$27 + 40 + \frac{10+20}{2} \times b'c' = 159, \quad \text{or} \quad b'c' = 6.1 \text{ ft.};$$

$$159 + 40 + \frac{20+30}{2} \times d'e' = 421, \quad \text{or} \quad d'e' = 8.9 \text{ ft.}$$

Hence, for the batter angles,

For Da, $\quad \tan\beta = \frac{5.4}{10} \quad$ or $\quad \beta = 28\frac{1}{2}°$;

For bc, $\quad \tan\beta = \frac{6.1}{10}, \quad$ or $\quad \beta = 31\frac{1}{2}°$;

For dA, $\quad \tan\beta = \frac{8.9}{10}, \quad$ or $\quad \beta = 41\frac{1}{2}°$.

(9) *Find the batter angle for a railway embankment 30 ft. high, 12 ft. top base. Let $\gamma_1 = 100$, $\phi = 34°$, $h_0 = 4$ and factor of safety $n = 2$. Let the locomotive weight be about 6000 pounds per linear ft. of track.*

ANS. The weight of locomotive is 6000 pounds on 12 square feet, or 500 pounds per sq. ft. This is equivalent to a mass of earth 5 feet high.

We take, then, $h = 35$ feet in equation (11), page 469, and have

$$\tan \frac{1}{2}\beta = \frac{1}{1.584}\left[0.829 + \sqrt{0.0286}\right] = 0.416, \quad \text{or} \quad \beta = 45°.$$

The embankment with this batter contains 47 cubic yards per lineal foot, while with the natural slope of 34° it would contain 62°. There will then be a saving in cost of construction if the expense of protecting the slope to preserve the cohesion is not greater than the saving in embankment.

(10) *A railway cut is made in material for which $\gamma_1 = 100$, $\phi = 34°$, $h_0 = 5$ ft. The depth of cut is $h = 40$ ft., and the road-bed is 16 ft. Find the batter angle for a factor of safety of 3.*

ANS. We find $\beta = 47°$. The cut with this batter is 87 cubic yards per linear foot. For the natural slope it would be 111. There is then a saving in cost if the expense of protecting the slope to preserve the cohesion is not greater than the saving in excavation.

STATICS OF ELASTIC SOLIDS.

CHAPTER I.

ELASTICITY AND STRENGTH FOR TENSION, COMPRESSION AND SHEAR.

Elasticity.—A body is said to be "elastic" when force is necessary to change either its volume or shape and when the continued application of that force is necessary to maintain such change, so that the body recovers more or less completely its initial volume and shape when the force is removed.

If the body recovers completely its initial volume, it has perfect elasticity of volume. If it recovers completely its original shape, it has perfect elasticity of shape. If the recovery is imperfect, the elasticity is imperfect.

All bodies have some elasticity of volume. If a body possesses no elasticity of shape, it is called a *fluid*, as air, water, etc. If a body possesses elasticity of shape as well as volume, it is called an elastic *solid*. We deal here only with elastic solids.

Prismatic Body.—We shall treat also only of *prismatic* solids. A prismatic body is one whose volume can be generated by the motion of a plane area, moving so that its centre of mass describes any curve, the generating plane being always at right angles to that curve.

This curve is the AXIS of the body, and the generating plane at right angles to the axis is the CROSS-SECTION. The area and form of this cross-section may be constant or variable.

If either the area or form of cross-section varies, the cross-section is said to be VARIABLE. If neither the area nor form varies, the cross-section is said to be uniform.

Stress and Force.—As we have seen, page 397, we distinguish three kinds of force and stress.

Tensile force, tending to pull the particles of a body apart in parallel straight lines in the direction of the force, and tensile stress, resisting such separation.

Compressive force, tending to push the particles of a body together in parallel straight lines in the direction of the body, and compressive stress, resisting such approach.

Shearing force, tending to make adjacent particles move past one another at right angles to the line joining the particles, and shearing stress, resisting such motion.

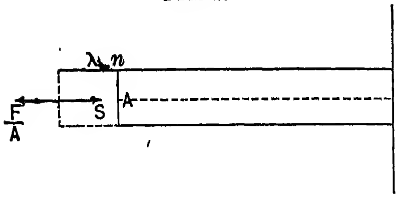
Other stresses with which we have to do are combinations of these.

We measure force and stress, then, in pounds, and unit force and unit stress in pounds per square inch.

STRAIN—The change of distance between two particles of a body, in a direction contrary to coexisting stress between those particles, is called STRAIN.

Thus if a tensile force F is applied to a straight homogeneous bar of uniform cross-section A in the axis of the bar, and this force F is uniformly distributed over the end area A, so that the unit force is $\frac{F}{A}$, we have a corresponding equal and opposite unit-stress S at every point of the cross-section, and the bar is extended a small amount λ in a direction opposite to coexisting stress. The strain λ between end particles is one of extension, or tensile strain.

FIG. 1.



If a compressive force F in the axis is uniformly distributed over the end area A, so that the unit-force is $\frac{F}{A}$, we have a corresponding equal and opposite unit-stress S, and the bar is compressed a small amount λ in a direction opposite to coexisting stress. The strain λ between end particles is one of compression, or compressive strain.

FIG. 2.

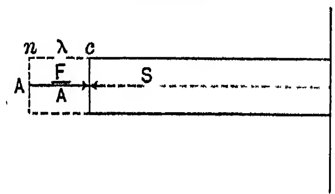
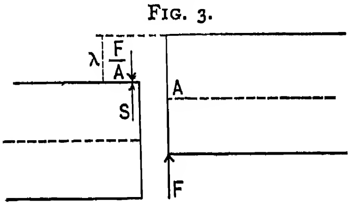


FIG. 3.



If a shearing unit-force $\frac{F}{A}$ is applied, we have a corresponding equal and opposite unit-stress S, and the cross-section slides on the adjacent one a small amount λ in a direction opposite to coexisting stress. The strain λ is one of shear, or shearing strain.

Strain, then, being a distance, is measured in units of length, as feet or inches, and the term "strain" is general and signifies always a change of distance between points of a body, *always in opposition to coexisting stress* between those points, without specifying whether that change of distance is due to extension, shortening, or sliding, or whether the form of the body is changed or not.*

It should be noted that when there is no coexisting *opposite* stress, there is no strain. Thus when the bar in Fig. 2 is compressed from n to c, let the force F be removed. The bar would expand from c to n, and during such expansion we have unbalanced stress in the direction of expansion. Such expansion is not, then, strain. It is not opposite to coexisting stress, and is simply displacement. The bar is not strained by such expansion, but, on the contrary, the original strain λ diminishes to zero at the neutral point n.

But after the end of the bar reaches the neutral point n, if it still continues to expand, as in Fig. 1, the stress will be opposite to the direction of expansion, and such expansion is strain. The bar is strained by such expansion.

Law of Elasticity.—When a force F is applied to a homogeneous elastic bar of uniform cross-section A, so that the unit-force is $\frac{F}{A}$ and acts to elongate, compress, or shear the bar, as in Figs. 1, 2, 3, a corresponding equal and opposite unit-stress S must always exist when there is equilibrium, and a corresponding strain λ is always observed in the direction of the force F, and therefore opposite to the coexisting unit-stress.

* It will be noticed, therefore, that strain does not necessarily imply distortion. In the two first cases above there is strain without distortion or change of shape. In the third there is strain with distortion. If a homogeneous sphere is equally compressed radially in all directions, it remains a sphere. It has strain, but no distortion. The word "distortion" is not, therefore, equivalent to shear, and in many cases would be misapplied if used as equivalent. The term "strain" in common language is used indifferently either for change of distance between points of a body, or for the force which causes this change. Our definition simply restricts its use to the first meaning.

So long as this unit-stress $S = \frac{F}{A}$ does not exceed a certain magnitude, all experiments prove that the unit-stress S and the corresponding strain λ are approximately proportional, so that

$$\frac{S}{\lambda} \quad \text{or} \quad \frac{F}{A\lambda} = \text{a constant}.$$

This is known as the *law of elasticity.*

The point where this law begins to fail is called the ELASTIC LIMIT, and the corresponding unit-stress at the elastic limit is therefore the elastic limit unit-stress. We denote it by S_e.

The law of elasticity is then expressed by saying that *for homogeneous bodies and within the elastic limit the strain and unit-stress are proportional.*

Beyond the elastic limit the strain increases more rapidly than the unit-stress, until finally rupture occurs.

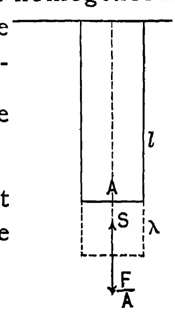
Set and Shock.—As we have seen, page 473, if when the force F is removed the strain λ entirely disappears, the bar is perfectly elastic. If the strain λ does not entirely disappear, the bar is imperfectly elastic, and the residual strain is called the SET.

For very small stresses the set, if any, is too small to be detected by measurement, and the bar may be regarded as practically perfectly elastic. As, however, no solid body is perfectly elastic, the point at which set is first observed depends upon accuracy of measurement. Theoretically there is a set for any stress.

The point at which set is first observed is often taken as marking the limit of elasticity. But in view of the preceding it should not be so taken, and experiments generally show a slight set before the limit of elasticity is reached.

A suddenly applied stress or shock is found by experiment to be more injurious than a steady stress or a stress gradually applied.

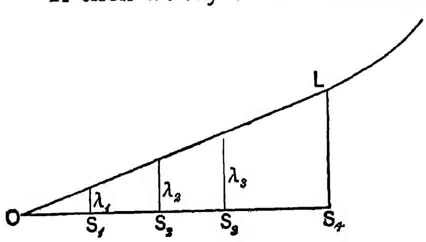
Determination of Elastic Limit and Ultimate Strength.—Let a straight homogeneous bar of given material have a length l and uniform cross-section A. Let a tensile force F be applied in the axis and be uniformly distributed over the end cross-section so that the unit-stress is $S = \frac{F}{A}$ opposite in direction to F. Let the observed strain of elongation be λ.



Now, according to the law of elasticity, provided the elastic limit is not exceeded, if we double F we shall observe a strain 2λ. If we apply $3F$, we shall observe a strain 3λ, and so on.

If then we lay off the successive unit stresses

$$S_1 = \frac{F}{A}, \qquad S_2 = \frac{2F}{A}, \qquad S_3 = \frac{3F}{A}, \quad \text{etc.,}$$



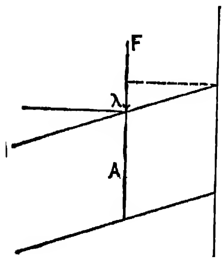
to scale along a horizontal line from O, and lay off to scale as ordinates the corresponding observed strains λ_1, $\lambda_2 = 2\lambda_1$, $\lambda_3 = 3\lambda_1$, etc., it is evident that within the elastic limit we have, by the law of elasticity, a straight line OL. The point L, where the straight line begins to curve, marks the elastic limit, and the corresponding unit-stress S_e is the elastic limit unit-stress.

As the straight line passes into a curve gradually, it is evident that the point L is not

very definite, and its location, even with very accurate experimental results, will vary more or less within certain limits.

When the elastic limit is passed the elongation increases more rapidly than the unit-stress, and the cross-section decreases, until the bar ruptures. The unit-stress at which rupture occurs is called the ULTIMATE STRENGTH for tension. We denote it by U_t.

We determine the elastic limit unit-stress S_e for compression, and the ultimate strength for compression U_c, in precisely the same way by experiment. The force F is axial and uniformly distributed over the end cross-section. The length of the test specimen should be small, not over five times its least diameter, in order to avoid bending. After the elastic limit is reached the strain increases more rapidly than the unit-stress, and the cross-section enlarges.



For shear the strain λ is as shown in the figure. The elastic limit unit-stress S_s and ultimate strength U_s are then theoretically determined in the same way as for tension and compression, by measuring λ for different values of $\frac{F}{A}$.

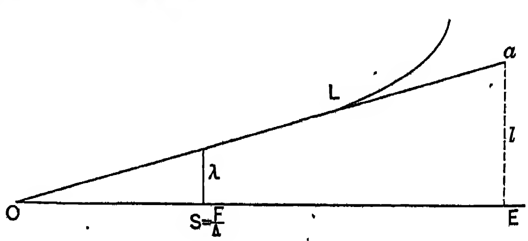
It is, however, difficult to measure λ for shear, and hence the experimental determination of S_s is more uncertain than for tension or compression, and reliable experiments are wanting.

The values of S_e, U_t, U_c and U_s vary considerably for the same material, according to grade and quality. Thus, for instance, for timber there are different kinds, and each kind varies according to soil, climate, season when cut, method and duration of seasoning, direction of fibres with reference to stress, etc. So also iron and steel vary according to quality, process of manufacture, whether in bars, plates or wire, etc. We give in the following table such average values as may be used for ordinary preliminary computations and estimates. For actual design, the necessary values for the material used should be accurately known by special tests.

TABLE OF ELASTIC LIMIT AND ULTIMATE STRENGTH—AVERAGE VALUES.

Material.	Elastic Limit Unit-stress in pounds per square inch.		Ultimate Strength in pounds per square inch.		
	Tension. S_e.	Compression. S_e.	Tension. U_t.	Compression. U_c.	Shear. U_s.
Wrought iron..........	25000	25000	55000	55000	50000
Steel (structural).......	40000	40000	100000	150000	70000
Cast iron...............	6000	....	20000	90000	20000
Timber.................	3000	3000	10000	8000	600 longitudinal. 3000 transverse.
Stone..................	....	....	....	6000	....
Brick..................	....	....	....	2500	....

Coefficient of Elasticity.—In the experiment described on page 475, we find for a homogeneous bar of given material, cross-section A and original length l, extended by an axial force F uniformly distributed over the end area, that within the elastic limit, according to the law of elasticity,



$$\frac{S}{\lambda} \text{ or } \frac{F}{A\lambda}$$

is constant. We see from the figure that this constant is the tangent of the angle which the straight line OL makes with the vertical.

But if we had taken a different length l_1, everything else being the same, we should have for the same loads a different constant or tangent,

$$\frac{F}{A\lambda_1}.$$

Now if the bar is homogeneous, the stress and hence the strain between two consecutive particles in the line of strain will be the same along the whole length. The total strain will then be the sum of all the strains between consecutive particles, or proportional to the length.

We have then for different lengths

$$\lambda_1 : \lambda :: l_1 : l, \quad \text{or} \quad \lambda_1 = \frac{l_1\lambda}{l}.$$

Substituting this, we have

$$\frac{Fl}{Al_1\lambda} = \text{a constant.}$$

If then λ is known for any length l by experiment, this expression will give the constant or tangent of the angle of OL with the vertical for any other length l_1.

Let the length l_1 be unity. Then we shall have for the constant in this case

$$\frac{Fl}{A\lambda}.$$

This latter constant is called the *coefficient of elasticity*, and we denote it by E. We have then

$$E = \frac{Fl}{A\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

Now $\frac{F}{A}$ is the unit-stress, and $\frac{\lambda}{l}$ is the strain per unit of length, or the unit-strain.

We can then define the coefficient of elasticity as *the ratio of the unit-stress to the unit-strain*, within the limit of elasticity.

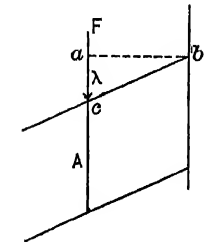
We have also, from (I),

$$\frac{F}{A} : \lambda :: E : l,$$

and from the preceding figure we see at once that we may also define E as that unit-stress which would cause an elongation equal to the original length of the bar, if the law of elasticity held good without limit.

The value of E thus determined by experiments well within the elastic limit is then a measure of the elasticity of the body. We determine the coefficient of elasticity E_e for

compression in precisely the same way, having regard to the precautions mentioned on page 476.



For shear we have the same expression for E_s as given by (I), if we put for l the distance ab, as shown in the figure, and for λ the distance ac. The direct measurement of λ is difficult and uncertain, but E_s can be determined indirectly by experiments on torsion and bending, as will be explained later (pages 511 and 555).

The values of E_t, E_c and E_s vary for the same material according to grade and quality, just as the values of S_e, for the reasons given on page 476. We give in the following table average values which may be used for ordinary preliminary computations. For actual design the values should be accurately known by special test.

Equation (I) then holds for tension, compression or shear, *within the elastic limit*, if we put E_t, E_c, E_s in place of E. In using (I), since E is always given in pounds per square inch, we should take l and λ in inches and A in square inches.

TABLE OF COEFFICIENT OF ELASTICITY—AVERAGE VALUES.

Material.	Coefficient of Elasticity in pounds per square inch.		
	Tension. E_t	Compression. E_c	Shear. E_s
Wrought iron..............	25 000 000	25 000 000	15 000 000
Steel (structural)............	30 000 000	30 000 000	7 000 000
Cast iron..................	15 000 000	15 000 000	6 000 000
Timber....................	1 500 000	1 500 000	400 000 longitudinal.
Stone.....................	....	6 000 000	....
Brick.....................	....	....	....

As an example, we give the following record of an actual series of experiments made upon a wrought-iron bar $\frac{3}{4}$ inches diameter and 12 inches long. The first four columns give the experimental record. In the fifth column we give the corresponding values of $\frac{\lambda}{S}$.

Total Stress in Pounds. F.	Stress pounds per square inch $S = \frac{F}{A}$.	Elongation λ		$\frac{\lambda}{S} = \frac{l}{E}$.
		Load on λ in inches.	Load off λ in inches.	
2245	5000	0.001	0.000	0 000 000 20
4490	10000	0.004	0.000	0 000 000 40
6375	15000	0.005	0.000	0.000 000 33
8980	20000	0.008	0.000	0.000 000 40
9878	22000	0.009	0.000	0.000 000 41
10776	24000	0.010	0.000	0.000 000 42
11674	26000	0.0105	0.000	0.000 000 40
12572	28000	0.011	0.000	0.000 000 39
13470	30000	0.013	0.000	0.000 000 43
14368	32000	0.014	0.000	0 000 000 44
15266	34000	0.015	0.002	0.000 000 44
16164	37000	0.022	0.007	0.000 009 61
17062	38000	0.416	0.3995	0.000 001 09
17960	40000	0.5445	0.523	0.000 013 61
25450	50000	1.740	1.707	0.000 034 80
23175	51600	2.468	....	....

Specimen broke at 51600 pounds per square inch.

We see that the ultimate tensile strength is $U_t = 51600$ pounds per square inch; also that the elastic limit unit stress S_e lies between 34000 and 36000 pounds per square inch. If we should judge by the beginning of noticeable set, the elastic limit would be between 32000 and 34000 pounds per square inch, and up to this point the elasticity appears to be perfect. The elasticity, however, is not perfect, and more accurate measurement would undoubtedly show the set beginning earlier. We cannot, then, take the beginning of observed set as the limit of elasticity. Since the elastic limit lies, then, between 34000 and 36000 pounds per square inch, we have for the average $\frac{\lambda}{S}$ up to this point $\frac{\lambda}{S} = 0.000\,000\,387\,3$, and hence the value of E_t given by this experiment is

$$E_t = \frac{lS}{\lambda} = \frac{12}{0.000\,000\,387\,3} = 30\,984\,000 \text{ pounds per square inch.}$$

Examples.—(1) *A steel rod 30 ft. long and 4 square inches cross-section is subjected to a tensile force of 40000 pounds. The elongation is observed to be $\frac{12}{100}$ of an inch. Find the coefficient of elasticity.*

ANS. The unit stress is 10000 pounds per square inch. From our table page 476 we see that this is well within the elastic limit unit-stress S_e, and equation (I) can therefore be applied and we have

$$E_t = \frac{Fl}{A\lambda} = \frac{40000 \times 30 \times 12}{4 \times \frac{12}{100}} = 30\,000\,000 \text{ pounds per square inch.}$$

(2) *A rectangular timber strut is 40 feet long and $13\frac{1}{3}$ inches deep. If E_c is 1 200 000 pounds per square inch, find the width so that the strain under a stress of 270000 pounds may be one inch, all lateral bending being prevented.*

ANS. We have, from equation (I),

$$A = \frac{Fl}{E\lambda} = \frac{270000 \times 40 \times 12}{1\,200\,000 \times 1} = 108 \text{ square inches.}$$

This gives 2500 pounds per square inch, which we see from our table page 476 is within the elastic limit. Equation (I) therefore applies, and we have then a width of $\frac{108}{13\frac{1}{3}} = 8.1$ inches.

Strain Due to Weight.—If a homogeneous straight prismatic body of uniform cross-section A has a considerable length l, the strain due to its own weight may be considerable.

Let δ be the density or weight of a cubic foot, For any length x the weight is then δAx, and the corresponding strain in an element of length dx is then, by equation (I),

$$d\lambda = \frac{\delta Ax \cdot dx}{AE} = \frac{\delta}{E} x\, dx.$$

Integrating between the limits $x = l$ and $x = 0$, we have for the entire strain

$$\lambda = \frac{\delta l^2}{2E} = \frac{\delta Al^2}{2AE},$$

or, since δAl is the entire weight W,

$$\lambda = \frac{Wl}{2AE},$$

or, as we see from (I), the strain is one-half as much as that due to the same weight at the end.

Example.—*How long must a homogeneous bar of wrought iron of uniform cross-section A be in order that when suspended vertically the greatest unit-stress may be equal to the elastic limit unit-stress, and what is the extension?*

ANS. The weight δAl must equal S_eA, or $\delta l = S_e$, or $l = \frac{S_e}{\delta}$. From our table page 476, $S_e = 25000$ pounds per square inch; and since a bar of wrought iron one square inch in cross-section and 3 feet long weighs 10 pounds (page 19), $\delta = \frac{10}{36}$ lbs. per cubic inch. We have then

$$l = \frac{25000 \times 36}{10 \times 12} = 7500 \text{ feet.}$$

The extension is, taking $E = 25\,000\,000$ pounds per square inch (page 478),

$$\lambda = \frac{S_e l}{2E} = \frac{25000 \times 7500}{2 \times 25\,000\,000} = 3.75 \text{ feet.}$$

Stress Due to Change of Temperature.—We have, from equation (I),

$$\lambda = \frac{Sl}{E},$$

where λ is the strain corresponding to the unit-stress $S = \frac{F}{A}$ for a bar of length l and uniform cross-section, the coefficient of elasticity being E.

If the bar is constrained so that it cannot change its length and then is exposed to change of temperature, there will be a unit-stress S equal to that due to a strain equal to the change of length for the same unconstrained bar under the same change of temperature.

Thus if ϵ is the coefficient of linear expansion for one degree of temperature, t the number of degrees change of temperature, and l the original length, the change of length of the unconstrained bar would be

$$\lambda = \epsilon tl.$$

The coefficient of expansion $\epsilon = \frac{\lambda}{tl}$ is then the strain per unit of length per degree.

If the bar is constrained so that its length cannot change, we have then a unit-stress

$$S = \frac{E\lambda}{l} = E\epsilon t.$$

which is independent of the length l. The total stress, if the area is A, is then

$$AS = AE\epsilon t.$$

We give the following average values of the coefficient of linear expansion ϵ for one degree Fahrenheit:

Wrought iron	$\epsilon = 0.0000067$
Steel	$\epsilon = 0.0000065$
Cast iron	$\epsilon = 0.0000062$
Stone and brick	$\epsilon = 0.0000050$

Example.—*A wrought-iron rod of length l and 2 square inches cross-section has its ends fixed rigidly when the temperature is 60° F. Taking the contraction at $0.0000067l$ for one degree, what tension will be exerted when the temperature is 20° F.?*

ANS. The shortening due to cooling is $\lambda = 0.0000067 \times 40l$. We have then, from equation (I),

$$F = \frac{EA\lambda}{l} = \frac{E \times 2 \times 0.0000067 \times 40l}{l} = 0.000536E.$$

If $E = 30\,000\,000$ pounds per square inch, $F = 16080$ pounds. This gives the unit-stress 8040 pounds per square inch, which we see, from our table page 476, is well within the elastic limit, and equation (1) therefore applies.

Working Stress—Factor of Safety.—The greatest unit-stress to which a material can safely be subjected in practice is called the WORKING STRESS. We denote it by S_w. The working stress should never equal the elastic limit unit-stress S_e, since if it exceeds that limit the elasticity is impaired. For security, then, we take a fraction of S_e. The number f by which S_e is thus divided is called the *factor of safety*. We have then the working stress

$$S_w = \frac{S_e}{f},$$

where f is the factor of safety.

For steady stress it is customary to take $f = 1.5$, and for repeated stress or suddenly applied stress $f = 3$.

As we have seen, page 476, the exact determination of S_e is difficult. It is therefore also customary to take S_w a certain fraction of the ultimate strength U_t, U_c, U_s for tension, compression or shear.

The following table gives the factors of safety usually adopted when this method is used:

FACTORS OF SAFETY FOR $S_w = \frac{U_t}{f}, \frac{U_c}{f}, \frac{U_s}{f}$.

	Steady Stress (Buildings).	Varying Repeated Stress (Bridges).	Shocks (Machines).
Wrought iron	4	6	10
Steel (structural)	5	7	10
Cast iron	6	10	15
Timber	8	10	15
Stone	15	25	30
Brick	15	25	30

In order to find the area of cross-section A for simple tension, compression or shear, we have then simply to divide the given total stress by the working stress S_w. We have then, *when flexure is not to be apprehended*, for steady or varying stress or shocks

$$A = \frac{\text{total stress}}{S_w}.$$

Sometimes we have *alternating stress*, i.e. tension and compression alternating, as in the connecting-rod or piston-rod of an engine. In such case we find the area of cross-section for each stress and take the greatest. Thus, *if flexure is not to be apprehended*,

$$A = \frac{\text{total tensile stress}}{S_{wt}} \quad \text{or} \quad \frac{\text{total compressive stress}}{S_{wc}},$$

whichever is the greatest.

When flexure of long columns is to be guarded against we must proceed as on page .

Variable Working Stress.—The fact that the working stress S_w, as determined in the preceding article, is constant in any given case, is by many engineers considered objectionable.

The total stress can in general be divided into two portions. The one portion is a steady stress always existing, such as that due to weight or dead load. The other portion is a repeated stress, such as that due to loads recurring at intervals of time.

Evidently when the ratio of the steady stress to the total stress is great, we should be able to use a greater working stress than when this ratio is small. Thus when the steady stress is equal to the total stress there is no repeated stress at all, and the working unit-stress should have its greatest value. On the other hand, when the steady stress is zero we have repeated stress only, and the working unit-stress should have its least value.

It is therefore customary to take for the working unit-stress, *when flexure is not to be apprehended*, for *repeated stress* of any kind

$$S_w = \frac{U_p}{f}\left(1 + \frac{U - U_p}{U_p} \cdot \frac{\text{steady stress}}{\text{total stress}}\right). \quad \ldots \ldots \ldots \quad (1)$$

From (1) we see that when the steady stress is equal to the total stress, that is, when there is no repeated stress, we have $S_w = \frac{U}{f}$, where U is accordingly the ultimate strength and f the factor of safety, just as in the preceding article.

But when the steady stress is zero, we have only repeated stress, and (1) gives us $S_w = \frac{U_p}{f}$. Hence U_p is the ultimate strength for repeated stress only, or the "repetition strength" as it may be called.

In like manner, when *flexure of long columns is not to be apprehended*, we have for the working unit-stress *for alternating stress*

$$S_w = \frac{U_p}{f}\left(1 - \frac{U_p - U_v}{U_p} \cdot \frac{\text{least of the two stresses}}{\text{greatest of the two stresses}}\right). \quad \ldots \ldots \quad (2)$$

From (2) we see that when the least of the two stresses is zero, we have $S_w = \frac{U_p}{f}$, as in the previous case, for steady stress zero.

But when the two alternating stresses are equal, we have $S_w = \frac{U_v}{f}$. Hence U_v is the ultimate strength for equal alternating stresses, or the "vibration strength" as it may be called.

The difficulty and uncertainty of determining U_p and U_v by experiment, and the few experiments thus far available, make the method of the preceding article the most generally accepted. The present method is, however, extensively used with assumed average values for U, U_p and U_v as given in the following table:

	$\frac{U_p}{f}$	$\frac{U - U_p}{U_p}$	$\frac{U_p - U_v}{U_p}$
Wood	400	2	$\frac{1}{2}$
Wrought iron	7500	1	$\frac{1}{2}$
Cast iron	10000	$\frac{4}{3}$	$\frac{2}{5}$
Steel (structural)	17800	1	$\frac{7}{15}$

These values are for direct tension or compression. For shear we take four-fifths of S_w as determined above.

In order to determine the area of cross-section A, we have, then, in all cases

$$A = \frac{\text{total maximum stress}}{S_w}.$$

When flexure of long columns is to be provided against we must proceed as on page 569.

Examples.—(1) *A wrought iron tie-rod in a roof-truss is under a stress of 20000 lbs. Find its area of cross-section.*

ANS. Taking the ultimate tensile strength (page 476) $U_t = 55000$ pounds per square inch, and factor of safety 4 (page 481), we have

$$S_w = \frac{U_t}{f} = \frac{55000}{4} = 13750 \text{ pounds per square inch,}$$

and hence

$$A = \frac{20000}{13750} = 1.45 \text{ square inches.}$$

Or again, from (1), page 482, if steady stress is equal to total stress, we have the same result.

(2) *Suppose the rod is a bridge member and the dead-load stress is 5000 pounds, the total stress being 20000 pounds as before.*

ANS. We have, as before, $U_t = 55000$, and (page 481) the factor of safety 6. Hence

$$S_w = \frac{55000}{6} = 9166 \quad \text{and} \quad A = \frac{20000}{9166} = 2.18 \text{ sq. in.}$$

Also by (1), page 482, we have

$$S_w = 7500\left(1 + \frac{5000}{20000}\right) = 9375 \quad \text{and} \quad A = \frac{20000}{9375} = 2.13 \text{ sq. in.}$$

(3) *The greatest steam-pressure upon the piston of a steam-engine is $p = 150$ pounds per square inch. If the area of the piston is $a = 200$ square inches, find the cross-section of the steel piston-rod, if lateral bending is prevented.*

ANS. The total pressure is pa. Taking the ultimate compressive strength $U_c = 150000$ and the ultimate tensile strength $U_t = 100000$ pounds per square inch (page 476), and the factor of safety 10 (page 481), we have the working stress

$$S_w = \frac{150000}{10} = 15000 \quad \text{or} \quad S_w = \frac{100000}{10} = 10000.$$

Taking the least of these, we have

$$A = \frac{pa}{S_w} = \frac{150 \times 200}{10000} = 3 \text{ square inches.}$$

Also, om (2), page 482, we have

$$S_w = 17800\left(1 - \frac{7}{15}\right) = 9500.$$

Hence by this method

$$A = \frac{150 \times 200}{9500} = 3.15 \text{ square inches.}$$

(4) *Find the height to which a brick wall of uniform thickness can be safely carried, the density being $\delta = 125$ pounds per cubic foot.*

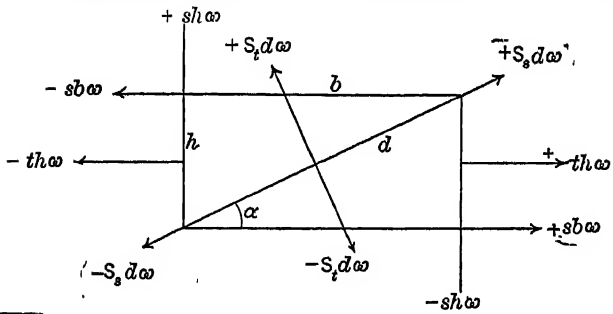
ANS. Taking the ultimate strength $U_c = 2500$ (page 476) and the factor of safety 15 (page 481), we have the working stress

$$S_w = \frac{2500 \times 144}{15} \text{ pounds per square foot.}$$

If h is the height, l the length and t the thickness, all in feet, the cubic contents is hlt cubic feet, and the weight δhlt pounds. The base is lt square feet, and hence the pressure on the base is $\frac{\delta hlt}{lt} = \delta h$ pounds per square foot. Putting this equal to S_w, we have

$$\delta h = S_w, \quad \text{or} \quad h = \frac{S_w}{\delta} = \frac{2500 \times 144}{15 \times 125} = 192 \text{ feet.}$$

Combined Tension or Compression and Shear.—Suppose an element of a body of breadth b, height h and width w, and let t be the unit stress of direct tension and s the unit stress of direct shear.



We have then the two equal and opposite tensile stresses $+thw$ and $-thw$, and the two equal and opposite shearing stresses $+shw$ and $-shw$. These last two stresses form a couple and can only be held in equilibrium by an equal and opposite couple $+sbw$ and $-sbw$.

Let d be the diagonal and α the angle of the diagonal with the direction of t. Let S_s be the unit shear along the diagonal, and S_t the unit tension normal to the diagonal. Then we have along the diagonal the components $+S_sdw$ and $-S_sdw$, and normal to the diagonal the components $+S_tdw$ and $-S_tdw$.

We have then for equilibrium

$$S_sdw - thw\cos\alpha - sbw\cos\alpha + shw\sin\alpha = 0,$$

$$S_tdw - thw\sin\alpha - sbw\sin\alpha - shw\cos\alpha = 0.$$

Now we have

$$\sin\alpha = \frac{h}{d} \quad \text{and} \quad \cos\alpha = \frac{b}{d};$$

hence, dividing these equations by dw, we have

$$S_s = t\sin\alpha\cos\alpha + s\cos^2\alpha - s\sin^2\alpha = \frac{t}{2}\sin 2\alpha + s\cos 2\alpha,$$

$$S_t = t\sin^2\alpha + 2s\cos\alpha\sin\alpha = \frac{t}{2} - \frac{t}{2}\cos 2\alpha + s\sin 2\alpha.$$

If we differentiate these equations and put the first differentials $\dfrac{dS_s}{d\alpha} = 0$ and $\dfrac{dS_t}{d\alpha} = 0$, we have, when S_s is a maximum,

$$\tan 2\alpha = \frac{t}{2s}, \quad \text{or} \quad \sin 2\alpha = \frac{t}{\sqrt{4s^2+t^2}} \quad \text{and} \quad \cos 2\alpha = \frac{2s}{\sqrt{4s^2+t^2}}; \quad . \quad (1)$$

and when S_t is a maximum,

$$\tan 2\alpha = -\frac{2s}{t}, \quad \text{or} \quad \sin 2\alpha = -\frac{2s}{\sqrt{4s^2+t^2}} \quad \text{and} \quad \cos 2\alpha = \frac{t}{\sqrt{4s^2+t^2}}. \quad . \quad (2)$$

Inserting these values, we have for the maximum combined unit shear

$$max.\ S_s = \sqrt{s^2 + \frac{t^2}{4}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the maximum combined unit tension

$$max.\ S_t = \frac{t}{2} + \sqrt{s^2 + \frac{t^2}{4}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If then the direct tensile and shearing unit stresses t and s are given, we have from (3) the maximum combined unit shear S_s, and from (1) its direction, or the angle α which it makes with the direction of t. We also have from (4) the maximum combined unit tension S_t, and from (2) its direction, or the angle α which it makes with the normal to t.

For compression and shear combined we have to substitute for t the direct compressive unit stress c, and then the same equations hold.

Example.—*A rivet $\frac{3}{4}$ inch in diameter is subjected to a tension of 2000 pounds, and at the same time to a shear of 3000 pounds. Find the combined maximum tensile and shearing unit stresses and the angles they make with the axis of the rivet.*

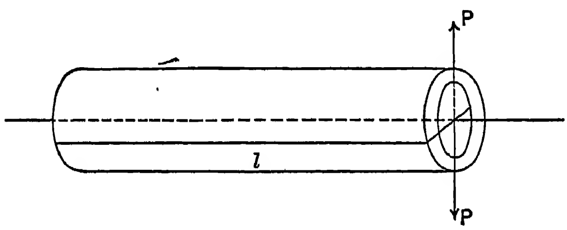
ANS. We have the area $\frac{\pi d^2}{4}$, where d is the diameter of the rivet. Hence the direct tensile unit stress is $t = \frac{2000 \times 4}{\pi d^2}$, and the direct shear is $s = \frac{3000 \times 4}{\pi d^2}$. From (3) we have $S_s = 7155$ pounds per square inch, making, from (1), an angle $\tan 2\alpha = \frac{t}{2s}$, or $\alpha = 9° \ 13'$, with the axis of the rivet, and from (4), $S_t = 9420$ pounds per square inch, making an angle $\tan 2\alpha = -\frac{2s}{t}$, or $54° \ 23'$, with the normal to the axis, or $35° \ 47'$ with the axis of the rivet.

CHAPTER II.

STRENGTH OF PIPES AND CYLINDERS. RIVETING.

Strength of Pipes and Cylinders.—A practical application of the principles of the preceding chapter is the determination of the size of pipes and cylinders, subjected to internal pressure.

Let p be the pressure per square inch on the interior surface of a pipe or cylinder, due to the pressure of water or steam or air, or other fluid. It is a well-known principle of physics that the pressure of a fluid in any direction is equal to its pressure on a plane at right angles to that direction.



Hence, in the figure, the pressure P, say in a vertical direction, is equal to the pressure on a horizontal plane of area ld, where l is the length and d the interior diameter. We have then

$$P = pld.$$

If S_w is the working stress for the material, and t is the thickness, we have then

$$pld = 2tlS_w, \quad \text{or} \quad t = \frac{pd}{2S_w}.$$

Pipes come in commercial sizes, and the preceding formula enables us to select the nearest commercial size for given unit-pressure, diameter and working stress.

If we consider the preceding figure as a closed cylinder, then the pressure on the head is $p \times \frac{\pi d^2}{4}$, and the area of cross-section is πtd. We have then

$$p \times \frac{\pi d^2}{4} = \pi tdS_w, \quad \text{or} \quad t = \frac{pd}{4S_w}.$$

Hence the thickness to resist longitudinal rupture is twice that necessary to resist end rupture. For water-pressure, if the head h is taken in feet, the pressure in pounds per square inch is $p = 0.434h$.

Examples.—(1) *A cast-iron water-pipe 12 inches diameter and $\frac{5}{8}$ inch thick is under a head of 300 feet. Find the factor of safety.*

ANS. The unit-pressure is $300 \times 0.434 = 130.2$ pounds per square inch. Hence the working stress is

$$S_w = \frac{pd}{2t} = \frac{130.2 \times 12}{2 \times \frac{5}{8}} = 1230 \text{ pounds per square inch.}$$

From our table page 476, the ultimate strength of cast iron for tension is 20000 pounds per square inch. The factor of safety is then $\frac{20000}{1230}$ = about 16.

(2) *Find the thickness of a cast-iron pipe 18 inches in diameter under a head of water of 300 feet, taking a factor of safety of 10.*

ANS. From our table page 476, we have the ultimate strength for cast iron in tension 20000 pounds per square inch. Our working stress is then $S_w = 2000$ pounds per square inch. Hence

$$t = \frac{pd}{2S_w} = \frac{300 \times 0.434 \times 18}{2 \times 2000} = 0.586 \text{ inch.}$$

(3) *A wrought-iron pipe 4.5 inches internal diameter weighs 12.5 pounds per linear foot. What pressure can it carry with a factor of safety of 8?*

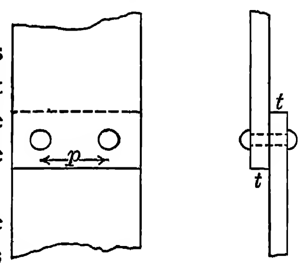
ANS. A bar of wrought iron 1 square inch in cross-section and 3 feet long weighs 10 pounds. Hence the area of cross-section of the pipe is $12.5 \times \frac{3}{10} = 3.75$ square inches. The thickness is then $t = \frac{3.75}{2\pi r} = \frac{1}{4}$ inch. We have, from our table page 481, the working stress $S_w = \frac{55000}{8}$. Hence

$$p = \frac{2tS_w}{d} = \frac{2 \times 55000t}{8d} = 763 \text{ pounds per square inch.}$$

Theory and Practice of Riveting.—Another important practical application of preceding principles is the determination of the size and number of the rivets with which plates are fastened together.

KINDS OF RIVETED JOINTS.—We may distinguish the following joints:

FIG. 1.



1st. *Simple "Lap" Joint, Single-riveted.*—Fig. 1 shows this joint front and side. The two plates overlie each other by an amount equal to the "*lap*," and are united by a single row of rivets. The distance p from centre to centre of a rivet is called the *pitch*. We denote the diameter of rivet by d, and the thickness of plate by t.

2d. "*Lap" Joint, Double-riveted.*—This joint is similar to the preceding, except that two rows of rivets are used. In both cases the rivets are in *single shear*.

In all cases where more than one row of rivets is used the rivets are "*staggered*," or so spaced that those in one row come midway between those in the next, as shown in Fig. 2.

Lap joints are used in tension only.

FIG. 2.

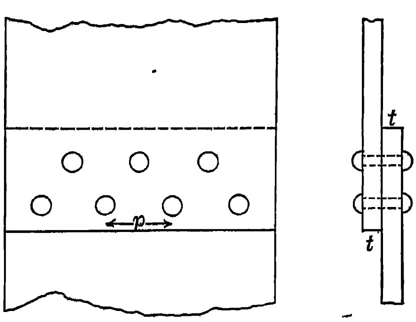
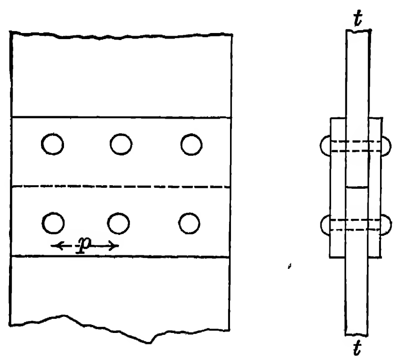


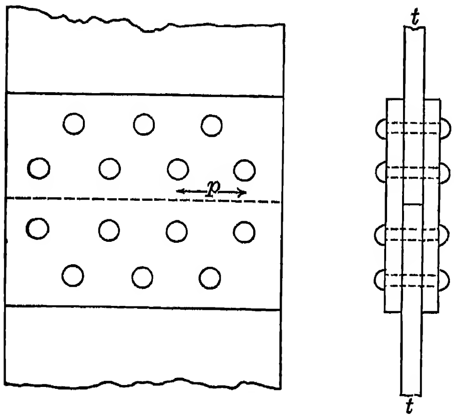
FIG. 3.



3d. "*Butt" Joint, Single-riveted, Two Cover-plates.*—Here the two plates are set end to end, making a "*butt*" *joint*, and a pair of "*cover-plates*" are placed on the back and front and riveted through by a single row of rivets on each side of the joint (Fig. 3). The plates in such a joint are in general not allowed to actually touch, and the entire stress, whether tensile or compressive, is therefore transmitted by the rivets. The thickness of the cover-

plates should not be less than half the thickness of the plates joined, except when this rule would give a thickness less than $\frac{1}{4}$ inch. Owing to deterioration of the metal by the action of the weather, *no plate is used in construction less than $\frac{1}{4}$ inch in thickness.* Hence if the plates joined are less than $\frac{1}{2}$ inch, the cover-plates should be $\frac{1}{4}$ inch.

FIG. 4.



4th. *"Butt" Joint, One Cover-plate, Single-riveted.*—This is the same as the preceding, except that one cover-plate only is used, of the same thickness as the plates themselves.

5th. *Double-riveted "Butt" Joint, Two Cover-plates.*—This joint is the same as case 3, except that we have two rows of rivets on each side of the joint.

The thickness of the cover-plates is determined by the same considerations as in case 3.

6th. *"Butt" Joint, One Cover-plate, Double-riveted.*—This is the same as the preceding case, except that there is only one cover-plate of the same thickness as the plates themselves.

7th. *Chain Riveting.*—When we have more than two rows of rivets on each side of a butt joint the system is called chain riveting. Such a disposition becomes necessary when the requisite number of rivets is so great that they cannot be disposed in two rows without unduly weakening the plates.

SIZE AND NUMBER.—In a riveted joint the resistance of the rivets should equal the strength of the plates joined. A rivet may fail by shearing across or by being crushed. The plate may fail by rupture between the rivets or by tearing through of the rivets at the edge of the plate. The rivets should be so proportioned and spaced that the strength for any method of failure may be equal and the plates weakened as little as possible.

Notation.—Let S_w be the working unit stress of the plates, either compression or tension, S_{wc} the working unit stress for compression, S_{ws} the working stress for shear, t the thickness of the plates, d the diameter of rivet, p the pitch of rivets in a row, or the distance from centre to centre in a row, and n the number of rivets.

Diameter of Rivets.—Then the area of a rivet is $\frac{\pi d^2}{4} = 0.7854d^2$. The shearing resistance of a rivet is $0.7854d^2S_{ws}$, and the total shearing resistance of n rivets is $0.7854nd^2S_{ws}$. The bearing surface of a rivet is dt, of n rivets ndt, and the resistance to crushing $ndtS_{wc}$. For equal strength of crushing and shearing we have for *single shear*, or lap joint,

$$0.7854nd^2S_{ws} = ndtS_{wc}, \quad \text{or} \quad d = \frac{tS_{wc}}{0.7854S_{ws}}. \qquad (1)$$

For double shear, or butt joint with two cover-plates, we have

$$1.5708nd^2S_{ws} = ndtS_{wc}, \quad \text{or} \quad d = \frac{tS_{uc}}{1.5708S_{ws}}. \qquad (2)$$

For threefold shear we have 3×0.7854 in place of 0.7854 in (1), and so on.

It is customary to take $S_{wc} = 12500$ lbs. per square inch and $S_{ws} = 7500$ lbs. per square inch for wrought-iron rivets in single shear.

We have then

$$\left.\begin{aligned} d &= 2.12t \text{ for single shear,} \\ d &= 1.06t \text{ for double shear.} \end{aligned}\right\} \qquad (3)$$

Practical Value of d.—Owing to risk of injury to the material in punching, the diameter of rivet *must always be at least as large as the thickness of the thickest plate through which it passes*, and the diameter as given by (1), (2) or (3) must be chosen with reference to this restriction. The least allowable thickness of a plate is $\frac{1}{4}$ inch. We should have then as a lower limit for double shear, $d = \frac{1}{4}$ inch. But rivets as small as this are rarely used. Usually $\frac{1}{2}$ inch is the least diameter allowable. A common practical rule is

$$d = 1\tfrac{1}{4}t + \frac{3}{16}, \qquad (4)$$

where d is the diameter of rivet, and t the thickness of the plate in inches. When this rule gives d greater than (1), (2) or (3), we use it; otherwise we use (1), (2) or (3), *unless considerations of pitch*, as given in what follows, prevent.

Pitch of Rivets.—The area of plate between two rivets is $(p - d)t$; and if S_w is the working unit stress of tension or compression for the plates, and S_{ws} the working unit stress for shear, we have for equal strength:

for single shear or lap joint

$$(p - d)tS_w = \frac{\pi d^2}{4}S_{ws}, \quad \text{or} \quad p = d\left(1 + 0.7854\frac{dS_{ws}}{tS_w}\right);$$

for double shear or butt joint

$$(p - d)tS_w = \frac{\pi d^2}{2}S_{ws}, \quad \text{or} \quad p = d\left(1 + 1.5708\frac{dS_{ws}}{tS_w}\right).$$

Since S_{ws} and S_w are nearly equal, we have practically, if A is the area of cross-section of a rivet,

for single shear

$$p = d\left(1 + 0.7854\frac{d}{t}\right) = d + \frac{A}{t};$$

for double shear

$$p = d\left(1 + 1.5708\frac{d}{t}\right) = d + \frac{2A}{t}. \qquad (5)$$

The plate section is reduced by punching from pt between two rivets to $(p - d)t$, so that in the case of a tension joint the strength is reduced in the ratio

$$\frac{p - d}{p} = \frac{1}{1 + \frac{4t}{\pi d}} \quad \text{or} \quad \frac{1}{1 + \frac{2t}{\pi d}}.$$

We see at once that for a given thickness t a large rivet gives a large pitch and less reduction in strength than a small rivet. Small rivets allow a less pitch at a sacrifice of strength. But the less the pitch the tighter the joint. When strength rather than tightness is desired, as in bridges and parts of buildings and machines, we should then use a large rivet. When tightness is essential, as in steam-boilers, we should use a small rivet with a sacrifice of strength.

Practical Restrictions.—Owing to risk of injury to the material in punching and liability to tear out, *rivets are not allowed a pitch of less than 3 diameters*, or if this distance is less

than 3 inches, as it usually is, *less than 3 inches.* *Rivets should not be spaced farther apart than 6 inches* in any case, or, when the plate is in compression, *16 times the thickness of the thinnest outside plate.* This last is to guard against buckling of the outside plate between rivets. With these restrictions we may apply (5).

Number of Rivets.—Guided by the preceding restrictions and rules, we can select in any case a suitable size of rivet. This done, we can easily determine the number required.

A rivet is considered as failing either by shearing across or by crushing. In any case, then, the diameter being chosen, we must take such a number as shall give security against these two methods of failure, choosing the greater number. In general the number to resist crushing will be more than enough to resist shear. Still we should try for both. The bearing area of a rivet is the projection of the hole upon the diameter, or dt.

The allowable compressive stress is about 12500 lbs. per square inch. The allowable shear is taken at 7500 lbs. per square inch for single shear.

In the following table we have given the safe shearing and bearing resistance for rivets of different sizes and for different thicknesses of plate. Having chosen, then, the size of rivet, an inspection of the table will give its resistance. The stress to be resisted being known, the number to resist this stress either by bearing or shearing is easily determined. The greatest of these two numbers is taken, with enough over in any case to complete the row or rows. As most practical cases are double shear, the greatest number will usually be determined by the bearing resistance.

Distance from End to Edge.—The distance between the end and edge of any plate and the centre of rivet-hole, or between rows, is fixed by practice *at never less than $1\frac{1}{4}$ inches,* and when practicable it should be at least 2 diameters for rivets over $\frac{5}{8}$ inch diameter.

Joints in Compression.—The size and number of rivets are determined for joints in compression precisely as for joints in tension, because the joints are not considered as in contact, and hence the rivets must transmit the stress in either case.

RIVET TABLE.

SHEARING AND BEARING RESISTANCE OF RIVETS.

Diameter of Rivet in inches.		Area of Rivet in square inches.	Single Shear at 7500 lbs per square inch.	Bearing Resistance in pounds for Different Thicknesses of Plate at 12500 lbs. per square inch = 12500 × dt.										
Fraction	Decimal.			$\frac{1}{4}''$	$\frac{5}{16}''$	$\frac{3}{8}''$	$\frac{7}{16}''$	$\frac{1}{2}''$	$\frac{9}{16}''$	$\frac{5}{8}''$	$\frac{11}{16}''$	$\frac{3}{4}''$	$\frac{13}{16}''$	$\frac{7}{8}''$
$\frac{3}{8}$	0.375	0.1104	828	1170	1465	1760								
$\frac{7}{16}$	0.4375	0.1503	1130	1370	1710	2050	2390							
$\frac{1}{2}$	0.5	0.1963	1470	1560	1950	2340	2730	3125						
$\frac{9}{16}$	0.5625	0.2485	1860	1760	2200	2640	3080	3520	3955					
$\frac{5}{8}$	0.625	0.3068	2300	1950	2440	2930	3420	3900	4390	4880				
$\frac{11}{16}$	0.6875	0.3712	2780	2150	2680	3220	3760	4290	4830	5370	5908			
$\frac{3}{4}$	0.75	0.4418	3310	2340	2930	3520	4100	4690	5270	5860	6440	7030		
$\frac{13}{16}$	0.8125	0.5185	3890	2540	3170	3800	4440	5080	5710	6350	6980	7620	8250	
$\frac{7}{8}$	0.875	0.6013	4510	2730	3420	4100	4780	5470	6150	6840	7520	8200	8890	9570
$\frac{15}{16}$	0.9375	0.6903	5180	2930	3660	4390	5130	5860	6590	7320	8050	8790	9520	10250
1	1	0.7854	5890	3125	3900	4690	5470	6250	7030	7810	8590	9370	10160	10940
$1\frac{1}{16}$	1.0625	0.8866	6650	3320	4150	4980	5810	6640	7470	8300	9130	9960	10790	11620
$1\frac{1}{8}$	1.125	0.9940	7460	3520	4390	5270	6150	7030	7910	8790	9667	10550	11420	12300
$1\frac{3}{16}$	1.1875	1.1075	8310	3710	4640	5570	6490	7420	8350	9280	10200	11130	12060	12990

Examples.—(1) *A boiler is to be made of wrought-iron plates ⅜ inch thick, united by single lap-joints. Find the size and pitch of rivets. If the boiler is 30 inches in diameter and carries a pressure of 100 lbs. per square inch above the atmosphere, find the factor of safety, taking the ultimate strength at 55000 lbs. per square inch.*

ANS. From (4), page 489, we have ⅝-in. rivets. But from (3), page 488, we have ¾-in. This size would be chosen for ordinary construction work. In this case we wish a tight joint, and therefore use a small rivet at sacrifice of strength. Let us take, then, ⅝-in. rivets. Then from (5), page 489, we find the pitch ¾ in. But this violates the practical restriction that rivets should not have a less pitch than three diameters. We take the pitch, then, 2 inches. The pressure on a length equal to the pitch is $30 \times 2 \times 100 = 6000$ lbs. If S is the unit stress, the resisting stress is $S\left(2 - \frac{5}{8}\right)t = \frac{33}{64}S$. Hence $S = \frac{64 \times 6000}{33} = 11640$ lbs. per square inch. The factor of safety is then about 5. If this is considered too small, we should use a less pitch or a larger rivet. A larger rivet would not be tight enough. For a less pitch the holes must be drilled and not punched.

(2) *Required to unite two ½-inch plates by a butt joint with two cover-plates; the stress to be transmitted being 40000 lbs. and the unit working stress 10000 lbs. per square inch.*

ANS. The area of the plates must then be 4 square inches *net* if the joint is in tension, *gross* if in compression. The cover-plates can be each ¼ inch thick. Our rule (4), page 489, gives for diameter of rivet $d = \frac{13}{16}$ inch. This is greater than given by (3), page 488, therefore we take it. From our table page 490 we have for the resistance to shear of a $\frac{13}{16}$-inch rivet 3890 lbs. The rivets are in double shear in a butt joint, hence we require $\frac{20000}{3890}$ = about 5 rivets. The bearing resistance from our table is 5080 lbs. We require, then, for bearing $\frac{40000}{5080}$ = about 8 rivets. This, then, is the number we should use.

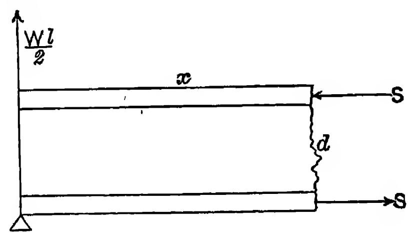
For the pitch we have from (5), page 489, 2.887 inches. This is less than 3 inches. We therefore take the pitch 3 inches. We must have at least 1¼ inches for distance from end and edge (page 490).

If the plates are 8⅓ inches wide, we must then have three rows of rivets, three in the first and last and two in the middle on each side of the joint. The cover-plates must then be 10 inches long. The student can now sketch the cover-plates with the rivet-holes properly spaced.

(3) *A plate girder is 17 feet long and 27 inches deep. The uniformly distributed load is 55000 pounds. The thickness of the web is ¼ inch and of the flange angle-irons $\frac{9}{16}$ inch. Find the size, number and spacing of the rivets to unite the web and flanges.*

ANS. Our rule (4), page 489, gives for diameter of rivet $d = \frac{7}{8}$ inch. This is greater than the size given by (3), page 488. We take the rivets, then, ⅞ inch diameter.

If we neglect the web, the stress S of compression in the upper flange or of tension in the lower, at any point distant x feet from the end, is given by



$$Sd - \frac{wl}{2}x + \frac{wx^2}{2} = 0, \quad \text{or} \quad S = \frac{wx}{2d}(l - x),$$

where l is the length in feet, d the depth in feet and w the uniform load per foot of length.

Inserting $l = 17$, $d = \frac{27}{12}$, $w = \frac{55000}{17}$,

$$S = \frac{55000x}{4.5}\left(1 - \frac{x}{17}\right).$$

If we take	$x = 0$	2.5 ft.	5 ft.	8.5 feet,
we have the stress at these points	$S = 0$	26062	43137	51944 pounds.

We have then for the first space of 2.5 feet the horizontal stress 26062 pounds or 13 tons to be taken by the rivets in that space; in the second space of 2.5 feet, $43137 - 26062 = 17075$ pounds or 8.5 tons; and in the third space of 3.5 feet we have $51944 - 43137 = 8807$ pounds or 4.4 tons to be taken by the rivets in that space.

For the shear at any point distant x feet from the end we have

$$\text{Shear} = \frac{wl}{2} - wx = \frac{55000}{2}\left(1 - \frac{2x}{17}\right).$$

If we take	$x = 0$	2.5 ft.	5 ft.	8.5 feet,
we have the shear at these points	27500	19400	11300	0 pounds.

We have then for the first space of 2.5 feet the shear 27500 — 19400 = 8100 pounds or 4 tons to be taken by the rivets in this space. In the second space of 2.5 feet we have 19400 – 11300 = 8100 pounds or 4 tons; and in the third space of 3.5 feet we have 11300 pounds or 5.65 tons.

Hence the *combined shear* (page 484) in the first space of 2.5 feet is

$$\sqrt{4^2+\frac{13^2}{4}} = 7.63 \text{ tons} = 15260 \text{ pounds.}$$

In the second space of 2.5 feet

$$\sqrt{4^2+\frac{8.5^2}{4}} = 5.9 \quad \text{tons} = 11800 \text{ pounds.}$$

In the third space of 3.5 feet

$$\sqrt{5.65^2+\frac{4.4^2}{4}} = \quad 6 \text{ tons} = 12000 \text{ pounds.}$$

The bearing resistance of a seven-eighths rivet is, from our table page 490, 2730 pounds. We require then, for bearing, in the first space of 2.5 feet, $\frac{15260}{2730} = 6$ rivets; in the next 2.5 feet, $\frac{11800}{2730} = 5$ rivets; in the third space of 3.5 feet, $\frac{12000}{2730} = 5$ rivets.

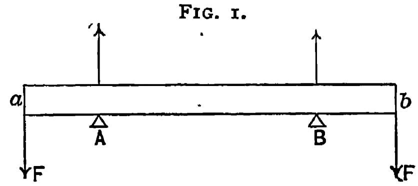
We must not pitch the rivets less than 3 inches nor more than 6 inches (page 490). A pitch of 4 inches for the first 2.5 feet, then 5 inches for the next 2.5 feet, and then 6 inches to the middle will therefore give more rivets than are necessary.

CHAPTER III.

STRENGTH OF BEAMS. TORSION.

Flexure or Bending Stress.—When a body is in equilibrium under equal and opposite couples in the same plane, the stress is one of bending or pure flexure in that plane. Thus if a beam, Fig. 1, supported at A and B, is loaded at the ends a and b with equal loads F, the upward pressure or reaction at the supports A and B will be F, and if the distances aA, bB are equal, the portion of the beam between A and B is in equilibrium under opposite and equal couples in the same plane, and is therefore under pure flexure or bending stress, and the beam bends as shown in Fig. 2.

FIG. 1.



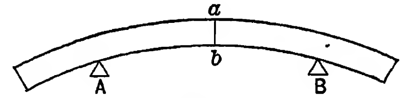


FIG. 2.

Beyond the supports A and B we have bending stress and shear combined; between the supports A and B we have pure bending stress, or flexure, only.

As already stated, we consider only prismatic bodies (page 473). We consider all such bodies as composed at any cross-section ab, Fig. 2, of an indefinitely large number of parallel filaments or *fibres* of indefinitely small cross-section.

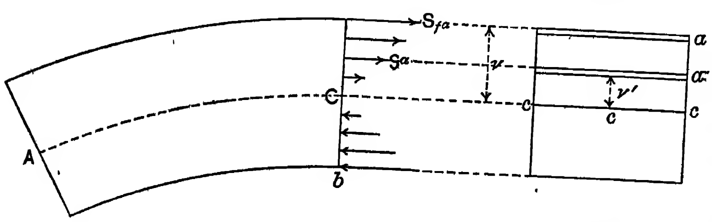
Assumptions upon which the Theory of Flexure is Based.—We assume in all that follows in our discussion of beams, first, that the coefficient of elasticity E is constant; second, that any section which is plane before bending remains plane after bending; third, that the deflection is very small compared to the length; fourth, that the elastic limit is not exceeded.

Upon these assumptions the theory of flexure rests. The comparison of its results with experiment shows them to be correct *so long as the elastic limit is not exceeded.*

The reader should therefore never apply the theoretical formulas to cases where the elastic limit is exceeded.

Neutral Axis of Cross-section and Body.—When a body is bent it is evident, as shown in Fig. 1, that for any cross-section ab the outer layer of fibres on the convex side at a is extended and the outer layer on the concave side at b is compressed. Between these two layers there must then exist a layer cc, Fig. 2, which is neither extended nor compressed. This layer is the *neutral axis of the cross-section.* It is at right angles to the plane of bending through some point C, Fig. 1.

FIG. 1. FIG. 2.



For any layer of fibres above or below cc, if we assume that any plane cross-section before bending remains plane after bending, the strain will be proportional to the distance

from cc, and, by the law of elasticity (page 475), if the elastic limit is not exceeded, the stress is proportional to the strain.

Let S_f be the unit stress in the most remote fibre at a distance v from cc. If a is the cross-section of this fibre, $S_f a$ is the stress. Let Sa be the stress in any other fibre at the distance v' from cc.

Then if the elastic limit is not exceeded and if any cross-section, plane before bending, remains plane after, we have

$$Sa : S_f a :: v' : v, \quad \text{or} \quad Sa = \frac{v'}{v} S_f a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now for equilibrium the algebraic sum of all the parallel fibre stresses must be zero, or

$$\Sigma \frac{v'}{v} S_f a = \frac{S_f}{v} \Sigma a v' = 0.$$

But if $\Sigma a v' = 0$, the fibre layer cc must pass through the centre of mass C of the cross-section, and this layer is not strained by bending.

Hence *if the elastic limit is not exceeded and if any cross-section, plane before bending, remains plane after, the neutral axis for any cross-section passes through the centre of mass of that cross-section at right angles to the plane of bending.*

If then we draw a line AC through the centre of mass of every cross-section, this line is not strained by bending and is called the *neutral axis of the body.*

Hence *if the elastic limit is not exceeded and if any cross-section, plane before bending, remains plane after, the neutral axis of the body passes through the centres of mass of all the cross-sections.*

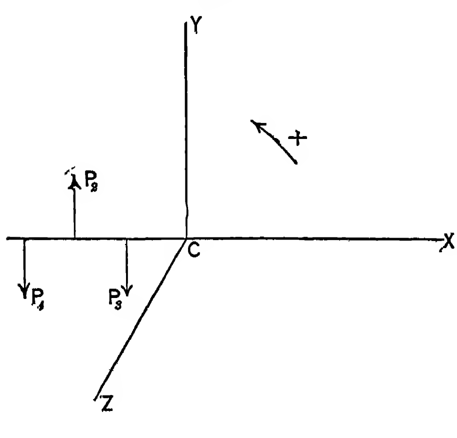
Bending Moment.—The resultant moment relative to the neutral axis cc of any cross-section, of all the external forces on the right or left of that cross-section, causes bending.

For reasons to be given later (page 497), we always take the resultant moment *on the left* of the cross-section, and call this the BENDING MOMENT for that cross-section. The resultant moment on the left of any cross-section is the algebraic sum of the moments of all the external forces on the left.

We define, then, the bending moment for any cross-section as the algebraic sum of the moments of all the external forces *on the left* of that cross-section, and denote it by M.

If all the external forces are co-planar, we take the plane of XY as the common plane, and the horizontal line through the centre of mass C of the cross-section as the axis of X.

FIG. 1.



Hence, as shown in Fig. 1, rotation counter-clockwise from X to Y is positive, upward forces and forces towards the right are positive, and distances on the right of C and above CX are positive.

Thus if we have any number of vertical forces, P_1, P_2, P_3, etc., on the left of C with the lever-arms x_1, x_2, x_3, etc., the bending moment at C is

$$M = P_1x_1 + P_2x_2 + P_3x_3 + \ldots = \Sigma Px,$$

where every term in the algebraic sum is to be taken with its proper sign. In Fig. 1, for instance, we should have, taking the forces as given,

$$M = +P_1x_1 - P_2x_2 + P_3x_3,$$

where the numerical values of the forces and lever-arms are to be inserted without sign.

CASE 1.—Thus for a beam of length l, Fig. 2, fixed horizontally at the right end, with a load P at the left end, the bending moment at any point C distant x from the left end is

$$M_x = +Px, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when the numerical values of P and x are to be inserted without sign.

FIG. 2.

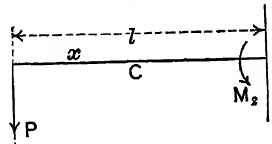
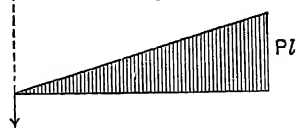
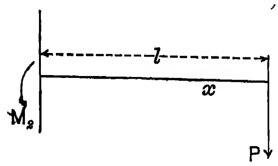


FIG. 3.



The bending moment is a maximum and equal to $M_2 = +Pl$ at the fixed end, and the bending moment at any point is given to scale by the ordinate to a straight line, as shown in Fig. 3.

FIG. 4.



If the beam is fixed at the left end, as shown in Fig. 4, the moment on the right is $-Px$; and since for equilibrium the moment on the left must be equal and opposite, we still have for moment *on left*, as before,

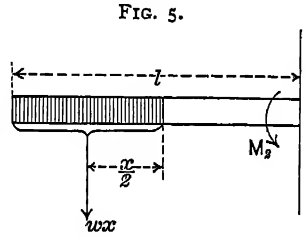
$$M_x = +Px,$$

where P and x are to be taken without sign.

CASE 2.—If the beam is covered with a uniformly distributed load of w per unit of length, Fig. 5, the load on the left of C is wx, and this load can be considered as acting at its centre. The bending moment is then

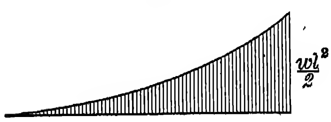
$$M_x = wx \times \frac{x}{2} = +\frac{wx^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

FIG. 5.



where the numerical values of w and x are to be inserted without sign. Here, again, the bending moment is a maximum at the fixed end and equal to $M_2 = +\frac{wl^2}{2}$, and the bending moment at any point is given to scale by the ordinates to a parabola whose vertex is at the left end, as shown in Fig. 6.

FIG. 6.

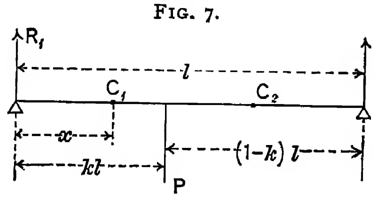


If Fig. 5 were reversed, we should have, as in Fig. 4, the moment on the right $-\frac{wx^2}{2}$. Hence, the moment on the left being equal and opposite, we should still have for the moment *on the left*

$$M_x = +\frac{wx^2}{2}.$$

CASE 3.—Let a beam of length l, Fig. 7, rest horizontally on two supports and have a load P at a distance kl from the left end and $(1-k)l$ from the right end, where k is any given fraction. Thus if the load is at the centre, $k = \frac{1}{2}$ and $kl = \frac{1}{2}l$. If the load is at $\frac{1}{4}l$ from the left end, $k = \frac{1}{4}$, etc. We have then the left reaction R_1 given by

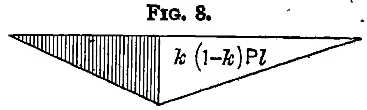
FIG. 7.



$$-R_1 l + P(1-k)l = 0, \quad \text{or} \quad R_1 = (1-k)P.$$

The bending moment for any point C_1 between the left end and the load is then

FIG. 8.



$$M_x = -R_1 x = -(1-k)Px. \quad . \quad . \quad . \quad (4)$$

For any point C_2 between the load and the right end the bending moment is

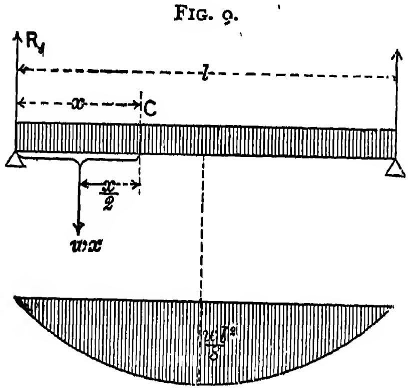
$$M_x = -R_1x + P(x - kl) = -kP(l - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In both cases the bending moment is a maximum at the load and equal to $-k(1-k)Pl$, and the bending moment at any point is given to scale by the ordinates to two straight lines as shown in Fig. 8.

If the load is at the centre, $k = \frac{1}{2}$, and we have $R_1 = \frac{1}{2}P$, $M = -\frac{Pl}{4}$.

CASE 4.—If the beam has a uniformly distributed load of w per unit of length, Fig. 9, the reaction at the left end $R_1 = \frac{wl}{2}$, and the bending moment at any point C is

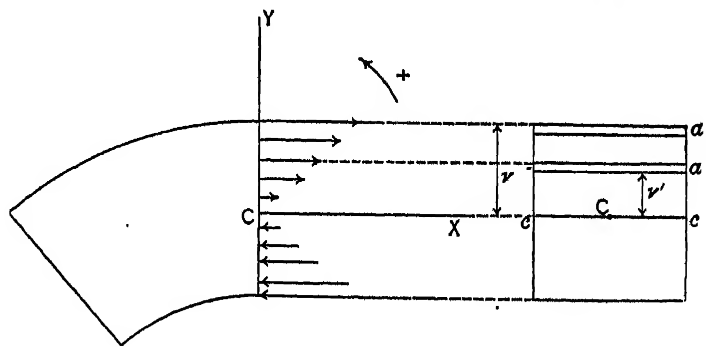
FIG. 9.



$$M_x = -\frac{wlx}{2} + \frac{wx^2}{2}.$$

The bending moment is a maximum at the centre and equal to $-\frac{wl^2}{8}$, and the bending moment at any point is given to scale by the ordinates to a parabola whose vertex is at the centre as shown in Fig. 10. (See page 420.)

Resisting Moment.—The resultant moment of all the fibre stresses at any cross-section, relative to the neutral axis of that cross-section, resists bending. It must therefore be equal and opposite to the bending moment. We call it therefore the *resisting moment*. Since we have taken for the bending moment the resultant moment of all external forces *on the left*, the resisting moment must be taken for all fibre stresses *on the right* of the cross-section.



Let S be the unit stress in any fibre of the cross-section, and a the area of cross-section of the fibre. The stress in the fibre is then Sa. If v' is the distance of the fibre from the neutral axis cc of the cross-section, we have for the moment of the fibre stress

$$-Sav'.$$

The resisting moment for the cross-section is then

$$-\Sigma Sav'.$$

But from equation (1), page 494, within the limit of elasticity, if any cross-section plane upon bending remains plane after,

$$Sa = \frac{v'}{v}S_f a,$$

where S_f is the unit stress in the most remote fibre at a distance v. Hence we have for the resisting moment

$$-\Sigma\frac{S_f a v'^2}{v} = -\frac{S_f}{v}\Sigma a v'^2.$$

But $\Sigma a v'^2$ is the moment of inertia I of the reaction relative to its neutral axis cc. Hence the resisting moment is

$$\text{resisting moment} = -\frac{S_f I}{v}.$$

Now if M is the resultant moment of all the external forces either on right or left of section relative to the neutral axis of the section, we have in general for equilibrium, within the limit of elasticity, if any cross-section plane before bending remains plane after,

$$M - \frac{S_f I}{v} = 0, \quad \text{or} \quad \frac{S_f I}{v} = M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

If in equation (II) we always take M *on the left of the section*, it is evident that S_f will be positive when the stress is tension and negative when the stress is compression. It is for this reason that we always take the bending moment M *on the left* of the section.

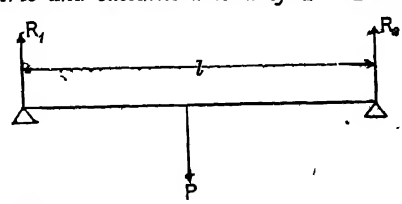
In applying (II), since S_f is always given in pounds per square inch, all distances should be taken in inches. In finding M the change of lever-arm due to bending is neglected. That is, *the deflection is assumed as very small compared to the length.*

By inserting for M its value as found on page 495, we can find in any case the load which will give any desired outer fibre stress S_f within the limit of elasticity.

For values of I for different cross-sections see page 42. We give here some of the most common.

Rectangular cross-section............	$I = \frac{1}{12}bd^3$,	$v = \frac{d}{2}$.	
Circular "	$I = \frac{\pi r^4}{4}$,	$v = r$.	
Triangular "	$I = \frac{bh^3}{36}$,	$v = \frac{2}{3}h$,	$\frac{1}{3}h$.

Examples.—(1) *A timber beam of length $l = 10$ feet and constant rectangular cross-section of breadth $b = 4$ inches and depth $d = 6$ inches, rests horizontally upon two supports and sustains a load of $P = 2000$ pounds at the centre. Find the maximum fibre stresses.*



ANS. We have $R_1 = \frac{P}{2}$. The maximum bending moment is at the centre and given by (page 496),

$$M = -R_1\frac{l}{2} = -\frac{Pl}{4}.$$

The maximum fibre stress S_f, if the elastic limit is not exceeded, is then given, from (II), by

$$S_f = \frac{Mv}{I}.$$

Now $I = \frac{1}{12}bd^3$; and since $v = +\frac{d}{2}$ for the top fibre, we have for the top fibre

$$S_f = -\frac{3Pl}{2bd^2} = -\frac{3 \times 2000 \times 120}{2 \times 4 \times 36} = -2500 \text{ pounds per square inch.}$$

The minus sign shows that the top fibre is in compression.

For the bottom fibre we have $v = -\frac{d}{2}$, and hence $S_f = +2500$ pounds per square inch tension. We see from our table page 476 that this is within the elastic limit.

(2) *Let the cross-section be triangular with the point up, the base of the triangle being 8 inches and the height 10 inches. Find the maximum fibre stresses.*

ANS. We have as before $M = -\frac{Pl}{4}$. But $I = \frac{bh^3}{36}$, and for the top fibre $v = +\frac{2}{3}h$. Hence for the top fibre

$$S_f = -\frac{6Pl}{bh^2} = -\frac{6 \times 2000 \times 120}{8 \times 100} = -1800 \text{ pounds per sq. in.}$$

For the bottom fibre, since $v = -\frac{1}{3}h$,

$$S_f = +\frac{3Pl}{bh^2} = +900 \text{ pounds per sq. in.}$$

The maximum fibre stresses are then 1800 pounds per square inch compression in the top fibre and 900 pounds per square inch tension in the bottom fibre, at the centre of the span.

We see from our table page 476 that the elastic limit is not exceeded.

(3) *Required the depth of a rectangular beam supported at the ends and carrying a load P at the middle in order that the elongation of the lowest fibre shall equal $\frac{1}{1400}$ of its original length.*

ANS. We have from (1), page 477, the elongation λ given by

$$\lambda = \frac{S_f l}{E} = \frac{l}{1400}. \qquad \text{Hence } S_f = \frac{E}{1400}.$$

Also from (II), page 497,

$$S_f = \frac{Mv}{I} = \frac{E}{1400}.$$

Now from (4), page 495, $M = -\frac{Pl}{4}$. Also in the present case $v = -\frac{d}{2}$, $I = \frac{1}{12}bd^3$. Hence

$$\frac{3Pl}{2bd^2} = \frac{E}{1400}, \quad \text{or} \quad d = \sqrt{\frac{2100Pl}{bE}}.$$

(4) *A cast-iron rectangular beam rests horizontally upon supports 12 feet apart, and carries a load of 2000 pounds at the centre. If the breadth is one half the depth, find the area of cross-section, so that the unit* stress *stress in the lower fibre may nowhere exceed 4000 pounds per square inch.*

ANS. The given unit stress 4000 is less than the elastic limit (page 476), so (II) applies. The greatest will be at the centre, where the bending moment is greatest.

We have then for tension in lower fibre

$$S_f = 4000 = \frac{Mv}{I}.$$

In the present case, from (4), page 495, $M = -\frac{Pl}{4}$, $v = -\frac{d}{2}$, $I = \frac{d^4}{24}$. Hence

$$\frac{3Pl}{d^3} = 4000, \quad \text{or} \quad d^3 = \frac{3 \times 2000 \times 12 \times 12}{4000}, \quad \text{or} \quad d = 6 \text{ inches}.$$

(5) *A wrought-iron plate girder $27\frac{1}{2}$ inches deep centre to centre of the flanges rests horizontally upon supports 26 feet apart. Its bottom flange is $a = 48$ square inches area of cross-section. Neglecting the web, find the load at the centre which would cause a unit stress of 15000 pounds per square inch in the bottom flange.*

ANS. The unit stress 15000 is less than the elastic limit (page 476), so (II) applies. We have then at centre

$$S_f = 15000 = \frac{Mv}{I}.$$

In the present case, from (4), page 495, $M = -\frac{Pl}{4}$, $v = -\frac{d}{2}$ and, *neglecting the web*, $I = ad^2$. Hence

$$\frac{Pl}{8ad} = 15000, \quad \text{or} \quad P = \frac{8 \times 15000 \times 48 \times 27.5}{26 \times 12} = 507692 \text{ pounds.}$$

(6) *A cast-iron plate girder, fixed horizontally at one end and free at the other, 8 feet long and 12 inches deep centre to centre of flanges, causes a uniformly distributed load of 16000 pounds. Find the area of the top flange, neglecting the web, so that the unit-stress shall not exceed 3000 pounds per square inch.*

ANS. From (3), page 495, the maximum moment is $M = \frac{wl^2}{2}$. From (II), $S_f = 3000 = \frac{Mv}{I}$. In the present case $v = +\frac{d}{2}$, neglecting the web $I = ad^2$, and $w = \frac{16000}{l}$. Hence

$$\frac{wl^2}{4ad} = S_f, \quad \text{or} \quad a = \frac{wl^2}{4dS_f} = \frac{16000 \times 8 \times 12}{4 \times 12 \times 3000} = 10\tfrac{2}{3} \text{ square inches.}$$

Designing and Strength of Beams—Crippling Load—Coefficient of Rupture.—Let A be the area of cross-section of a beam at any point, V the shear at that point or algebraic sum of all the vertical external forces between the point and *left end* (see page 398), and S_{ws} the working unit stress for shear.

Then for safety, as regards shear, we must have at every point

$$S_{ws}A \geqq V. \quad \ldots \ldots \ldots \quad (1)$$

From (II) we have

$$M = \frac{S_f I}{v}, \quad \ldots \ldots \ldots \quad (2)$$

where S_f is the unit stress within the elastic limit in the most remote fibre of any cross-section at a distance v from the neutral axis of that cross-section, I is the moment of inertia of the cross-section relative to that neutral axis, M is the bending moment as defined and found on page 495, that is the algebraic sum of the moments of all the external forces *on the left of the section.*

If in (2) we replace S_f by the elastic limit unit stress S_e, and M by the maximum bending moment, we have

$$\text{max. } M = \frac{S_e I}{v}. \quad \ldots \ldots \ldots \quad (3)$$

Equation (3), if we put for maximum M in any case its value as found page 495, will give the load which will strain the beam to its elastic limit. We call this load the CRIPPLING LOAD, since it cannot be exceeded without overstraining. The working load may be taken a suitable fraction of this.

But, as we have seen (page 476), S_e is difficult of exact determination by experiment.

The customary method of estimating the strength of a beam, therefore, is to put equation (3) in the form

$$\text{max. } M = \frac{RI}{v}, \quad \ldots \ldots \ldots \quad (4)$$

where R is determined by direct experiments carried to the point of rupture. The value of R thus found by experiment is called the COEFFICIENT OF RUPTURE.

This use of (3) is of course applying it beyond the elastic limit, and (4) is then a purely empirical formula whose form only is dictated by theory.

When satisfactory experimental values of R are not at hand, the best we can do is to replace R by the ultimate tensile strength U_t or the ultimate compressive strength U_c, using whichever one gives the least value for the breaking load. For the safe load we then take a suitable factor of safety (page 481).

We give in the following table average values of R for different materials.

	R. Pounds per square inch.
Steel (structural)	120000
Wrought iron	55000
Cast iron	35000
Timber	9000
Stone	2000

If then we put in (4) for max. M its value as given page 495, we have the breaking load in the various cases there given:

CASE 1.—For beam fixed horizontally at one end with load P at free end

$$P = \frac{RI}{vl}. \qquad (5)$$

CASE. 2.—For same beam uniformly loaded

$$P = wl = \frac{2RI}{vl}. \qquad (6)$$

CASE 3.—For beam resting horizontally on two supports with load at a distance kl from left end and $(1 - k)l$ from right end, where k is any given fraction of l,

$$P = \frac{RI}{vk(1 - k)l}. \qquad (7)$$

CASE 4.—For the same beam uniformly loaded

$$P = wl = \frac{8RI}{vl}. \qquad (8)$$

If in these equations R is not known, we can replace it by U_t or U_c, whichever gives the least value for P.

If we replace R by S_e, we obtain the crippling load.

If we replace R by S_f, we obtain the load which will cause any desired maximum outer fibre stress S_f within the elastic limit.

Examples.—(1) *A timber beam of length $l = 10$ feet and uniform rectangular cross-section of breadth $b = 4$ inches and depth $d = 6$ inches, rests horizontally upon two supports. Find the crippling and breaking load at the centre, and the working load.*

ANS. From (7) we have for breaking load, since $I = \frac{1}{12}bd^3$, $v = \frac{d}{2}$, $k = \frac{1}{2}$

$$P = \frac{4RI}{vl} = \frac{2Rbd^2}{3l},$$

and for crippling load

$$P_e = \frac{2S_e bd^2}{3l}.$$

If we take $S_e = 3000$ pounds per square inch as given by our table page 476, we have the crippling load

$$P_e = \frac{2 \times 3000 \times 4 \times 36}{3 \times 120} = 2400 \text{ pounds.}$$

If we take $R = 9000$ pounds per square inch as given by our table page 500, we have the breaking load

$$P = 7200 \text{ pounds.}$$

If we take a factor of safety of 8 (page 481), we have for the working load

$$P = 900 \text{ pounds.}$$

If we take $U_c = 8000$, as given by our table page 476, in place of R, we should have for the breaking load $P = 6400$ pounds, and a working load of 800 pounds.

(2) *Let the same beam have a uniform triangular cross-section with the point up, the base $b = 8$ inches and the height $h = 10$ inches.*

ANS. We have from (7) as before, since $I = \frac{bh^3}{36}$, $v = \frac{2}{3}h$, $k = \frac{1}{2}$ for breaking load,

$$P = \frac{4RI}{vl} = \frac{Rbh^2}{6l},$$

and for crippling load

$$P_e = \frac{S_e bh^2}{6l}.$$

Taking $S_e = 3000$ as before, we have for the crippling load

$$P_e = \frac{3000 \times 8 \times 100}{6 \times 120} = 3333\tfrac{1}{3} \text{ pounds.}$$

Taking $R = 9000$ as before, we have the breaking load

$$P = 10000 \text{ pounds,}$$

or for a factor of safety of 8 (page 481) a working load of 1250 pounds.

If R were unknown, we should have, taking the ultimate strength for tension and compression U_t and U_c, in the place of R, from our table page 476, and $v = \frac{1}{3}h$ for the lower or tension fibre and $\frac{2}{3}h$ for the upper or compression fibre,

$$P = \frac{4U_tI}{\frac{1}{3}hl} = \frac{12 \times 10000I}{hl},$$

or

$$P = \frac{4U_cI}{\frac{2}{3}hl} = \frac{6 \times 8000I}{hl} = \frac{8000 \times 8 \times 100}{6 \times 120} = 8888\tfrac{8}{9} \text{ pounds.}$$

The second gives P the least value and should be taken.

(3) *A beam of yellow pine, 14 inches wide, 15 inches deep, resting horizontally upon supports 10 feet 6 inches apart, broke under a uniformly distributed load of 121940 pounds, rupture beginning in the lowest fibre at the centre. Find the coefficient of rupture.*

ANS. We have, from (8),

$$R = \frac{Pvl}{8I} = \frac{3Pl}{4bd^2} = \frac{3 \times 121940 \times 126}{4 \times 14 \times 15 \times 15} = 3658.2 \text{ pounds per sq. in.}$$

(4) *A wrought-iron beam 4 inches deep and $1\frac{1}{2}$ inches wide, fixed horizontally at one end and loaded with 7000 pounds at the free end, ruptured at a point 2 feet 8 inches from the load. Find the coefficient of rupture R.*

ANS. From (5),

$$R = \frac{Pvl}{I} = \frac{6Pl}{bd^2} = \frac{6 \times 7000 \times 32}{\frac{3}{2} \times 4 \times 4} = 56000 \text{ pounds per sq. in.}$$

(5) *A wrought-iron beam 2 inches wide and 4 inches deep rests horizontally upon supports 12 feet apart. Find the uniformly distributed load it will carry in addition to its own weight for a factor of safety of 4.*

ANS. From our table page 500, $R = 55000$ pounds per square inch.

A bar of iron 3 feet long and one square inch in cross-section weighs 10 pounds. The weight of the beam is then 320 pounds.

We have then, from (8), for the breaking load

$$P + 320 = \frac{8RI}{vl}.$$

Hence for a factor of 4

$$P + 320 = \frac{8RI}{4vl} = \frac{Rbd^2}{3l} = \frac{55000 \times 2 \times 16}{3 \times 144} = 4074 \text{ pounds},$$

and $P = 3754$ pounds.

(6) *Find the length of a beam of ash 6 inches square which would break of its own weight when supported horizontally at the ends, the weight of the timber being 30 pounds per cubic foot and $R = 7000$ pounds per square inch.*

ANS. The weight per cubic inch is $\frac{30}{1728}$; and since the cross-section is 36 square inches, the weight per inch in length is $w = \frac{30 \times 36}{1728} = \frac{5}{8}$. We have then, from (8),

$$l = \frac{8RI}{wvl} = \frac{32Rbd^2}{15l}, \quad \text{or} \quad l = \sqrt{\frac{32Rbd^2}{15}}.$$

Hence

$$l = \sqrt{\frac{32 \times 7000 \times 6 \times 36}{15}} = 1796 \text{ inches} = 149\tfrac{2}{3} \text{ feet}.$$

(7) *A wrought-iron beam 12 ft. long, 2 inches wide and 4 inches deep is supported at the ends. The beam weighs $\frac{1}{4}$ pound per cubic inch. Taking R at 54000 pounds per square inch, find the uniform load it can carry to rupture.*

ANS. Without the weight of beam, 16000 pounds.
Besides the weight of beam, 15712 pounds.

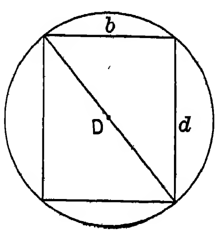
Comparative Strength.—We see at once from (5), (6), (7) and (8), page 500, that, taking the semi-beam of Case 1 as unity, the same beam with uniform load will carry twice as much, the beam on two supports with load in centre will carry four times as much, and the same beam uniformly loaded eight times as much.

Examples.—(1) *A round and a square beam are equal in length and have the same loading. Find the ratio of the diameter to side of square so that the two may be of equal strength, R being the same for both.*

ANS. Since P, R, l are the same in both cases, we see from our equations page 500 that $\frac{I}{v}$ must be equal in both cases. Hence $\frac{\text{Diameter}}{\text{Side}} = 2\sqrt[3]{\frac{2}{3\pi}}$.

(2) *Compare the relative strengths of a cylindrical beam and the strongest rectangular beam that can be cut from it.*

ANS. Let D be the diameter of the cylinder, and b and d the breadth and depth of the rectangle. The strength of the rectangle is proportional to $\frac{I}{v} = \frac{bd^2}{6}$, or to bd^2. But we have



$$b^2 + d^2 = D^2, \quad \text{or} \quad d^2 = D^2 - b^2.$$

Hence

$$bd^2 = bD^2 - b^3.$$

If we differentiate for b and put the differential coefficient equal to zero, we have for maximum strength

$$D^2 - 3b^2 = 0, \quad \text{or} \quad b = \frac{D}{\sqrt{3}}, \text{ and hence } d = D\sqrt{\frac{2}{3}},$$

For the strongest rectangular beam we have then

$$\frac{I}{v} = \frac{bd^2}{6} = \frac{D^3}{9\sqrt{3}}.$$

For the cylinder we have

$$\frac{I}{v} = \frac{\pi D^3}{32}.$$

We have then

$$\frac{\text{Strength of cylindrical}}{\text{Strongest rectangular}} = \frac{9\pi\sqrt{3}}{32}.$$

(3) *Compare the relative strengths of a square beam to that of the inscribed cylinder.*

ANS. $\frac{16}{3\pi} = 1.7.$

(4) *Compare the strength of a square beam with its sides vertical to that of the same beam with a diagonal vertical.*

ANS. $\frac{\text{Side vertical}}{\text{Diagonal vertical}} = \sqrt{2} = 1.414.$

Beams of Uniform Strength.—If the unit stress S_f in the outer fibre is the same at every cross-section, the beam is of uniform strength. We have then, from (II), for the condition for uniform strength

$$S_f = \frac{Mv}{I} = \text{a constant quantity.}$$

CASE 1.—For beam fixed horizontally at one end with load P at the free end, we have then, since $M = Px$, for S_f at any cross-section distant x from the free end

$$S_f = \frac{Pvx}{I} = \text{constant.}$$

At the end cross-section let $x = l$, $v = v_1$ and $I = I_1$. Then, since S is constant, we have

$$\frac{Plv_1}{I_1} = \frac{Pvx}{I}, \quad \text{or} \quad \frac{vx}{I} = \frac{v_1 l}{I_1}. \qquad (1)$$

If, for instance, the beam is rectangular, we have the breadth and height at the fixed end b_1 and h_1, and at any point b and h. Hence $I = \frac{1}{12}bh^3$, $I_1 = \frac{1}{12}b_1h_1^2$ and $v = \frac{h}{2}$, $v_1 = \frac{h_1}{2}$. We have then for uniform strength, from (1),

$$\frac{x}{bh^2} = \frac{l}{b_1h_1^2}. \qquad (2)$$

If the height is constant, $h = h_1$, and we have for the breadth at any point distant x from the free end

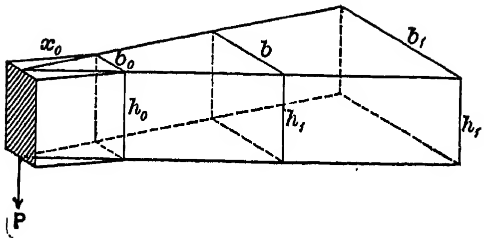
$$b = \frac{b_1}{l}x. \qquad (3)$$

The breadth then varies as the ordinates to a straight line, from b_1 at the fixed end to zero, theoretically, at the free end. Practically the breadth cannot be zero at the free end, but must have a value b_0 such that the area $A = b_0h_1$ at the free end may resist the shear safely.

We have then from (1), page 499,

$$b_0h_1 = \frac{V}{S_{ws}} = \frac{P}{S_{ws}}, \quad \text{or} \quad b_0 = \frac{P}{h_1S_{ws}}.$$

Substituting this value of b_0 for b in (3), we find that the cross-section must be constant and equal to $b_0 h_1 = \frac{P}{S_{ws}}$ for a distance x_0 from the free end equal to



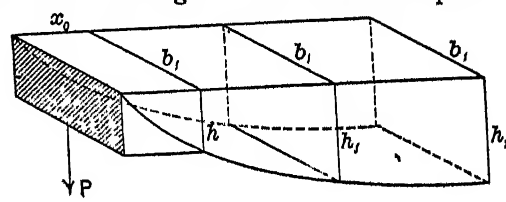
$$x_0 = \frac{Pl}{b_1 h_1 S_{ws}}.$$

For any value of x greater than x_0 the breadth is given by (3).

If the breadth is constant, we have, in (2), $b = b_1$ and hence

$$h^2 = \frac{h_1^2}{l} x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The height then varies as the ordinates to a parabola from b_1 at the fixed end to zero, theoretically, at the free end. Here, again, we must have the height at the free end equal to



$$h_0 = \frac{P}{b_1 S_{ws}}.$$

Substituting this for h in (4), we find that the cross-section must be constant and equal to $b_1 h_0 = \frac{P}{S_{ws}}$ for a distance x_0 from the free end equal to

$$x_0 = \frac{P^2 l}{h_1^2 b_1^2 S_{ws}^2}.$$

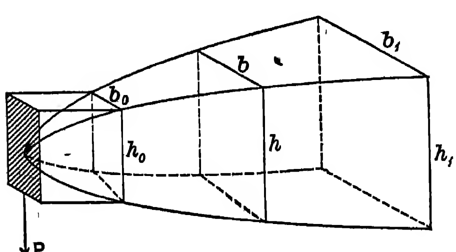
For any value of x greater than x_0 the height is given by (4).

If both b and h vary, but the cross-section at every point is rectangular, we have

$$b_1 : h_1 :: b : h, \quad \text{or} \quad b = \frac{b_1 h}{h_1}, \qquad h = \frac{h_1 b}{b_1}.$$

Substituting these in (2), we have

$$h^3 = \frac{h_1^3}{l} x, \qquad b^3 = \frac{b_1^3}{l} x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$



The height and breadth vary, then, as the ordinates to a cubic parabola from h_1 and b_1 at the fixed end to zero, theoretically, at the free end. The area at any point is then

$$bh = b_1 h_1 \sqrt[3]{\frac{x^2}{l^2}}.$$

The area at the free end should be, then,

$$A = b_0 h_0 = \frac{P}{S_{ws}}.$$

The cross-section should therefore be constant and equal to $b_0 h_0 = \frac{P}{S_{ws}}$ for a distance x_0 from the free end, given by

$$x_0 = \frac{Pl}{h_1 b_1 S_{ws}} \sqrt{\frac{P}{h_1 b_1 S_{ws}}}.$$

In a similar way we can find the shape for uniform strength for any other form of cross-section by substituting in (1) the values of I, I_1, v and v_1.

CASE 2.—For beam fixed horizontally at the end and loaded uniformly, we have, since $M = \frac{wx^2}{2}$,

$$\frac{wvx^2}{2I} = \frac{wv_1 l^2}{2I_1}, \quad \text{or} \quad \frac{vx^2}{I} = \frac{v_1 l^2}{I_1}. \quad \ldots \ldots \quad (1)$$

For rectangular cross-section we have $I = \frac{1}{12}bh^3$, $v = \frac{h}{2}$, $I_1 = \frac{1}{12}b_1h_1^3$, $v_1 = \frac{h_1}{2}$, and hence

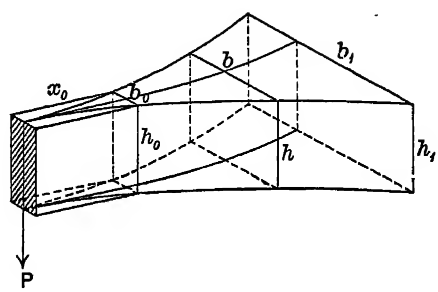
$$\frac{x^2}{bh^2} = \frac{l^2}{b_1h_1^2}. \quad \ldots \ldots \quad (2)$$

For constant height $h = h_1$ and

$$b = \frac{b_1}{l^2}x^2. \quad \ldots \ldots \quad (3)$$

The breadth then varies as the ordinates to a parabola from b_1 to zero, theoretically, at the free end, and to provide against strain the cross-section must be constant for the distance x_0 from the free end equal to

$$x_0 = \frac{wl^2}{b_1h_1S_{ws}}.$$



For any value of x greater than this the breadth is given by (3). For constant breadth $b = b_1$ in (2), and

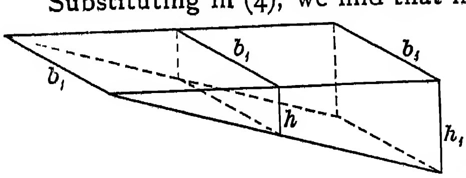
$$h = \frac{h_1}{l}x. \quad \ldots \ldots \quad (4)$$

To provide against shear, we have for the height h_0 at the distance x_0

$$h_0 = \frac{wx_0}{b_1S_{ws}}.$$

Substituting in (4), we find that in order to resist shear the end cross-section

$$A_1 = b_1h_1 = \frac{wl}{S_{ws}}.$$



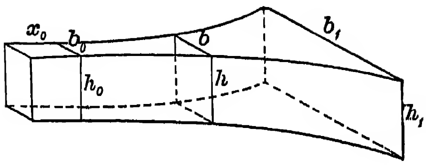
If then the end cross-section is safe for shear, every cross-section is safe, and for any value of x the height is given by (4). The height then varies as the ordinates to a straight line from h_1 at the fixed end to zero.

If both b and h vary, we have for rectangular cross-section

$$b_1 : h_1 :: b : h, \quad \text{or} \quad b = \frac{b_1h}{h_1}, \qquad h = \frac{bh_1}{b_1}.$$

Substituting in (2), we have

$$h^3 = \frac{h_1^3}{l^2}x^2, \qquad b^3 = \frac{b_1^3}{l^2}x^2. \quad . \quad . \quad . \quad . \quad . \quad (5)$$



To provide against shear we must have

$$b_0 h_0 = \frac{wx_0}{S_{ws}}, \quad \text{or} \quad b_0^3 h_0^3 = \frac{w^3 x_0^3}{S_{ws}^3}.$$

Hence from (5) the cross-section must be constant and equal to $b_0 h_0 = \frac{wx_0}{S_{ws}}$ for a distance x_0 from the free end given by

$$x_0 = \frac{w^3 l^4}{b_1^3 h_1^3 S_{ws}^3}.$$

For any value of x greater than x_0 the height and breadth are given by (5).

In a similar way we can find the shape for uniform strength for any other form of cross-section by substituting in (1) the values of I, v, I_1 and v_1.

Theory of Pins and Eyebars.—A direct application of our principles is to the designing of pins and eyebars. A pin is a round beam subjected to bending and shear. It is also subjected to crushing. Hence its bearing resistance should equal the greatest pressure upon it due to any plate through which it passes.

BEARING.—If d is the diameter of pin, t the thickness of any plate through which it passes, then dt is the bearing area. Let S_{wc} be the working unit-stress for compression, then dtS_{wc} is the bearing resistance of the pin. This should equal the stress transmitted by the plate, or

$$dtS_{wc} = \text{stress}.$$

We may take S_{wc} at 6.25 tons per square inch for wrought-iron pins. The stress transmitted is always known. For a transmitted stress of *one ton* the required bearing area is then

$$dt = \frac{1}{6.25},$$

and hence we have

$$\textit{lineal bearing on pin per ton of stress} = \frac{1}{6.25d}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From (1), having given the diameter d, we can find the corresponding lineal bearing or thickness of plate for every ton of transmitted stress. We have only to multiply this by the number of tons transmitted stress in any case to find the requisite thickness of the plate.

DIAMETER OF PIN.—Let t be the thickness of plate or eyebar, and h its depth, then th is the area of cross-section of plate or eyebar. If S_{wt} is the working unit stress for tension, then thS_{wt} is the transmitted stress. Now if d is the diameter of the pin, and the thickness of the eyebar *head* is equal to the thickness of the bar, we have td for the bearing area of pin, and tdS_{wc} for its bearing resistance. We must have, then, for equal strength

$$tdS_{wc} = thS_{wt}, \quad \text{or} \quad d = \frac{S_{wt}}{S_{wc}}h.$$

We can take the ratio $\frac{S_{wt}}{S_{wc}} = \frac{3}{4}$. Hence the *least diameter of pin* is

$$d = \frac{3}{4}h. \qquad (2)$$

The diameter of pin may need to be greater than this, but it cannot be less, unless the thickness of eyebar *head* is made greater than the thickness of the bar itself.

When this is the case, if t_1 is the thickness of the bar and t the thickness of the head, we have for the least diameter of pin

$$tdS_{wc} = t_1 hS_{wt}, \quad \text{or} \quad d = \frac{3}{4}\frac{t_1}{t}h, \qquad (3)$$

and for the thickness of head

$$t = \frac{3ht_1}{4d}. \qquad (4)$$

For a beam subjected to flexure we have from (II), page 497,

$$\text{max. } M = \frac{S_f I}{r},$$

where r is the radius of the pin, and S_f is the unit-stress in the outer fibre. Now $I = \frac{\pi r^4}{4}$. Hence

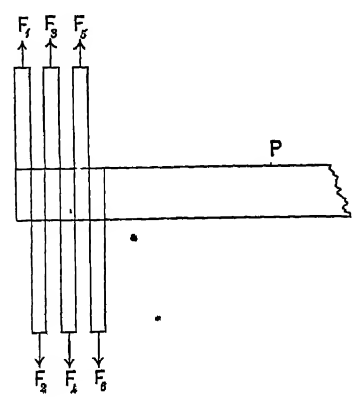
$$\text{max. } M = \frac{\pi S_f d^3}{32}, \qquad (5)$$

where max. M is the maximum bending moment. The usual value taken for S_f is 15000 pounds per square inch for iron and 20000 pounds per square inch for steel.

We have then in any case to find the maximum bending moment, and then from (5) we can find d.

Maximum Bending Moment.—In general for any pin we must resolve the stress in every bar through which the pin passes into its vertical and horizontal components. The stress in each bar is considered as acting along the centre line or axis, and hence the point of application of each vertical and horizontal component is at the centre of the bearing of the corresponding bar.

Let M_h be the maximum bending moment of all the horizontal and M_v of all the vertical forces. Then the resultant maximum bending moment is



$$M_{\text{max.}} = \sqrt{M_h^2 + M_v^2}.$$

From (5) we then find the diameter d of the pin.

Let the parallel horizontal or vertical components on one side of the centre of pin be F_1, F_2, F_3, F_4, etc., the odd indices F_1, F_3, etc., acting in one direction, and the even indices F_2, F_4, etc., acting in the other. Let l_1 be the distance between centres of bearing F_1 and F_2, l_2 the distance between F_2 and F_3, etc. We can now easily find the maximum moment by trial.

Thus the moment at F_2 is $F_1 l_1$. Add to this $(F_1 - F_2)l_2$ and we have the moment at F_3. Add again $(F_1 - F_2 + F_3)l_3$ and we have the moment at F_4, and so on. The greatest of all these is the moment required.

Since all the forces F_1, F_3, F_5, etc., on one side are equal to all on the other, F_2, F_4, F_6, etc., they reduce to a couple on each side of centre of the pin, and hence the moment at any point P beyond the last force, as F_6, is constant. We have then only to find the greatest moment M_h or M_v by trial as directed.

PRACTICAL SIZES FOR PINS.—Pins are furnished in sizes differing by $\frac{1}{8}$ inch, and all sizes are an even number of sixteenths. A pin must always be ordered at least one sixteenth larger than the hole it is to fit, in order that it may be turned down to fit. We must then add $\frac{1}{16}$ inch to the calculated size, and if this gives an even number of sixteenths it can be ordered; if not, add $\frac{1}{16}$ more.

Thus if the size of a pin is $4\frac{3}{8}$ inches by calculation, it should be ordered at least $4\frac{7}{16}$; but since only even sixteenths are furnished, we should order $4\frac{1}{2}$ and turn down to fit the hole.

Examples.—(1) *A pin 3 inches diameter passes through the web of a channel bar three fifths of an inch thick. The transmitted stress is 55500 lbs. Find the thickness of re-enforcing plate necessary to give sufficient bearing on the pin.*

ANS. The thickness for each ton (page 506) is

$$\frac{1}{6.25d} = \frac{1}{6.25 \times 3} = 0.0533 \text{ inches.}$$

For 55500 lbs. = 27.75 tons we should have a thickness of $0.0533 \times 27.75 = 1.48$ inches.

The channel web is only $\frac{3}{5} = 0.6$ inch thick. In order to have the proper thickness for safe bearing on the pin, we must then increase the thickness by $1.48 - 0.6 = 0.88$ inch. Two re-enforcing plates on each side of the web, each 0.44 inch thick, or about $\frac{7}{16}$ inch each, will then give the required thickness.

(2) *If the depth of an eyebar is 10 inches, find the least diameter of pin which can be used without having the thickness of the head greater than that of the bar.*

ANS. (Page 507.) $d = 7\frac{1}{2}$ inches.

(3) *A bar 8 in. by $\frac{7}{8}$ in. has a pin $4\frac{5}{8}$ inches diameter passing through it. Find the thickness of bar head.*

ANS. The least diameter without having the head thicker than bar is 6 inches. As the pin is less than this, the head must be thicker than the bar and equal to

$$t = \frac{3ht_1}{4d} = \frac{3 \times 8 \times \frac{7}{8}}{4 \times 4\frac{5}{8}} = 1\frac{5}{37} \text{ inches.}$$

(4) *In a panel of a bridge truss we have at each end of the pin two eyebars on one side, 4 in. by $1\frac{3}{16}$ in., and on the other side one eyebar 4 in. by $1\frac{7}{16}$ in. Also one tie on each side of centre of pin $1\frac{9}{16}$ in. thick. The tie is packed close to the vertical post, which consists of two channels of $\frac{1}{2}$ in. thickness. The bars are packed snug. The vertical compression in the half post is 40000 lbs. The working unit-stress of the bars is 10000 lbs. per square inch. Find the size of pin required.*

ANS. We have here on one side acting horizontally

$$F_1 = F_3 = 4 \times 1\tfrac{3}{16} \times 10000 = 47500 \text{ lbs.,}$$

and on the other side

$$F_2 = 4 \times 1\tfrac{7}{16} \times 10000 = 47500 \text{ lbs.}$$

The horizontal component of the tie-stress is

$$F_4 = 2 \times 47500 - 57500 = 57500 \text{ lbs.}$$

The distances are

$$l_1 = l_2 = \frac{1}{2}(1\tfrac{3}{16} + 1\tfrac{7}{16}) = 1\tfrac{5}{16} \text{ inches},$$

$$l_3 = \frac{1}{2}(1\tfrac{3}{16} + 1\tfrac{9}{16}) \times \frac{7}{8} = 2\tfrac{1}{2} \text{ inches}.$$

We have then at F_2 the moment $F_1 l_1 = 47500 \times 1\frac{5}{16} = 62344$ inch-lbs.,
at F_3 we have $62344 + (F_1 - F_2)l_2 = 49219$ inch-lbs.,
at F_4 we have $49219 + (F_1 - F_2 + F_3)l_3 = 133594$ inch-lbs.

The maximum horizontal bending moment is then

$$M_h = 133594 \text{ inch-lbs.} = 66.797 \text{ inch-tons.}$$

The vertical compression in post is 40000 lbs. Its lever-arm is

$$\frac{1}{2}\left(1\tfrac{9}{16} + \frac{7}{8}\right) = 1\tfrac{7}{32}.$$

Hence

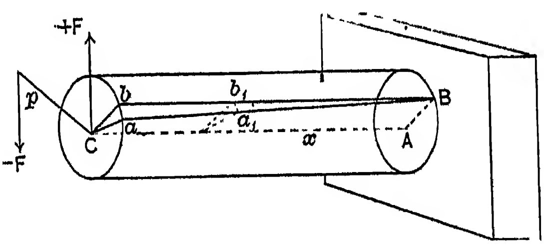
$$M_v = 40000 \times 1\tfrac{7}{32} = 48750 \text{ inch-lbs.} = 24.375 \text{ inch-tons.}$$

The resultant maximum bending moment is then

$$M_{\max.} = \sqrt{M_h^2 + M_v^2} = \sqrt{66.8^2 + 24.4^2} = 71.11 \text{ inch-tons} = 142220 \text{ inch-lbs.}$$

We have then for size of pin about $4\frac{3}{8}$ inches diameter, or $4\frac{1}{2}$ commercial size. The least allowable diameter is $\frac{3}{4}h = 3$ inches. Hence the bearing is abundant.

Torsion.—Torsion occurs when the external forces acting upon a body tend to twist it, so that each section turns on the next adjacent section about a common axis at right angles to the plane of section.



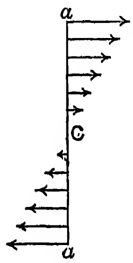
Let a horizontal shaft of length l be fixed at one end, and let a force couple $+F$, $-F$ act at the free end whose moment about the axis AC is Fp.

The shaft will be twisted about the axis AC so that any radial line, as aC, moves to bC through the angle $aCb = \theta$.

If the elastic limit is not exceeded, any longitudinal plane $aBAC$ before twisting remains plane after, as $bBAC$, and when the couple $+F$, $-F$ is removed the line bC returns to its original position aC. Also, the angle aCb is proportional to F and to the distance $AC = l$ of the cross-section from the fixed end. Thus if θ is the angle aCb at the distance l from the fixed end, the angle $a_1C_1b_1$ at the distance x from the fixed end is $\frac{x}{l}\theta$. If the elastic limit is exceeded, this proportionality does not hold, the line bC does not return to its original position when the couple $+F$, $-F$ is removed, and if the twist is great enough we have rupture.

NEUTRAL AXIS.—Consider the shaft to be made up of an indefinitely great number of parallel fibres. Since, within the elastic limit, stress is proportional to strain, as one cross-section of the shaft turns about the axis and slides upon the adjacent cross-section, the strain and therefore the shearing stress on each fibre of a cross-section is *proportional to its distance from the axis AC.* For the fibre at the axis AC there is then no shearing stress. The axis AC is then the NEUTRAL AXIS.

POSITION OF THE NEUTRAL AXIS.—Let a be the cross-section of any fibre, and S_s the unit shearing stress within the elastic limit for that fibre in any cross-section *most remote from the neutral axis* at the distance v. Then the shearing stress for the most remote fibre in any cross-section at the distance v is $S_s a$, and for any other fibre in that cross-section, at



the distance r, it is $\frac{r}{v}S_s a$. The sum of all the fibre stresses of any section in any straight line perpendicular to the axis is then $\frac{S_s}{v}\Sigma ra$.

But the sum of the external forces $+F$, $-F$ is zero, hence for equilibrium we must have $\Sigma ar = 0$.

Therefore the neutral axis AC *must pass through the centre of mass of the cross-sections.*

TWISTING AND RESISTING MOMENT.—All the external forces acting upon the shaft reduce to a couple $+F$, $-F$, as shown in the figure, whose moment Fp with reference to the neutral axis is the TWISTING MOMENT M_t. This moment is the same at every point of the neutral axis AC, and therefore tends to make each cross-section turn on its adjacent cross-section nearest the fixed end, about the axis AC, so that there must be for equilibrium between every two cross-sections an equal and opposite RESISTING MOMENT due to the shearing stress between these two cross-sections.

Since for any cross-section the shearing stress for any fibre at a distance r from the neutral axis is $\frac{r}{v}S_s a$, the moment of that stress about the neutral axis is $\frac{S_s}{v}ar^2$, and the sum of the moments of all the stresses for any cross-section about the axis, or the resisting moment, is then $\frac{S_s}{v}\Sigma ar^2$.

For equilibrium this is balanced by the twisting moment M_t.

But Σar^2 is the *polar moment of inertia I_s of the cross-section* with reference to the axis through the centre of mass (page 34).

We have then for equilibrium, without reference to direction of rotation,

$$\frac{S_s I_s}{v} = M_t, \qquad (1)$$

where S_s is the unit shearing stress within the limit of elasticity in the most remote fibre of any cross-section at the distance v from the neutral axis, I_s is the polar moment of inertia of the cross-section with reference to that axis, which always passes through its centre of mass, and M_t is the twisting moment.

The reader will note the similarity of this equation with that for flexure, page 497.

From (1) we can find M_t for any given S_s when I_s and v are known and the elastic limit is not exceeded, or, inversely, for given M_t and S_s we can find the dimensions.

Coefficient of Rupture for Torsion.—Equation (1) holds within the elastic limit, or for all values of S_s less than S_e. If we consider it as holding without limit, and find the value of S_s in this case from experiments *carried to the point of rupture*, we call the value of S_s thus obtained the coefficient of rupture, and denote it by R. We have then for rupture

$$M_t = \frac{RI_s}{v} \qquad (2)$$

From equation (2), then, if R is known, we can compute the couple $\pm F$ which, acting with the lever-arm p, will rupture a given shaft, or the dimensions of a shaft to resist rupture with any desired factor of safety. If R is in pounds per square inch, P should be taken in pounds, p in inches, and all dimensions in inches. The distance v in inches is the distance to the remotest fibre of the cross-section.

We have, page 38, for

circular cross-section of radius r............. $I_s = \frac{\pi r^4}{2}, \quad v = r;$

rectangle with sides b and d................ $I_s = \frac{bd^3}{12} + \frac{db^3}{12}, \quad v = \frac{1}{2}\sqrt{b^2 + d^2};$

square cross-section........................ $I_s = \frac{d^4}{6}, \quad v = \frac{d}{\sqrt{2}}.$

We give here also average values of the coefficient of rupture for different materials:

	R. Pounds per square inch.
Steel (structural)	75000
Wrought iron	50000
Cast iron	25000
Timber	2000

Coefficient of Elasticity for Shear determined by Torsion.—Let the length of shaft be l, and let the angle of torsion be θ in radians, and the twisting moment be M_t. Then, within the limit of elasticity, the strain of the outer fibre for the end cross-section is $v\theta$, and the strain per unit of length is $\frac{v\theta}{l}$. The shearing unit stress of the outer fibre of the end cross-section is S_t. Then from page 477, since the coefficient of elasticity is the ratio of the unit stress to the unit strain,

$$E_s = \frac{S_s}{\frac{v\theta}{l}} = \frac{lS_s}{v\theta},$$

where v is the distance of the outer fibre of the end cross-section from the neutral axis, and θ is taken in radians.

If we substitute for S_s its value from (1), we have

$$E_s = \frac{lM_t}{\theta I_s}, \qquad (3)$$

from which E_s can be computed if the other quantities are known and the elastic limit is not exceeded. The angle θ is taken in radians.

Inversely we have

$$\frac{E_s\theta I_s}{l} = M_t. \qquad (4)$$

From (4) we can find M_t for any given θ in radians, when E_s, I_s and l are given and the elastic limit is not exceeded.

Work of Torsion.—If θ is the angle of torsion in radians for any cross-section, the strain of any fibre in that cross-section at a distance r from the neutral axis is $r\theta$, and the stress for that fibre is $\frac{r}{v}S_s a$. The work is then one half the product of the stress and strain (page 515), or $\frac{S_s\theta}{2v}ar^2$.

The work of all the fibres is then $\frac{\theta S_s}{2v}\Sigma ar^2$, or, since $\Sigma ar^2 = I_s$, we have, from (4) and (1), for the work

$$W = \frac{\theta S_s I_s}{2v} = \frac{M_t\theta}{2} = \frac{EI_s\theta^2}{2l} = \frac{M_t^2 l}{2EI_s}, \qquad (5)$$

where θ is taken in radians.

Transmission of Power by Shafts.—Work is the product of a force by the distance through which it acts, and is measured therefore in foot-pounds. Power is time-rate of work, and is measured in foot-pounds per minute or per second. A horse-power is 33000 ft.-pounds per minute or 550 ft.-pounds per second. If a shaft makes n revolutions per minute and the twisting force is F with a lever-arm p, then $2\pi p \times n$ is the distance and $2\pi npF$ is the work per minute, and the horse-power, if p is taken in inches, is

$$\text{H.P.} = \frac{2\pi nFp}{33000 \times 12}.$$

But $Fp = M_t = \frac{S_s I_s}{v}$. Hence

$$\text{H.P.} = \frac{\pi n S_s I_s}{198000\, v}, \qquad (6)$$

where n is the number of revolutions per minute, I_s and v are to be taken in inches and S_s in pounds per square inch.

Combined Flexure and Torsion.—We have for bending, from equation (II), page 497,

$$S_f = \frac{Mv}{I},$$

and for torsion, from equation (1), page 510,

$$S_s = \frac{M_t v}{I_s}.$$

Hence for the combined unit stresses of shear and compression or tension we have, from equations (1) and (2), page 484,

$$S = \sqrt{S_s^2 + \frac{S_f^2}{4}}. \qquad (7)$$

$$S_t \text{ or } S_c = \frac{S_f}{2} + \sqrt{S_s^2 + \frac{S_f^2}{4}}. \qquad (8)$$

Examples.—(1) *An iron shaft 5 feet long and 2 inches diameter is twisted through an angle of 7 degrees by a couple of ± 5000 pounds with a leverage of 6 inches, and on the removal of the couple returns to its original position. Find the value of E_s for shear.*

ANS. From (3), page 511, we have, since $l = 5 \times 12 = 60$ inches, $r = 1$ inch, $M_t = 5000 \times 6 = 30000$ inch-pounds, $I_s = \frac{\pi r^4}{2} = \frac{\pi}{2}$, $\theta = \frac{7\pi}{180}$ radians,

$$E_s = \frac{60 \times 30000 \times 2 \times 180}{\pi \times 7\pi} = 9\,390\,000 \text{ pounds per square inch.}$$

(2) *What is the couple which, acting with a lever-arm of 12 inches, will rupture a steel shaft 1.4 inches diameter, the coefficient of rupture by torsion being 75000 pounds per square inch?*

ANS. From (2), page 510, since $p = 12$, $R = 75000$, $v = 0.7$, $I_s = \frac{\pi \times 1.4^4}{64}$,

$$F \times 12 = \frac{75000 \times \pi \times 1.4^4}{64 \times 0.7}, \quad \text{or} \quad F = 1683 \text{ pounds.}$$

(3) *A circular shaft twisted by a couple of 90 pounds with a lever-arm of 27 inches has a unit shearing stress of 2000 pounds per square inch. If the same shaft is twisted by a couple of 40 pounds with a lever-arm of 57 inches, what is the unit shearing stress?*

ANS. From (1), page 510, since $S_s = 2000$, $M_t = 90 \times 27$, we have $\frac{I_s}{v} = \frac{M_t}{S_s} = \frac{90 \times 27}{2000}$. Hence for $M_t = 40 \times 57$ we have

$$S_s = \frac{40 \times 57 \times 2000}{90 \times 27} = 1877 \text{ pounds per square inch.}$$

(4) *An iron shaft 5 feet long and 2 inches diameter is twisted by a couple of 5000 pounds with a leverage of 6 inches. If E_s is 9 390 000 pounds per square inch, find the angle of twist.*

ANS. From (4), page 511, $\theta = \frac{5000 \times 6 \times 60}{9\,390\,000 \times \pi}$ radians, or $\theta = \frac{5000 \times 6 \times 60 \times 180}{9\,390\,000 \times \pi^2} = 7$ degrees.

(5) *Compare the strength of a square shaft with that of a circular shaft of equal area.*

ANS. From (2), page 510, we see that the strength is proportional to $\frac{I_s}{v}$. For a square shaft $\frac{I_s}{v} = \frac{d^3}{3\sqrt{2}}$. For a circular shaft $\frac{I_s}{v} = \frac{\pi r^3}{2}$. The ratio is then $\frac{\sqrt{2}d^3}{3\pi r^3}$. For equal areas we have $\pi r^2 = d^2$. Hence the ratio is $\frac{\sqrt{2\pi}}{3}$.

(6) *A circular shaft 2 feet long is twisted through an angle of 7 degrees by a couple of 200 pounds with a lever-arm of 6 inches. Find the angle for the same shaft 4 feet long when twisted by a couple of 500 pounds with a lever-arm of 18 inches.*

ANS. 105 degrees.

(7) *Find the combined unit stresses for a wrought-iron shaft 3 inches diameter and 12 feet long, resting on bearings at each end, which transmits 40 horse-power while making 120 revolutions per minute, upon which a load of 800 pounds is brought by a belt and pulley at the centre.*

ANS. The unit stress for flexure is

$$S_f = \frac{Mr}{I} = \frac{Plr}{4I} = \frac{Pl}{\pi r^3} = \frac{800 \times 12 \times 12 \times 8}{\pi \times 27} = 10800 \text{ pounds per square inch.}$$

The unit stress for torsion is, from (6), page 512,

$$S_s = \frac{198000 \times 40 \times r}{\pi \times 120 I_s} = \frac{198000 \times 40 \times 2 \times 8}{\pi^2 \times 120 \times 27} = 4000 \text{ pounds per square inch.}$$

The combined stresses, then, from (7) and (8), are, for shear, $S_s = 6700$ pounds per square inch, and for tension or compression, S_t or $S_c = 12100$ pounds per square inch.

(8) *A vertical shaft weighing with its loads 6000 pounds is subjected to a twisting moment by a force couple of 300 pounds acting with a leverage of 4 feet. If the shaft is of wrought iron 4 feet long and 2 inches diameter, find its unit stress, provided flexure is prevented.*

ANS. The unit stress of compression is $\frac{6000}{\pi r^2} = 1910$ pounds per square inch. The unit shearing stress for torsion is

$$S_s = \frac{M_t v}{I_s} = \frac{300 \times 48 \times 2}{\pi} = 9172 \text{ pounds per square inch.}$$

From equations (1) and (2), page 484, for combined compression and shear, we have for the shearing unit stress

$$S_s = \sqrt{9172^2 + \frac{1910^2}{4}} = 9215 \text{ pounds per square inch,}$$

and for the compressive unit-stress

$$S_c = \frac{1910}{2} + \sqrt{9172^2 + \frac{1910^2}{4}} = 10170 \text{ pounds per square inch.}$$

(9) *Find the diameter of a short vertical steel shaft to carry a load of 6000 pounds when twisted by a force of 300 pounds with a leverage of 4 feet, taking unit stress for shear at 7000 and for compression at 10000 pounds per square inch.*

ANS. About 2.5 inches.

CHAPTER IV.

WORK OF STRAINING. DEFLECTION OF FRAMED STRUCTURES. PRINCIPLE OF LEAST WORK. REDUNDANT MEMBERS. BEAMS FIXED HORIZONTALLY AT ENDS.

Work of Straining.—If the force F is gradually applied, increasing from zero up to F, the average force is $\frac{F}{2}$; and if λ is the strain, the work done by F is

$$\text{work} = \frac{F}{2}\lambda.$$

Since the stress is equal and opposite to F, the work of straining, within the elastic limit, is one half the product of the stress and strain.

Now, from (I),

$$\lambda = \frac{Fl}{EA}.$$

Hence we have for the work done by the force or against the stress, within the elastic limit,

$$\text{work} = \frac{F^2 l}{2EA}. \quad \ldots\ldots\ldots\ldots \quad \text{(III)}$$

Since E is always taken in pounds per square inch, if we take A in square inches, F in pounds and l in feet, the result will be foot-pounds.

Work and Coefficient of Resilience.—If the unit force $\frac{F}{A}$ is equal to the elastic limit unit stress S_e, so that

$$\frac{F}{A} = S_e, \quad \text{or} \quad F = S_e A,$$

we have, from (II), for the work done in straining up to the limit of elasticity

$$\text{work} = \frac{S_e^2 A l}{2E}.$$

If we regard the body as practically perfectly elastic up to the elastic limit, this is the work which the body can do in returning to its original dimensions when the force is removed. It is therefore called the WORK OF RESILIENCE.

Since, for uniform A, Al is equal to the volume V, we can write

$$\text{work of resilience} = \frac{S_e^2}{2E} \cdot Al = \frac{S_e^2}{2E} \cdot V.$$

The coefficient $\frac{S_e^2}{2E}$, or the work per unit of volume, is called the coefficient of resilience.

The work of resilience measures the ability of the body to withstand shock or suddenly applied force due to the impact of another body. To bring such a body to rest requires work. If this work is not greater than the work of resilience, the elastic limit is not exceeded.

From our tables, pages 476 and 478, for average values of S_e and E, we can compute the following average values of the coefficient of resilience:

	Coefficient of Resilience.
Wrought iron....................	12.5 inch-pounds per cubic foot
Steel (structural).................	26.6 " " " "
Cast iron.......................	1.2 " " " "
Timber.........................	3 " " " "

Deflection of a Framed Structure.—Equations (I) and (III) find direct application in the computation of the deflection of a framed structure.

Thus let S be the stress in any member due to the *actual loading*, and l and a the length and area of cross-section of the member. Then, from (I), the strain of the member is

$$\lambda = \frac{Sl}{aE}.$$

Now let s be the stress in the same member due to any arbitrary load p, supposed to rest at the point where the deflection is required. The work due to this load p is

$$\frac{s\lambda}{2} = \frac{Ssl}{2aE}.$$

The total work in all the members due to this load p is then

$$\Sigma\frac{Ssl}{2aE}.$$

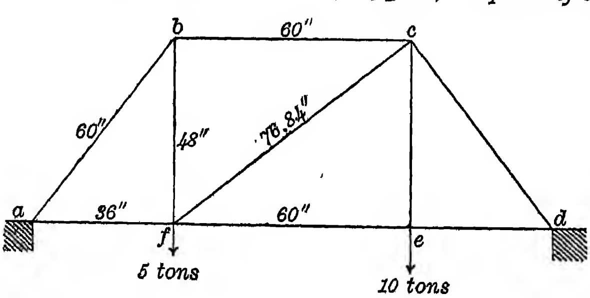
Now if Δ is the deflection at the point where p acts, the work done by p is $\frac{p\Delta}{2}$. Hence we have

$$\frac{p\Delta}{2} = \Sigma\frac{Ssl}{2aE},$$

or

$$\Delta = \frac{1}{pE}\Sigma\frac{Ssl}{a}.$$

Example.—*Suppose a truss (see figure) composed of two inclined rafters of length 60 inches, two vertical ties of length 48 inches, an upper chord of length 60 inches and a lower tie of length 132 inches, the two end panels 36 inches and the centre 60 inches. Let there be a diagonal strut cf whose length is 76.84 inches. Suppose a load of 5 tons at f and 10 tons at e. Required the deflection at e, the areas of cross-section being known.*



ANS. Let the coefficient of elasticity $E = 12500$ tons per square inch, and the areas of cross-section of the members as given in the following table.

Member.	Length l in inches.	E in tons per sq. in.	S in tons.	s in tons.	Cross-section a in sq. ins.	$\frac{l}{pE}$.	$\frac{Ss}{a}$.	Δ in inches.
ab..............	60	12500	$-$ 7.9545	$-$ 3.4091	1.85	$\frac{3}{6250}$	$+$ 14.6582	
bc..............	60	12500	$-$ 4.7727	$-$ 2.0454	1.00	$\frac{3}{6250}$	$+$ 9.7621	0.0372
cd..............	60	12500	$-$ 10.7954	$-$ 9.0909	1.85	$\frac{3}{6250}$	$+$ 53.0436	
de..............	36	12500	$+$ 6.4772	$+$ 5.4545	1.5	$\frac{9}{31250}$	$+$ 23.5532	
ef..............	60	12500	$+$ 6.4772	$+$ 5.4545	1.5	$\frac{3}{6250}$	$+$ 23.5532	0.0199
af..............	36	12500	$+$ 4.7727	$+$ 2.0454	1.5	$\frac{9}{31250}$	$+$ 6.5080	
bf..............	48	12500	$+$ 6.3636	$+$ 2.7272	2.0	$\frac{6}{15265}$	$+$ 8.6777	
ce..............	48	12500	$+$ 10.0000	$+$ 10.0000	2.0	$\frac{6}{15265}$	$+$ 50.0000	0.0308
cf..............	76.84	12500	$-$ 2.1829	$-$ 4.3658	0.75	$\frac{76.84}{125000}$	$+$ 12.7067	
							$\Delta =$	0.0879 in. at e

We take for the value of p the load of 10 tons at e and find the stresses s in every member due to this single load. We also find the stresses S in every member due to the actual loading. In the product Ss these stresses must be taken with their proper sign. Thus if s is compression or minus and S is also compression or minus, the product Ss is positive. If one is tension or positive and the other compression or negative, the product is negative. If the signs of S and s are carefully observed, the signs of the products Ss will thus take care of themselves.

If we take E in pounds or tons per square inch, S, s and p must be taken in pounds or tons, and l in inches and a in square inches.

We have taken p at e equal to 10 tons, or the load actually acting there. But if there were no load acting there, we could still assume $p = 10$ tons or 1 ton or any convenient amount, and proceed as before.

The stresses S due to actual loading are, strictly speaking, affected by the change of shape. This can, however, be disregarded without perceptible error, as the deflection in all practical cases is very small compared to the span.

Remarks on the Preceding Example.—In our example we assume E as constant for all members. We also assume that every member has its exact length and area of cross-section, that all pins fit perfectly tight, and all adjustable members, if any, are accurately adjusted.

A truss after erection may then be tested by calculating the deflection at the centre for a given loading and comparing with the actual observed deflection due to this loading.

A good agreement is thus a test of the close fit of all pins, of the proper adjustment of all adjustable members, of the agreement of the lengths and cross-sections of members with those called for by the design, of the constant value of E and its proper assumption as to magnitude, and finally of the fact that *the elastic limit is not exceeded.*

It is evident, however, that when so many conditions must concur, a discrepancy between the observed and the calculated deflection has little practical significance. The last-mentioned fact, that the elastic limit is not exceeded, is the most important, and this is proved, not by any close agreement between actual and calculated deflections, but by observing whether the deflection is constant under repeated applications of the same loading after the structure has attained its permanent set.

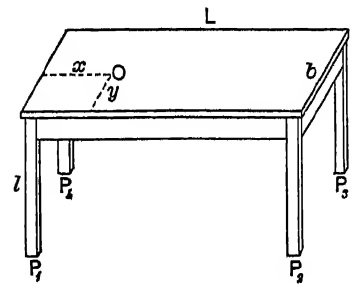
Computations of deflection are then of little value as a means of testing framed structures, and the calculated result cannot be expected to agree very closely with the actual deflection.

Principle of Least Work.—We have seen, page 311, that for stable equilibrium of a material system the conditions of static equilibrium must be fulfilled, and also that the

potential energy is a minimum. When an elastic body is strained by external forces within the elastic limit, the work done by the external forces is the work the body can do when released, or the potential energy is equal to the work of the external forces.

Hence for stable equilibrium of an elastic body *the work of the external forces must be the least possible consistent with the conditions of static equilibrium.* This is called the *principle of least work.*

As an illustration of the application of this principle, suppose a rectangular table of length L and breadth b to have four legs, all of equal length l and uniform cross-section A, one at each corner.



Let a load P rest on the table, and let x and y be the co-ordinates of its point of application.

Let P_1, P_2, P_3, P_4 be the loads carried by the legs. We have for the conditions of static equilibrium

$$P_1 + P_2 + P_3 + P_4 - P = 0,$$
$$P_3L + P_2L - Px = 0,$$
$$P_3b + P_4b - Py = 0.$$

From these equations we obtain

$$P_1 = P - \frac{Px}{L} - P_4,$$

$$P_2 = \frac{Px}{L} - \frac{Py}{b} + P_4,$$

$$P_3 = \frac{Py}{b} - P_4,$$

where P_1, P_2 and P_3 are given in terms of P_4. But P_4 is unknown. We have four unknown quantities and only three equations of condition. We need another equation of condition. This is furnished by the principle of least work.

Thus from equation (III) we have for the work of compressing the legs, assuming the floor and table-top to be rigid,

$$\text{work} = \frac{l}{2AE}[P_1^2 + P_2^2 + P_3^2 + P_4^2].$$

If in this we substitute the values of P_1, P_2, P_3, we have

$$\text{work} = \frac{l}{2AE}\Big[P^2\Big(1 - \frac{x}{L}\Big)^2 - 2P\Big(1 - \frac{x}{L}\Big)P_4 + P^2\Big(\frac{x}{L} - \frac{y}{b}\Big)^2 + 2P\Big(\frac{x}{L} - \frac{y}{b}\Big)P_4$$
$$+ \frac{P^2y^2}{b^2} - \frac{2Py}{b} \cdot P_4 + 4P_4^2\Big].$$

We thus have the work given in terms of P_4. Now P_4 must have such a value that the work shall be a minimum. Therefore putting the differential of the work relative to P_4 equal to zero, we have

$$\frac{d\,(\text{work})}{dP_4} = 0 = -2P\Big(1 - \frac{x}{L}\Big) + 2P\Big(\frac{x}{L} - \frac{y}{b}\Big) - \frac{2Py}{b} + 8P_4.$$

Hence

$$P_4 = \frac{P}{4} - \frac{Px}{2L} + \frac{Py}{2b} \quad \text{and} \quad P_3 = -\frac{P}{4} + \frac{Px}{2L} + \frac{Py}{2b}.$$

$$P_2 = \frac{P}{4} - \frac{Py}{2b} + \frac{Px}{2L},$$

$$P_1 = \frac{3P}{4} - \frac{Px}{2L} - \frac{Py}{2b}.$$

If P is at the centre, $x = \frac{L}{2}$, $y = \frac{b}{2}$ and

$$P_1 = P_2 = P_3 = P_4 = \frac{1}{4}P.$$

If P is at the middle of a side l, $x = \frac{L}{2}$, $y = 0$ and

$$P_1 = \frac{P}{2}, \qquad P_2 = \frac{P}{2}, \qquad P_3 = 0, \qquad P_4 = 0.$$

If P is over leg three, $x = L$, $y = b$ and

$$P_1 = -\frac{P}{4}, \qquad P_2 = +\frac{P}{4}, \qquad P_3 = +\frac{3}{4}P, \qquad P_4 = +\frac{P}{4}.$$

The minus sign for P_1 shows that the stress is reversed and leg one must be fastened down. If not, it is lifted off the floor and we have the table supported on three legs only.

Example — *Take the same table with a fifth leg at the centre, and show that the load carried by this leg is always* $\frac{1}{5}P$ *no matter where the load P may be placed.*

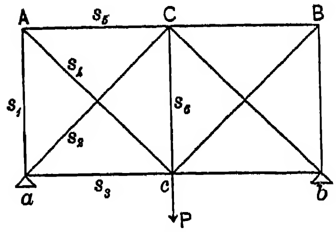
Remarks on the Preceding.—The preceding problem of the four-leg table illustrates the principle of least work. It also illustrates much more—*it furnishes an example of theory misapplied.* The theory is sound, and the results are therefore correct provided the assumptions are realized in fact. But these assumptions are not realized by any actual table. For instance, we have assumed the floor and table-top rigid, every leg of precisely equal length and the same uniform cross-section. Such a table is an ideal which has no physical existence. The theory is then misapplied, since the assumptions do not correspond to fact. A very small discrepancy in the length of the legs alone would entirely change the results.

The reader, then, must regard the problem simply as an illustration of the principle of least work, and should note that even sound principles need care in application, and that for proper application the assumptions should accord with reality, otherwise the results are worthless.

Thus in the present case the legs should be designed for three only. Then if any others are desired, they can be added of the same dimensions. This practical solution is not only simpler, but it is actually more accurate and scientific.

Redundant Members.—The principle of least work is also illustrated by the calculation of the stresses in a framed structure with redundant members. The method of procedure is the same as for the table with four legs.

FIG. 1.



Thus take the simple truss shown in Fig. 1, resting on supports at a and b with a load P at the centre. Let the inclined braces make the angle $\theta = 45°$ with the vertical, so that the lengths of all horizontal and vertical members are equal to the height h of the truss. The length of the inclined members is then $h\sqrt{2}$, and $\tan\theta = 1$, $\sec\theta = \sqrt{2}$.

Let the stress and area of cross-section of aA, aC, ac, Ac, AC, Cc be s_1 and a_1, s_2 and a_2, s_3 and a_3, etc., as indicated in Fig. 1. This truss consists of two statically determinate trusses, as shown in Fig. 2 and Fig. 3, superposed upon each other.

FIG. 2.

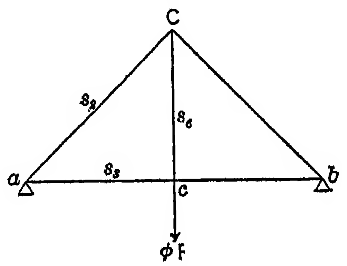
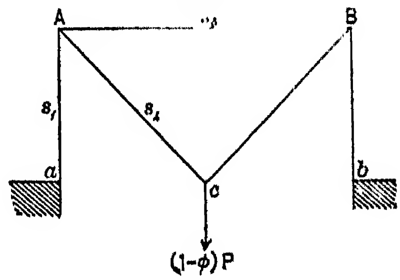


FIG. 3.



Let the truss of Fig. 2 carry a certain fraction ϕP of the load, and the truss of Fig. 3 carry the rest, or $(1-\phi)P$.

Then we have the stresses (page 409)

$$s_1 = -\frac{(1-\phi)}{2}P, \qquad s_2 = -\frac{\phi P}{2}\sec\theta = -\frac{\phi P}{\sqrt{2}},$$

$$s_3 = \frac{\phi P}{2}\tan\theta = \frac{\phi P}{2}, \qquad s_4 = \frac{(1-\phi)}{2}P\sec\theta = \frac{(1-\phi)P}{\sqrt{2}},$$

$$s_5 = -(1-\phi)P\tan\theta = -(1-\phi)P, \qquad s_6 = P\phi.$$

The value of ϕ must be that which makes the work a minimum. We have for the work, from equation (II), since the stresses and areas in the left portion of Fig. 1 are the same as the right,

$$\text{work} = \frac{1}{2E}\left[\frac{2s_1^2h}{a_1} + \frac{2\sqrt{2}s_2^2h}{a_2} + \frac{2s_3^2h}{a_3} + \frac{2\sqrt{2}s_4^2h}{a_4} + \frac{2s_5^2h}{a_5} + \frac{s_6^2h}{a_6}\right],$$

or, substituting the values of the stresses as found,

$$\text{work} = \frac{hP^2}{2E}\left[\frac{1-2\phi+\phi^2}{2a_1} + \frac{\sqrt{2}\phi^2}{a_2} + \frac{\phi^2}{2a_3} + \frac{\sqrt{2}(1-2\phi+\phi^2)}{a_4} + \frac{2(1-2\phi+\phi^2)}{a_5} + \frac{\phi^2}{a_6}\right].$$

If we differentiate with reference to ϕ and put the value of $\frac{d\,(\text{work})}{d\phi} = 0$, we have for the value of ϕ which makes the work a minimum

$$\phi = \frac{\dfrac{1}{a_1} + \dfrac{2\sqrt{2}}{a_4} + \dfrac{4}{a_5}}{\dfrac{1}{a_1} + \dfrac{2\sqrt{2}}{a_2} + \dfrac{1}{a_3} + \dfrac{2\sqrt{2}}{a_4} + \dfrac{4}{a_5} + \dfrac{2}{a_6}}.$$

We can therefore find s_1, s_2, etc., by inserting this value of ϕ in the equations for the stresses already given.

It will be noted that the cross-section a_1, a_2, etc., must be known for each member in advance.

If the cross-sections are all equal, we have

$$\phi = \frac{5 + 2\sqrt{2}}{8 + 4\sqrt{2}} = 0.57.$$

Evidently the same remarks apply here as in the case of the table with four legs. Every member must be of absolutely true length and of the exact cross-section assigned. Any variation from ideal conditions invalidates the result. As such ideal conditions do not and cannot exist, the actual stresses in any given case will not agree with the computed stresses.

We see, then, that the use of redundant members in a structure not only makes the calculation of stresses very involved and laborious, but also that the results obtained hold only for an ideal structure under ideal conditions and are by no means the actual stresses.

No Economy due to Redundant Members.—It remains to inquire whether there can be any compensating advantages in economy in the use of redundant members to offset the objections already noted.

This inquiry is directly answered by the preceding article. We see that Fig. 1 is composed of two statically determinate trusses, Fig. 2 and Fig. 3, superposed upon each other. Each of these carries its own proportion of the loading. It is also evident that one of these trusses is more economical than the other. The combined truss of Fig. 1 must therefore have an economy intermediate between the two, and therefore must necessarily be less economical than one of the two.

The same holds for any structure with redundant members. We may consider it as formed by the superposition of a series of statically determinate trusses. The economy of the combination must be less than the economy of some one of this series. Hence any structure with redundant members is less economical than some statically determinate structure included in the redundant structure.

There is, then, no gain of economy by the use of redundant members.

Work of Bending.—As before, let S_f be the unit stress in the most remote fibre of any cross-section at a distance v from the neutral axis of that cross-section; then the unit stress S in any fibre at a distance v' is, from equation (1), page 494,

$$S = \frac{v'}{v} S_f, \quad \text{hence} \quad S_f = \frac{v}{v'} S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, if is the bending moment, or moment of all the external forces on the left of the cross-section (page 494), we have, from (II), page 497,

$$\frac{S_f I}{v} = M.$$

Inserting the value of S_f from (1), we have for the unit stress S

$$S = \frac{Mv'}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If ds is the distance measured along the neutral axis between two consecutive cross-sections, and a the area of the cross-section of the fibre, we have, from (I), page 477, putting Sa for F, a for A, and ds for l, for the strain of any fibre

$$\lambda = \frac{Sds}{E},$$

or, inserting the value of S from (2),

$$\lambda = \frac{Mv'ds}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now the work on the fibre is half the product of the stress and strain (page 515), or, from (2) and (3),

$$Sa \cdot \frac{\lambda}{2} = \frac{M^2av'^2ds}{2EI^2};$$

and since $\Sigma av'^2 = I$, the work on all the fibres of the cross-section is

$$\frac{M^2ds}{2EI}.$$

For the total work, then, for all the cross-sections we have

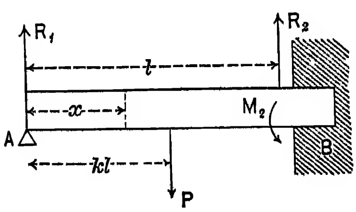
$$W = \int_0^s \frac{M^2ds}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

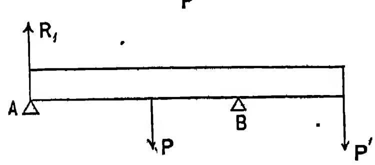
Equation (IV) is general whatever the shape of the beam before flexure. If the beam is straight before flexure, we have dx in place of ds, and l in place of s. Hence

$$W = \int_0^l \frac{M^2dx}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV')$$

Beams Fixed Horizontally at the Ends.—By means of equation (IV′) and the principle of least work, we can solve all cases of beams fixed horizontally at the ends.

CASE I. BEAM OF UNIFORM CROSS-SECTION FIXED HORIZONTALLY AT ONE END AND SUPPORTED AT THE OTHER—CONCENTRATED LOAD.—*Reaction.*—Let R_1 be the reaction at the supported end A on left, and let the load P be at a distance kl from the supported end and $(1-k)l$ from the fixed end B. Then for any point of the neutral axis distant x from the supported end we have the bending moment,





when $x < kl$, $\qquad M_x = -R_1x$;

when $x > kl$, $\qquad M_x = -R_1x + P(x - kl)$.

The beam may be fixed at the end B either by letting it into a wall or by continuing it over the support at B and applying a load P'. In either case the tangent to the deflected beam *must be horizontal at the fixed end B*, and the value of R_1 must be that which makes the work of bending *from A to B* a minimum. The work on the remainder of the beam, if any, can be neglected.

From (IV′), then, we have for the work of bending

$$\text{work} = \int_0^{kl} \frac{R_1^2 x^2 dx}{2EI} + \int_{kl}^{l} [-R_1 x + P(x - kl)]^2 \frac{dx}{2EI}.$$

If we differentiate with respect to R_1 and put $\frac{d(\text{work})}{dR_1} = 0$, we have for the value of R_1 which makes the work of bending from A to B a minimum, since E and I are constant,

$$\int_0^{l} R_1 x^2 dx + \int_{kl}^{l} - P(x - kl)x\, dx = 0.$$

Performing the integrations, we obtain

$$R_1 = \frac{P}{2}(1 - k)^2(2 + k).$$

If the load is at the centre, we have $k = \frac{1}{2}$ and $R_1 = \frac{5}{16}P$.

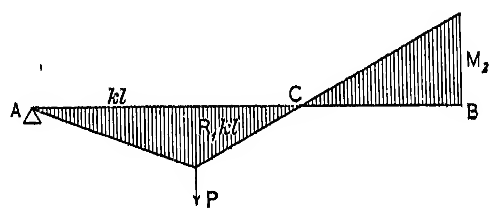
Moment.—The moment M_2 at the fixed end is

$$M_2 = -R_1 l + P(1 - k)l = +\frac{Pl}{2}(k - k^3)$$

and is positive. Hence $R_1 l$ is less than $P(1 - k)l$. The moment at the load is $-R_1 kl$ and is negative. If we subtract this from M_2, we have

$$-R_1 l(1 - k) + Pl(1 - k),$$

which is positive since P is greater than R_1. The moment at the fixed end is therefore the greatest, and the moment at any point is given to scale by the ordinates to two straight lines as shown in the figure.



Point of Inflection.—We see that at the point C in the figure the bending moment is zero. This point, as we shall see (page 547), is the point at which the curvature of the deflected beam changes from concave to convex. We can easily find its position. Thus, from the figure, if x_1 is the distance of C from the supported end A, we have

$$\frac{M_2}{l - x_1} = \frac{R_1 kl}{x - kl}, \quad \text{or} \quad x_1 = \frac{M_2 kl + R_1 kl^2}{M_2 + R_1 kl}.$$

Substituting the values of M_2 and R_1 already found,

$$x_1 = \frac{2l}{3 - k^2}.$$

If the load is at the centre, $k = \frac{1}{2}$ and $x_1 = \frac{8}{11}l$.

Breaking Load.—Since, as we have proved, the bending moment M_2 at the fixed end is greatest, we have, from equation (4), page 499,

$$M_2 = \frac{RI}{v},$$

or, inserting the value of M_2, we have for the breaking weight

$$P = \frac{2RI}{vlk(1 - k^2)}.$$

For the load at the centre $k = \frac{1}{2}$ and $P = \frac{16RI}{3vl}$, or $\frac{4}{3}$ as much as for the same beam supported at both ends (page 500).

The moment M_2 is a maximum for $k = \sqrt{\frac{1}{3}} = 0.5774$, that is when the load P is distant $0.5774l$ from the supported end.

For this position we have the least breaking load

$$P = \frac{3\sqrt{3}RI}{vl}.$$

CASE 2. BEAM OF UNIFORM CROSS-SECTION FIXED HORIZONTALLY AT ONE END AND SUPPORTED AT THE OTHER—UNIFORM LOAD.—*Reaction.*—In the preceding case we have found for concentrated load

$$R_1 = \frac{P}{2}(1 - k)^2(2 + k).$$

If for P we put wdx, and for k we put $\frac{x}{l}$, we have in the present case

$$R_1 = \int_0^l w\left(1 - \frac{3x}{2l} + \frac{x^3}{2l^3}\right)dx.$$

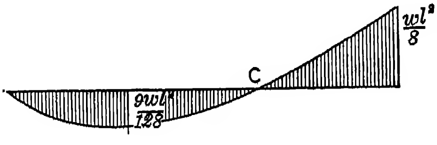
Performing the integration, we have

$$R_1 = \frac{3}{8}wl.$$

Moment.—The moment M_2 at the fixed end is then

$$M_2 = -\frac{3wl^2}{8} + \frac{wl^2}{2} = +\frac{wl^2}{8}.$$

The moment at any point distant x from the supported end is then



$$M_x = -R_1x + \frac{wx^2}{2} = -\frac{3wlx}{8} + \frac{wx^2}{2}.$$

This is the equation of a parabola whose vertex is at $\frac{3}{8}l$ from the supported end as shown in the figure, the moment at this point being $-\frac{9wl^2}{128}$.

Point of Inflection.—Here again the moment is zero and we have a point of inflection C at a distance x_1 from the supported end given by

$$x_1 = \frac{3}{4}l.$$

Breaking Load.—Since the moment M_2 at the fixed end is the greatest, we have, from equation (4), page 499,

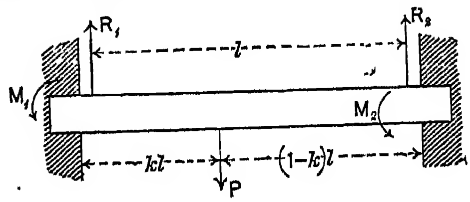
$$M_2 = \frac{RI}{v}, \quad \text{or} \quad \frac{wl^2}{8} = \frac{RI}{v}.$$

Hence the breaking load is

$$wl = \frac{8RI}{v},$$

or $\frac{3}{2}$ as much as for the same load at the centre and just the same as for the same beam supported at the ends (page 500).

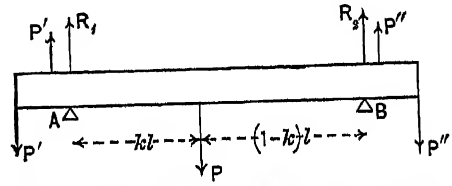
CASE 3. BEAM OF UNIFORM CROSS-SECTION FIXED HORIZONTALLY AT BOTH ENDS—CONCENTRATED LOAD.—In this case we have the bending moment for



$$x < kl, \qquad M_x = M_1 - R_1x;$$

and for

$$x > kl, \qquad M_x = M_1 - R_1x + P(x - kl).$$



The beam may be fixed at the ends either by letting it into a wall at each end, or by having it project beyond the supports and applying loads P', P'' at the ends. In either case the tangent to the deflected beam must be *horizontal at the ends*, and the values of R_1 and M_1 must be such as make the work of bending from A to B a minimum. The work on the remainder of the beam, if any, can be neglected.

From (IV′), then, we have for the work of bending

$$\text{work} = \int_0^{kl} [M_1 - R_1x]^2 \frac{dx}{2EI} + \int_{kl}^{l} [M_1 - R_1x + P(x - kl)]^2 \frac{dx}{2EI}.$$

Differentiating with respect to R_1 and M_1 and putting $\frac{d(\text{work})}{dM_1} = 0$ and $\frac{d(\text{work})}{dR_1} = 0$, we have for the values of R_1 and M_1 which make the work a minimum

$$\int_0^{kl} (M_1 - R_1x)dx + \int_{kl}^{l} P(x - kl)dx = 0,$$

$$\int_0^{l} (-M_1x + R_1x^2)dx + \int_{kl}^{l} - P(x - kl)x\, dx = 0.$$

Perorfming the integrations, we obtain

$$2M_1 - R_1l + Pl(1 - 2k + k^2) = 0,$$

$$-3M_1 + 2R_1l - Pl(2 - 3k + k^3) = 0.$$

From these two equations we have

$$R_1 = P(1-k)^2(1+2k), \qquad M_1 = Plk(1-k)^2.$$

If the load is at the centre, $k = \frac{1}{2}$ and

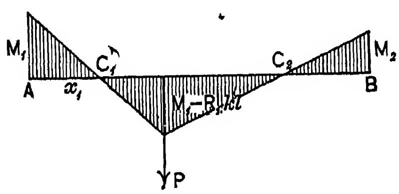
$$R_1 = \frac{1}{2}P, \qquad M_1 = \frac{Pl}{8}.$$

The reaction at the right end is

$$R_2 = P - R_1 = Pk(1-k)(1+2k),$$

and the moment M_2 *on the left* of the right end is

$$M_2 = M_1 - R_1 l + Pl(1-k) = Plk^2(1-k).$$



The moment at any point is given to scale by the ordinates to two straight lines, as shown in the figure, the moment at the load being

$$M_1 - R_1 lk = -2Plk^2(1-k)^2.$$

The greatest moment will then be at the nearest fixed end to the load.

Points of Inflection.—The moment is zero, and we have a point of inflection at C_1 and C_2. If x_1 is the distance of C_1 from the end A, we have at once, from the figure,

$$\frac{M_1}{x_1} = \frac{-M_1 + R_1 kl}{kl - x_1}, \quad \text{or} \quad x_1 = \frac{M_1}{R_1} = \frac{kl}{1+2k}.$$

For the distance x_2 of C_2 from B we have

$$\frac{M_2}{x_2} = \frac{-M_1 + R_1 kl}{(1-k)l + x_2}, \quad \text{or} \quad x_2 = \frac{M_2(1-k)l}{M_2 - M_1 + R_1 kl} = \frac{(1-k)l}{3-2k}.$$

If the load is at the centre, $k = \frac{1}{2}$ and $x_1 = x_2 = \frac{l}{4}$.

Breaking Load.—The greatest moment is at the end nearest to the load. Let this be the left end. Then, from equation (4), page 499,

$$M_1 = \frac{RI}{v}, \quad \text{or} \quad P = \frac{RI}{vlk(1-k)^2}.$$

For the load at the centre $k = \frac{1}{2}$ and $P = \frac{8RI}{vl}$, or twice as much as the same beam supported at the ends. The moment M_1 is a maximum for $k = \frac{1}{3}$. That is, the greatest

moment at the nearest end occurs when the load is distant $\frac{1}{3}l$ from that end. The value of this greatest moment is $M_1 = \frac{4Pl}{27}$. Hence the least breaking weight is given by

$$\frac{4Pl}{27} = \frac{RI}{v}, \quad \text{or} \quad P = \frac{27RI}{4vl},$$

or $\frac{27}{16}$ as great as for the same beam supported at the ends.

CASE 4. BEAM OF UNIFORM CROSS-SECTION FIXED HORIZONTALLY AT BOTH ENDS —UNIFORM LOAD.—*Reaction.*—In the preceding case we have found for concentrated load

$$R_1 = P(1 - k)^2(1 + 2k).$$

If for P we put wdx, and for k we put $\frac{x}{l}$, we have in the present case

$$R_1 = \int_0^l w\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right)dx.$$

Performing the integration, we have

$$R_1 = \frac{wl}{2}.$$

Moment.—We have found in the preceding case

$$M_1 = Plk(1 - k)^2.$$

If for P and k we put wdx and $\frac{x}{l}$, we have

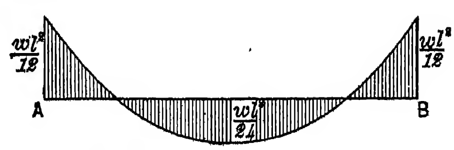
$$M_1 = \int_0 wl\left(\frac{x}{l} - \frac{2x^2}{l^2} + \frac{x^3}{l^3}\right)dx.$$

Performing the integration, we have

$$M_1 = \frac{wl^2}{12}.$$

The moment M_2 on the left of the right end is the same.
The moment at any point distant x from the left end is

$$M_x = M_1 - R_1x + \frac{wx^2}{2} = \frac{wl^2}{12} - \frac{wx}{2}(l - x).$$



This is the equation of a parabola whose vertex is at $\frac{1}{2}l$ as shown in the figure, the moment at this point being $-\frac{wl^2}{24}$. The moment at the fixed ends is the greatest.

Breaking Load.—Since the moment is greatest at the fixed end, we have, from equation (4), page 499,

$$\frac{wl^2}{12} = \frac{RI}{v}, \quad \text{or} \quad wl = \frac{12RI}{v}.$$

Bending and Tension Combined.—A beam may sometimes be subjected to bending and at the same time to tension. Thus, for instance, a lower chord panel of a bridge truss may be in tension and at the same time sustain loads applied by means of cross-ties between the panel points.

In such a case let M be the maximum bending moment, y the deflection, T the tensile stress and A the area of cross-section, so that $\frac{T}{A}$ is the tensile unit stress. Then we have the maximum moment

$$M + Ty,$$

and from equation (II), page 497, the unit stress in the outer tensile fibre is

$$\frac{T}{A} + \frac{(M + Ty)v}{I}.$$

If S_w is the working unit stress adopted, we have then

$$S_w = \frac{T}{A} + \frac{(M + Ty)v}{I}.$$

The neutral axis is now no longer at the centre of mass of the cross-section, and a strict solution leads to results of great complexity. If, however, we disregard the small deflection y, we have in all practical cases

$$S_w = \frac{T}{A} + \frac{Mv}{I}, \quad \text{hence} \quad M = \frac{\left(S_w - \frac{T}{A}\right)I}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we put for I its value $A\kappa^2$, where κ is the radius of gyration of the cross-section (page 32), we have

$$A = \frac{Mv}{S_w\kappa^2} + \frac{T}{S_w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we can find in any case the load for a given S_w, T and cross-section A, and from (2) the cross-section for a given load.

Bending and Compression Combined.—This case is just the same as the preceding, except that we must put the compressive stress C in place of T and take S_w the working stress for compression. If flexure is to be apprehended, we must take S_w as given on page 569.

Examples.—(1) *Find the area of cross-section for a square beam of 12 ft. span which sustains a load of 300 pounds at the centre and has at the same time a direct longitudinal tension of 2000 pounds, the working stress being taken at 1000 pounds per square inch.*

ANS. We have $S_w = 1000$, $T = 2000$, $v = \frac{d}{2}$, $\kappa^2 = \frac{d^2}{12}$, $A = d^2$, $M = 150 \times 6 \times 12$. Hence, from (2),

$$A = d^2 = \frac{324}{5d} + 2, \quad \text{or} \quad d = 4.18 \text{ inches.}$$

(2) *Find the area of cross-section for a square beam of 12 ft. span which sustains a load of 50 pounds per foot uniformly distributed and has at the same time a direct longitudinal tension of 2000 pounds, the working unit stress being taken at 1000 pounds per square inch.*

ANS. We have $S_w = 1000$, $T = 2000$, $v = \frac{d}{2}$, $\kappa^2 = \frac{d^2}{12}$, $A = d^2$, $M = \frac{wl^2}{8} = \frac{50 \times 12 \times 12 \times 12 \times 12}{12 \times 8}$. Hence, from (2),

$$A = d^2 = \frac{324}{5d} + 2, \quad \text{or} \quad d = 4.18 \text{ inches.}$$

(3) *A rectangular iron beam 12 feet long and 2 inches wide has a longitudinal tension of 20000 pounds and supports a load of 5000 pounds at the centre. Find the depth in order that the unit stress shall not exceed 10000 pounds per square inch.*

ANS. We have $S_w = 10000$, $T = 20000$, $v = \frac{d}{2}$, $\kappa^2 = \frac{d^2}{12}$, $A = 2d$, $M = \frac{5000 \times 6 \times 12}{2}$. Hence, from (2),

$$A = 2d = \frac{108}{d} + 2, \quad \text{or} \quad d = 7.86 \text{ inches.}$$

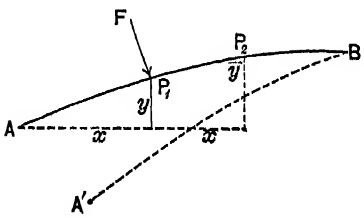
CHAPTER V.

DEFLECTION OF BEAMS.

Deflection of a Beam.—Let ds be the distance measured along the neutral axis between any two consecutive cross-sections, and v' the distance of any fibre from the neutral axis; then the strain λ of the fibre is, from equation (3), page 522,

$$\lambda = \frac{Mv'ds}{EI},$$

where M is the bending moment of all the external forces on the left.



Take the origin at the left end A, and let x, y be the co-ordinates of any point P_1. At this point let a force F be supposed to act, and let its moment relative to any point P_2 *on the right* of P_1 given by the co-ordinates $\bar{x}$, $\bar{y}$ be m.

Then from equation (2), page 521, the stress due to this force is

$$\frac{mav'}{I}.$$

The work of this force on any fibre of the cross-section at P_2 is then half the product of the stress and strain (page 515), or

$$\text{work} = \frac{mMav'^2ds}{2EI^2}.$$

On all the fibres of the cross-section, since $\Sigma av'^2 = 0$, we have then

$$\text{work} = \frac{mMds}{2EI},$$

and for all the cross-sections between the right end B and P_1, if $AB = s$ and $AP_1 = s_1$,

$$\text{work} = \int_{s_1}^{s} \frac{mMds}{2EI}. \quad \quad (1)$$

This equation is general whatever the shape of the beam, whether straight or curved, or the direction of F.

Let F be vertical. Then its moment m at P_2 is

$$m = F(\bar{x} - x),$$

and, from (1), its work is

$$\text{work} = \int_{s_1}^{s} \frac{F(\bar{x} - x)Mds}{2EI}.$$

But if Δ_y is the vertical deflection at P_1, the work of F is $\frac{F\Delta_y}{2}$, and this must be equal and opposite in sign to the work done against the fibre stresses, or

$$\frac{F\Delta_y}{2} = \int_{s_1}^{s} \frac{F(\bar{x} - x)Mds}{2EI}.$$

Hence the vertical deflection at any point P_1 is given by

$$\Delta_y = \int_{s_1}^{s} \frac{M(\bar{x} - x)ds}{EI} = \int_{s_1}^{s} \frac{Mxds}{EI} - x\int_{s_1}^{s} \frac{Mds}{EI}. \quad \text{(V)}$$

Again, let F be horizontal. Then its moment m at P_2 is

$$m = -F(\bar{y} - y),$$

and, from (1), its work is

$$\text{work} = -\int_{s_1}^{s} \frac{F(\bar{y} - y)Mds}{2EI}.$$

If Δ_x is the horizontal deflection at P_1, we have then

$$\frac{F\Delta_x}{2} = -\int_{s_1}^{s} \frac{F(\bar{y} - y)Mds}{2EI}.$$

Hence the horizontal deflection at any point P_1 is given by

$$\Delta_x = -\int_{s_1}^{s} \frac{Myds}{EI} + y\int_{s_1}^{s} \frac{Mds}{EI} \quad \text{(VI)}$$

Equations (V) and (VI) are general whatever the shape of the beam, whether straight or curved.

For a straight beam $y = 0$ and the horizontal deflection $\Delta_x = 0$. The vertical deflection is given by (V) if we put $ds = dx$, $s_1 = x$ and $s = l$. Hence for a straight beam, calling the vertical deflection y, we have

$$y = \int_{x}^{l} \frac{Mx\,dx}{EI} - x\int_{x}^{l} \frac{Mdx}{EI} \quad \text{(VII)}$$

If we differentiate this, we obtain

$$dy = \frac{Mx\,dx}{EI} - \frac{Mx\,dx}{EI} - dx\int_{x}^{l} \frac{Mdx}{EI},$$

or

$$\frac{dy}{dx} = -\int_{x}^{l} \frac{Mdx}{EI}. \quad (2)$$

If we differentiate again, we have

$$EI\frac{d^2y}{dx^2} = -M. \quad \text{(VIII)}$$

Equation (VIII) is the differential equation of the curve of deflection for a *straight beam* By integrating it we obtain (2), which gives the inclination at any point. Another integration gives (VII), the deflection y at any point.

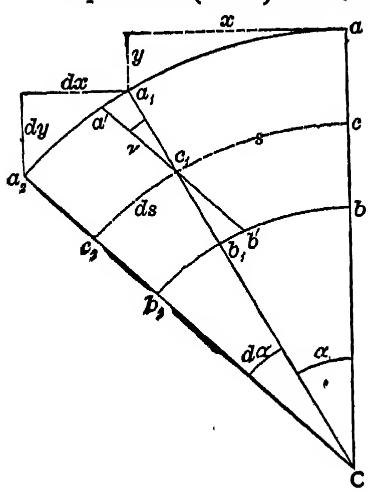
When the deflection is very small compared to the length, we have, from the Calculus,

$$\frac{d^2y}{dx^2} = \frac{1}{\rho},$$

where ρ is the radius of curvature. Hence we can write

$$\frac{EI}{\rho} = -M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IX)$$

Equations (VIII) and (IX) can also be directly deduced as follows:



Let ab, a_1b_1 and a_2b_2 be three cross-sections all parallel before bending, the distance between the first two being x, and the last two being consecutive at the distance dx.

After bending let these cross-sections be still plane, and intersect at some point C. Let cc_1c_2 be the neutral axis, so that $cC = c_1C = c_2C = \rho$. Let the angle $aCa_1 = \alpha$ *be very small.* Then the angle $a_1Ca_2 = d\alpha$, and the distance $cc_1 = s$ will be equal to x, and $c_1c_2 = ds$, to dx, approximately.

Through c_1 draw $a'b'$ parallel to a_2b_2. Then the angle $a_1c_1a' = d\alpha$.

For any fibre at a distance v from the neutral axis, then, the strain is

$$\lambda = vd\alpha.$$

Since the original length is dx, if the area of cross-section is a, we have from I), page 477, the stress of the fibre

$$S = \frac{Ea\lambda}{dx} = \frac{Eavd\alpha}{dx}.$$

The moment relative to c_1 is then

$$vS = \frac{Eav^2d\alpha}{dx}.$$

For all the fibres of the cross-section, then, since $\Sigma av^2 = I$, we have the moment

$$\frac{EId\alpha}{dx}.$$

This moment must be equal and opposite to the bending moment M. Hence

$$EI\frac{d\alpha}{dx} = -M.$$

Now for very small deflection $\alpha = \frac{dy}{dx}$ and hence $d\alpha = \frac{d^2y}{dx}$. Substituting, we have

$$EI\frac{d^2y}{dx^2} = -M,$$

which is equation (VIII).

Again, from similar triangles, we have

$$vd\alpha : v :: as : \rho,$$

or, since $ds = dx$,

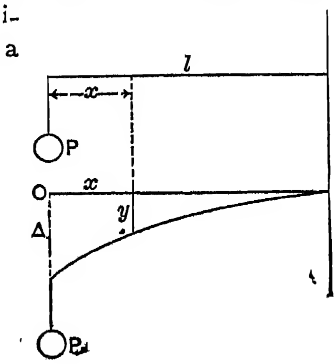
$$\frac{da}{dx} = \frac{1}{\rho}.$$

Substituting this, we have

$$\frac{EI}{\rho} = -M,$$

which is equation (IX).

Beam Fixed Horizontally at One End and Loaded at the Other.—(*a*) UNIFORM CROSS-SECTION.—For uniform cross-section I is constant. Take the origin at the free end. Then for any point of the neutral axis at a distance x the bending moment is



$$M = +Px.$$

We have then, from (VIII),

$$EI\frac{d^2y}{dx^2} = -Px.$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1,$$

where C_1 is the constant of integration.

Since the beam is fixed horizontally at the right end, the tangent at the right end to the curve of deflection is horizontal, and hence $\frac{dy}{dx} = 0$ when $x = l$. Hence $C_1 = +\frac{Pl^2}{2}$. We have then

$$EI\frac{dy}{dx} = \frac{Pl^2}{2} - \frac{Px^2}{2}.$$

Integrating again,

$$EIy = \frac{Pl^2x}{2} - \frac{Px^3}{6} + C_2,$$

where C_2 is the constant of integration.

Since the deflection at the fixed end is zero, we have $y = 0$ when $x = l$, and hence $C_2 = -\frac{Pl^3}{3}$.

We have then

$$EIy = \frac{Pl^2x}{2} - \frac{Px^3}{6} - \frac{Pl^3}{3}.$$

This equation gives the deflection at any point. The deflection is evidently greatest at the free end. Making, then, $x = 0$, we have the maximum deflection

$$\Delta = -\frac{Pl^3}{3EI}.$$

The minus sign shows that the deflection is downwards.

If the cross-section is rectangular, $I = \frac{1}{12}bd^3$ and we have

$$\Delta = -\frac{4Pl^3}{Ebd^3}.$$

(*b*) UNIFORM STRENGTH.—For uniform strength the cross-section varies and hence I is variable. We have then in general for rectangular cross-section and uniform strength

$$\frac{d^2y}{dx^2} = -\frac{Px}{EI} = -\frac{12Px}{Ebd^3},$$

where b and d are variable.

Constant Depth.—If the depth is constant and always equal to d_1, then, as we have seen (page 503), $b = \frac{x}{l}b_1$, where b_1 is the breadth at the fixed end. Hence for rectangular cross-section

$$\frac{d^2y}{dx^2} = -\frac{12Pl}{Eb_1d_1^3}.$$

As we have seen (page 532), $\frac{d^2y}{dx^2} = \frac{1}{\rho}$, where ρ is the radius of curvature. Hence for constant depth and uniform strength the radius of curvature ρ is constant and *the curve of deflection is a circle.*

Integrating, since $\frac{dy}{dx} = 0$ for $x = l$,

$$\frac{dy}{dx} = -\frac{12Plx}{Eb_1d_1^3} + \frac{12Pl^2}{Eb_1d_1^3}.$$

Integrating again, since $y = 0$ for $x = l$,

$$y = -\frac{6Plx^2}{Eb_1d_1^3} + \frac{12Pl^2x}{Eb_1d_1^3} - \frac{6Pl^3}{Eb_1d_1^3}.$$

For the deflection Δ at the free end $x = 0$ and

$$\Delta = -\frac{6Pl^3}{Eb_1d_1^3},$$

or $\frac{3}{2}$ as much as for same beam of uniform cross-section.

Constant Breadth.—For constant breadth b_1 we have (page 504) $d^2 = \frac{x}{l}d_1^2$. Hence

$$\frac{d^2y}{dx^2} = -\frac{12Pl\sqrt{l}}{Eb_1d_1^3\sqrt{x}}.$$

Integrating, since $\frac{dy}{dx} = 0$ for $x = l$,

$$\frac{dy}{dx} = -\frac{24Pl\sqrt{l}x^{\frac{1}{2}}}{Eb_1d_1^3} + \frac{24Pl^2}{Eb_1d_1^3}.$$

Integrating again, since $y = 0$ for $x = l$,

$$y = -\frac{16Pl\sqrt{l}x^{\frac{3}{2}}}{Eb_1d_1^3} + \frac{24Pl^2a}{Eb_1d_1^3} - \frac{8Pl^3}{Eb_1d_1^3}.$$

For the deflection Δ at the free end $x = 0$ and

$$\Delta = -\frac{8Pl^3}{Eb_1d_1^3},$$

or twice as much as for same beam of uniform cross-section.

For similar cross-sections we have (page 504) $d^3 = \frac{x}{l}d_1^3$. Hence $bd^3 = \frac{b_1d_1^3x^{\frac{4}{3}}}{l^{\frac{4}{3}}}$ and

$$\frac{d^2y}{dx^2} = -\frac{12Pl^{\frac{4}{3}}x^{-\frac{1}{3}}}{Eb_1d_1^3}.$$

Integrating, since $\frac{dy}{dx} = 0$ for $x = l$,

$$\frac{dy}{dx} = -\frac{18Pl^{\frac{4}{3}}x^{\frac{2}{3}}}{Eb_1d_1^3} + \frac{18Pl^2}{Eb_1d_1^3}.$$

Integrating again, since $y = 0$ for $x = l$,

$$y = -\frac{54Pl^{\frac{4}{3}}x^{\frac{5}{3}}}{5Eb_1d_1^3} + \frac{18Pl^2x}{Eb_1d_1^3} - \frac{36Pl^3}{5Eb_1d_1^3}.$$

For the deflection Δ at the free end $x = 0$ and

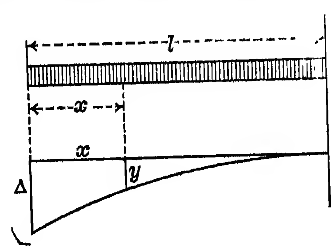
$$\Delta = -\frac{36Pl^3}{5Eb_1d_1^3},$$

or $\frac{9}{5}$ as much as for same beam of uniform cross-section.

The maximum deflections are then as $\frac{3}{2}$, 2 and $\frac{9}{5}$, or as 15, 20 and 18.

If we call the volume of the beam of constant cross-section V, then in the first case the volume is $\frac{1}{2}V$, in the second case $\frac{2}{3}V$, in the third case $\frac{3}{5}V$, or the volumes are as 15, 20 and 18. The maximum deflections, then, for a beam of uniform strength in three cases are as the volumes.

Beam Fixed Horizontally at One End and Uniformly Loaded.—(*a*) UNIFORM CROSS-SECTION.—If w is the load per unit of length, we have for any point of the neutral axis at a distance x from the free end, the bending moment



$$M = +\frac{wx^2}{2}.$$

Hence, from (VIII),

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}.$$

Integrating, since $\frac{dy}{dx} = 0$ for $x = l$,

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6}.$$

Integrating again, since $y = 0$ for $x = l$,

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8}.$$

The deflection Δ at the end is for $x = 0$

$$\Delta = -\frac{wl^4}{8EI},$$

or only $\frac{3}{8}$ as much as for an equal load at the end.

(*b*) UNIFORM STRENGTH. — We have then in general for uniform strength and rectangular cross-section

$$\frac{d^2y}{dx^2} = -\frac{6wx^2}{Ebd^3}.$$

Constant Depth.—For constant depth and uniform strength, as we have seen (page 505), $b = \frac{x^2}{l^2}b_1$ and $d = d_1$. Hence

$$\frac{d^2y}{dx^2} = -\frac{6wl^2}{Eb_1d_1^3}.$$

As we have seen (page 532), $\frac{d^2y}{dx^2} = \frac{1}{\rho}$, where ρ is the radius of curvature. Hence for constant depth and uniform strength the radius of curvature ρ is constant and *the curve of deflection is a circle.*

Integrating, since $\frac{dy}{dx} = 0$ for $x = l$,

$$\frac{dy}{dx} = -\frac{6wl^2x}{Eb_1d_1^3} + \frac{6wl^3}{Eb_1d_1^3}.$$

Integrating again, since $y = 0$ for $x = l$,

$$y = -\frac{3wl^2x^2}{Eb_1d_1^3} + \frac{6wl^3x}{Eb_1d_1^3} - \frac{3wl^4}{Eb_1d_1^3}.$$

The deflection Δ at the end is then for $x = 0$

$$\Delta = -\frac{3wl^4}{Eb_1d_1^3},$$

or twice as much as for a beam of constant cross-section.

Constant Breadth.—If the breadth is constant, we have (page 505) $d = \frac{x}{l}d_1$ and $b = b_1$; hence

$$\frac{d^2y}{dx^2} = -\frac{6wl^3}{Eb_1d_1^3x}.$$

Integrating as before,

$$\frac{dy}{dx} = \frac{12wl^3(\log l - \log x)}{Eb_1d_1^3}.$$

Integrating again

$$y = \frac{12wl^3x}{Eb_1d_1^3}\left[1 + \log\frac{l}{x}\right] - \frac{12wl^4}{Eb_1d_1^3}.$$

For $x = 0$ the deflection $\varDelta$ at the free end is

$$\varDelta = -\frac{12wl^4}{Eb_1d_1^3},$$

or eight times as much as for a beam of constant cross-section.

For similar cross-sections we have (page 505) $d^3 = \frac{x^2}{l^2}d_1^3$, $b = b_1\sqrt[3]{\frac{x^2}{l^2}}$. Hence

$$\frac{d^2y}{dx^2} = -\frac{6wl^{\frac{8}{3}}x^{-\frac{2}{3}}}{Eb_1d_1^3}.$$

Integrating, since $\frac{dy}{dx} = 0$ for $x = l$,

$$\frac{dy}{dx} = -\frac{18wl^{\frac{8}{3}}x^{\frac{1}{3}}}{Eb_1d_1^3} + \frac{18wl^3}{Eb_1d_1^3}.$$

Integrating again, since $y = 0$ for $x = l$,

$$y = -\frac{27wl^{\frac{8}{3}}x^{\frac{4}{3}}}{2Eb_1d_1^3} + \frac{18wl^3x}{Eb_1d_1^3} - \frac{9wl^4}{2Eb_1d_1^3}.$$

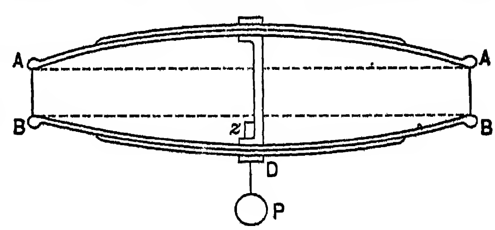
For $x = 0$ the deflection $\varDelta$ at the free end is

$$\varDelta = -\frac{9wl^4}{2Eb_1a_1^3},$$

or three times as much as for beam of constant cross-section.

Application to Metal Springs.—The most common examples of bodies of uniform strength are metal springs, such as dynamometer-springs and wagon-springs.

In the figure we have a spring dynamometer made of two parabolic springs AA and BB, united at their ends A by the links AB. Such a dynamometer measures the force P applied at D by the deflection indicated by the pointer at Z, which is equal to the sum of the deflection of the two springs.



Since the springs are of uniform breadth, and the depth varies as the ordinates to a parabola, they are (page 504) beams of uniform strength, fixed at the end with a load $\frac{P}{2}$ at the free end.

For such beams we have found (page 535) the deflection

$$\frac{8Pl^3}{Eb_1d_1^3}.$$

In the present case, then, if we take the length $l = AA$, we have to insert $\frac{P}{2}$ for P and $\frac{l}{2}$ for l, and we have for the deflection of each spring under a load P

$$\frac{Pl^3}{2Eb_1d_1^3}.$$

The total deflection, then, measured by the pointer at Z is twice this, or

$$\varDelta = \frac{Pl^3}{Eb_1d_1^3},$$

where b_1 and d_1 are the breadth and depth at the centre C and D. Hence

$$P = \frac{b_1d_1^3E}{l^3}\varDelta,$$

where $\varDelta$ is measured by the pointer at Z.

Example.—Let a steel-spring dynamometer such as described have a length $l = 3$ ft., a breadth $b_1 =$ 2 inches and depth $d_1 = 1$ inch.

For steel the limit of elasticity S_e is about 40000 pounds per square inch (page 476).

We have then, from equation (II), page 497, for the crippling load

$$M = \frac{S_eI}{v}.$$

In the present case $M = \frac{Pl}{4}$. Hence the crippling load is

$$P = \frac{4S_eI}{vl} = \frac{4 \times 40000 \times 2 \times 2}{36 \times 12} = 1480 \text{ pounds.}$$

The capacity of the instrument is about 1400 pounds, and *it should not be used to measure greater forces.* If we take $E = 30\,000\,000$ (page 478), we have for load 1400 pounds the deflection

$$\varDelta = \frac{Pl^3}{b_1d_1^3E} = \frac{1400 \times 36 \times 36 \times 36}{2 \times 30\,000\,000} = 1.08 \text{ inches.}$$

The graduation should not extend, then, over 1 inch.

Finally, by direct experiment, suppose we find that with a load of 1000 pounds the deflection $\varDelta$ observed is 0.8 inch. Then the coefficient

$$\frac{b_1d_1^3E}{l^3} = \frac{P}{\varDelta} = \frac{1000}{0.8} = 1250.$$

Hence for this instrument we have the equation

$$P = 1250\varDelta,$$

where $\varDelta$ is the reading of the pointer.

If instead of parabolic springs we have springs of uniform strength, of constant depth, and therefore triangular shape on top (page 504), we have found for such beams (page 534) the deflection

$$\frac{6Pl^3}{Eb_1d_1^3}.$$

In the present case, then, putting $\frac{P}{2}$ for P and $\frac{l}{2}$ for l, we have for the deflection of each spring

$$\frac{6Pl^3}{16Eb_1d_1^3}.$$

The total deflection measured by the pointer would be, then,

$$\varDelta = \frac{3Pl^3}{4Eb_1d_1^3}$$

and we have

$$P = \frac{4Eb_1d_1^3}{3l^3}\varDelta,$$

or one third greater than in the preceding case.

Wagon-springs are usually formed of a number of springs laid one over another.

If we have thus a compound spring composed of n springs of *constant rectangular cross-section*, laid one upon another, we have when the breadth, depth and length for each spring is b, d and l, and the load at the end of the entire spring is P, from page 534, the deflection

$$\varDelta = \frac{4Pl^3}{nEbd^3}.$$

For the crippling load we have (page 499)

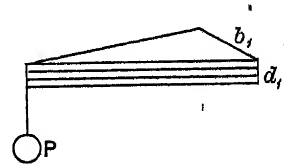
$$\frac{Pl}{n} = \frac{S_eI}{v}, \qquad \text{or} \qquad P = \frac{nS_ebd^2}{6l}.$$

Hence, also,

$$\varDelta = \frac{2S_el^2}{3Ed}, \qquad \text{or} \qquad \frac{\varDelta}{l} = \frac{2S_el}{3Ed}.$$

The ratio $\frac{\varDelta}{l}$ measures the flexibility of the spring.

If the spring is composed of n springs of uniform strength and constant depth, and therefore of triangular shape on top (page 504), as shown in the figure, the deflection, as given page 534, is



$$\varDelta = \frac{6Pl^3}{nEb_1d_1^3},$$

while the crippling load P is unchanged and given by

$$P = \frac{nS_eb_1d_1^2}{6l}.$$

Hence

$$\varDelta = \frac{S_el^2}{Ed_1}, \qquad \text{or} \qquad \frac{\varDelta}{l} = \frac{S_el}{Ed_1},$$

or the flexibility for the same strength is $\frac{3}{2}$ as much as for uniform cross-section. Hence beams of uniform strength are preferable for springs, since the object is to get with maximum strength maximum flexibility.

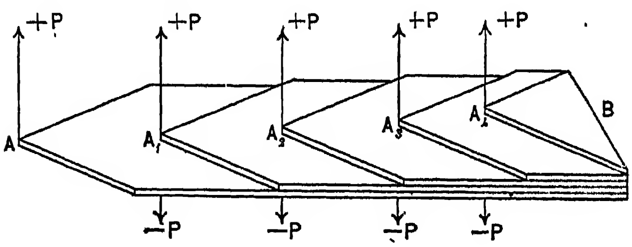
If the unit stress in the outer fibre is S_f, we have in this case for the total end cross-section

$$b_1d_1 = \frac{6Pl}{S_f d_1}.$$

Hence the amount of material is

$$b_1d_1 \times \frac{l}{2} = \frac{3Pl^2}{S_f d_1}.$$

A common form of spring consists, as shown in the figure of n flat superposed bars, each of uniform depth d and breadth b and different lengths, *triangular at the ends*, so placed that the point of each triangle just reaches the base line of the one below, the last bar, A_4B, being triangular only.



The portion AA_1 is a beam of uniform strength (page 503) which bends in a circle (page 534). If its length is $\frac{l}{n}$, the radius of curvature is (page 534)

$$\rho = \frac{nEbd^3}{12lP}.$$

Every other triangular portion, A_1A_2, A_2A_3, bends in a circle with the same radius of curvature and is of uniform strength, since AA_1 in bending exerts a pressure $+P$ at A_1, A_1A_2 the same pressure at A_2, etc.

As to the rectangular portions, we have from A_1 to B a constant moment $M = \frac{Pl}{n}$ due to the couple $+P, -P$. Hence $\frac{M}{I}$ is constant for each rectangular portion, which is therefore also of uniform strength and bends in a circle of the same radius of curvature as the triangular portions. The deflection $\varDelta$ of the entire spring is then

$$\varDelta = \frac{l^2}{2\rho} = \frac{6Pl^3}{nEb_1d_1^3}.$$

The crippling load is, as before,

$$P = \frac{nS_e bd^2}{6l}.$$

Hence

$$\varDelta = \frac{S_e l^2}{Ed}, \quad \text{or} \quad \frac{\varDelta}{l} = \frac{S_e l}{Ed}.$$

The flexibility is then just the same as in the preceding case of a number of superposed bars of the same length and uniform depth, triangular on top.

The end cross-section of each bar in the present case is

$$bd = \frac{6Pl}{nS_f d}.$$

The amount of material in the first bar is then

$$\frac{6Pl}{nS_f d}\left[\frac{l}{2n} + l - \frac{l}{n}\right];$$

for the next bar,

$$\frac{6Pl}{nS_f d}\left[\frac{l}{2n} + l - \frac{2l}{n}\right];$$

for the next,

$$\frac{6Pl}{nS_f d}\left[\frac{l}{2n} + l - \frac{3l}{n}\right];$$

and so on up to the last, which is

$$\frac{6Pl}{nS_f d}\left[\frac{l}{2n} + l - \frac{nl}{n}\right].$$

We have then the total amount of material,

$$\frac{6Pl}{nS_f d}\left[\frac{nl}{2n} + nl - \frac{l}{n}(1 + 2 + \ldots n)\right].$$

But $1 + 2 + \ldots n = \frac{n(n+1)}{2}$. Hence the total amount of material is

$$\frac{6Pl}{nS_f d}\left[\frac{l}{2} + nl - \frac{l(n+1)}{2}\right] = \frac{3Pl^2}{S_f d}.$$

The amount of material is then just the same as in the preceding case of a number of superposed bars of the same length and uniform depth, triangular on top. There is, then, no gain either in material or flexibility.

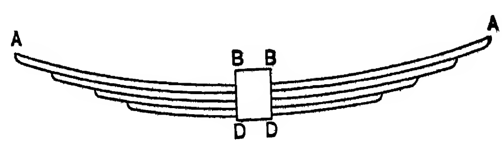
It is not necessary to make the ends of the bars triangular. We can have any other form, provided the radius of curvature is constant.

Thus in the general expression

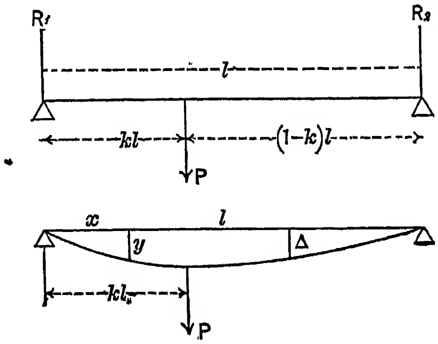
$$\frac{EI}{\rho} = M, \quad \text{or} \quad \rho = \frac{EI}{M} = \frac{Ebd^3}{12Px},$$

if we make the breadth constant and equal to b_1, and the depth variable, so that $d = d_1\sqrt[3]{\frac{nx}{l}}$ up to the next bar and then uniform, ρ will be constant still.

Such a spring is shown in the figure. The crippling load and flexibility are as before, the length l being measured from the ends BD, BD, and not from the centre.



Beam Supported at Both Ends—Uniform Cross-section—Concentrated Load.—Let the load P be distant from the left end a distance kl, where k is any given fraction. Then the distance from the right end is $(1 - k)l$, and the reaction at the left end is $P(1 - k)$.



Take the origin at the left end. Then for any value of x less than kl we have

$$\text{when } x < kl \qquad M = -P(1 - k)x;$$

and for x greater than kl we have

$$\text{when } x > kl \qquad M = -P(1 - k)x + P(x - kl).$$

Hence from (VIII) we have

$$\text{when } x < kl \qquad EI\frac{d^2y}{dx^2} = P(1 - k)x;$$

$$\text{when } x > kl \qquad EI\frac{d^2y}{dx^2} = P(1 - k)x - P(x - kl).$$

Integrating, we have

$$EI\frac{dy}{dx} = \frac{P(1 - k)x^2}{2} + C_1 \quad \text{and} \quad EI\frac{dy}{dx} = \frac{P(1 - k)x^2}{2} - \frac{Px^2}{2} + Pklx + C_2.$$

For $x = kl$ these two values of $\frac{dy}{dx}$ are equal, and hence

$$C_2 = C_1 - \frac{Pk^2l^2}{2}.$$

Inserting this value of C_2 and integrating again, we have

$$EIy = \frac{P(1-k)x^3}{6} + C_1x + C_3 \quad \text{and} \quad EIy = \frac{P(1 - k)x^3}{6} - \frac{Px^3}{6} + \frac{Pklx^2}{2} - \frac{Pk^2l^2x}{2} + C_1x + C_4.$$

In the first of these equations, when $x = 0$, $y = 0$, and hence $C_3 = 0$. For $x = kl$ these two equations are equal, hence $C_4 = \frac{Pk^3l^3}{6}$. For $x = 0$, y in the second equation is zero, hence

$$C_1 = -\frac{Pl^2k(1 - k)(2 - k)}{6},$$

and therefore

$$C_2 = -\frac{Pl^2k(2 + k^2)}{6}.$$

Substituting these constants, we have

$$\text{when } x < kl \qquad y = -\frac{P(1 - k)x}{6EI}(2kl^2 - k^2l^2 - x^2);$$

$$\text{when } x > kl \qquad y = -\frac{Pk(l - x)}{6EI}(2lx - k^2l^2 - x^2).$$

If we make $x = kl$, we have the deflection at the load

$$y = -\frac{Pl^3k^2(1-k)^2}{3EI}.$$

If we insert the values of C_1 and C_2 in the equations for $\frac{dy}{dx}$ and put $\frac{dy}{dx} = 0$, we have for the value of x which makes the deflection a maximum

$$\text{when } x<kl \qquad x = l\sqrt{\frac{k(2-k)}{3}};$$

$$\text{when } x>kl \qquad x = l - l\sqrt{\frac{1-k^2}{3}}.$$

When $x = kl$ in these equations the maximum deflection will be at the load and will be the greatest possible. Placing, therefore, $x = kl$, we obtain from both these equations the condition

$$k = \frac{1}{2}.$$

That is, the greatest maximum deflection is at the load when the load is at the centre. For any other position of the load the maximum deflection is on the right of the load when $k<\frac{1}{2}$, and on the left of the load when $k>\frac{1}{2}$. That is, *the maximum deflection is always between the load P and the farthest end.*

Inserting, then, these values of x in the values for y when $x>kl$, or when $x<kl$, we have the maximum deflection in either case,

$$\varDelta = -\frac{Pl^3k(1-k)(2-k)}{27EI}\sqrt{3k(2-k)}.$$

If the load is at the centre, we have $k = \frac{1}{2}$, and the equation of the curve of deflection is

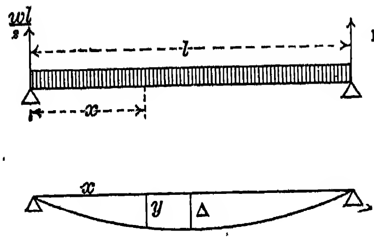
$$y = -\frac{Px}{48EI}(3l^2 - 4x^2),$$

and the maximum deflection in this case is

$$\varDelta = -\frac{Pl^3}{48EI},$$

or only $\frac{1}{16}$ as much as for a beam of same length fixed at one end and loaded at the other.

Beam Supported at Both Ends—Uniform Cross-section—Uniform Load.—For a load w per unit of length the bending moment at any point of the neutral axis distant x from the left end is



$$M = -\frac{wl}{2}x + \frac{wx^2}{2}.$$

Hence, from (VIII),

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}.$$

Integrating, since $\frac{dy}{dx} = 0$ for $x = \frac{l}{2}$, we have

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}.$$

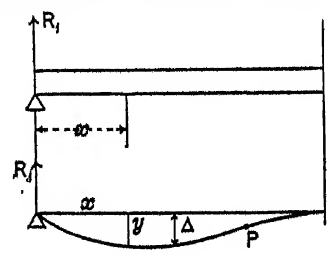
Integrating again, since for $x = 0$, $y = 0$, we have

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}.$$

The maximum deflection Δ is at the centre, and hence

$$\Delta = -\frac{5wl^4}{384EI}.$$

Beam Supported at One End and Fixed Horizontally at the Other—Uniform Cross-section—Uniform Load.—In this case we have for the moment at any point of the neutral axis distant x from the supported end



$$M = -R_1x + \frac{wx^2}{2}.$$

Hence, from (VIII),

$$EI\frac{d^2y}{dx^2} = R_1x - \frac{wx^2}{2}.$$

Integrating, since for $x = l$, $\frac{dy}{dx} = 0$, we have

$$EI\frac{dy}{dx} = \frac{R_1x^2}{2} - \frac{wx^3}{6} - \frac{R_1l^2}{2} + \frac{wl^3}{6}.$$

Integrating again, since for $x = 0$, $y = 0$,

$$EIy = \frac{R_1x^3}{6} - \frac{wx^4}{24} - \frac{R_1l^2x}{2} + \frac{wl^3x}{6}.$$

Since for $x = l$, $y = 0$, we have

$$\frac{R_1l^3}{6} - \frac{wl^4}{24} - \frac{R_1l^3}{2} + \frac{wl^4}{6} = 0, \quad \text{or} \quad R_1 = \frac{3}{8}wl.$$

This is the same value we have already found (page 524) by the principle of least work. Inserting this value of R_1, we have

$$EI\frac{dy}{dx} = \frac{3wlx^2}{16} - \frac{wx^3}{6} - \frac{wl^3}{48}.$$

$$EIy = \frac{3wlx^3}{48} - \frac{wx^4}{24} - \frac{wl^3x}{48}.$$

If we put the first of these equations equal to zero, we find for the point at which the deflection is a maximum

$$x = \frac{1+\sqrt{33}}{16}l, \quad \text{or} \quad x = 0.4215l.$$

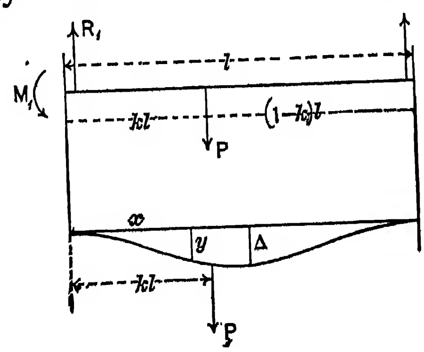
Inserting this value of x in the second equation, we have for the maximum deflection

$$\varDelta = -\frac{39+55\sqrt{33}}{16.}\cdot\frac{wl^4}{EI}.$$

If we put $\frac{d^2y}{dx^2} = 0$, we have for the point at which the moment is zero, or $\frac{d^2y}{dx^2}$ changes sign, that is for the point P of inflection,

$$-\frac{3wlx}{8} + \frac{wx^2}{2} = 0, \quad \text{or} \quad x = \frac{3}{4}l.$$

Beam Supported at One End and Fixed Horizontally at the Other—Uniform Cross-section—Concentrated Load.—Let the load P be distant from the supported end a distance kl, where k is any given fraction. Then the distance from the fixed end is $(1-k)l$. Take the origin at the supported end. Then for any value of x less than kl we have



$$\text{for } x<kl \qquad M = -R_1x,$$

and for x greater than kl we have

$$\text{for } x>kl \qquad M = -R_1x + P(x-kl).$$

Hence, from (VIII),

$$\text{for } x<kl \qquad EI\frac{d^2y}{dx^2} = R_1x, \qquad\qquad \text{for } x>kl \qquad EI\frac{d^2y}{dx^2} = R_1x - P(x-kl).$$

Integrating we have

$$EI\frac{dy}{dx} = \frac{R_1x^2}{2} + C_1 \quad \text{and} \quad EI\frac{dy}{dx} = \frac{R_1x^2}{2} - \frac{Px^2}{2} + Pklx + C_2.$$

For $x = kl$ these two values of $\frac{dy}{dx}$ are equal, and hence

$$C_2 = C_1 - \frac{Pk^2l^2}{2}.$$

Inserting this value of C_2 and integrating again, we have

$$EIy = \frac{R_1x^3}{6} + C_1x + C_3 \quad \text{and} \quad EIy = \frac{R_1x^3}{6} - \frac{Px^3}{6} + \frac{Pklx^2}{2} - \frac{Pk^2l^2x}{2} + C_1x + C_4.$$

In the first of these equations, when $x = 0$, $y = 0$ and hence $C_3 = 0$.

For $x = kl$ these two equations are equal, hence $C_4 = \frac{Pk^3l^3}{6}$.

For $x = l$, y in the second equation is zero, hence

$$C_1 = -\frac{R_1l^2}{6} + \frac{Pl^3}{6}(1 - k)^3.$$

Also for $x = l$, $\frac{dy}{dx} = 0$, and hence

$$2C_1 = -R_1l^2 + Pl^2(1 - k)^2.$$

Hence

$$C_1 = -\frac{Pl^2k(1 - k)^2}{4}, \quad \text{and} \quad R_1 = \frac{P}{2}(1 - k)^2(2 + k).$$

This is the same value for R_1 that we have already found (page 526) by the principle of least work.

We have then $C_2 = -\frac{Pl^2k(1 + k^2)}{4}$, and all the constants are determined.

If we insert the values of C_1 and C_2 in the equations for $\frac{dy}{dx}$ and put $\frac{dy}{dx} = 0$, we have for the value of x which makes the deflection a maximum

when $x < kl$ $\qquad x = l\sqrt{\frac{k}{2 + k}}$;

when $x > kl$ $\qquad x = \frac{l(1 + k^2)}{3 - k^2}$.

When $x = kl$ in these equations the maximum deflection will be at the load and will be the greatest possible. Putting, therefore, $x = kl$, we obtain from both these equations the condition

$$k = \sqrt{2} - 1 = 0.414213.$$

That is, the greatest maximum deflection is at the load when the load is at a distance of $\sqrt{2} - 1 = 0.414213$ of the span from the supported end. For any other position of the load the maximum deflection is between the load and the supported end when $k > \sqrt{2} - 1$, and between the load and the fixed end when $k < \sqrt{2} - 1$.

Inserting, then, these values of x in the values of y, we have for the maximum deflection in general

when $k > \sqrt{2} - 1$ $\qquad \Delta = -\frac{P(1 - k)^2kl^3}{6EI}\sqrt{\frac{k}{2 + k}}$;

when $k < \sqrt{2} - 1$ $\qquad \Delta = -\frac{P(1 - k)^3kl^3(1 + k)^3}{3EI(3 - k^2)^2}$.

Both of these are equal and have their greatest value when $k = \sqrt{2} - 1$. Inserting this value of k, we have for the greatest maximum deflection at the load

$$\text{when } k = \sqrt{2} - 1 \quad \Delta = -\frac{Pl^3(17 - 12\sqrt{2})}{3EI} = -\frac{5888}{600000}\frac{Pl^3}{EI},$$

or only about $\frac{47}{100}$ as much as for beam supported at the ends.

If the load is at the centre of the span, $k = \frac{1}{2}$ and

$$R_1 = \frac{5}{16}P;$$

and since $k > \sqrt{2} - 1$, we have the maximum deflection in this case

$$\text{when } k = \frac{1}{2} \qquad \Delta = -\frac{Pl^3}{48EI} \times \frac{1}{\sqrt{5}},$$

and this maximum deflection is at a point between the load and the supported end given by

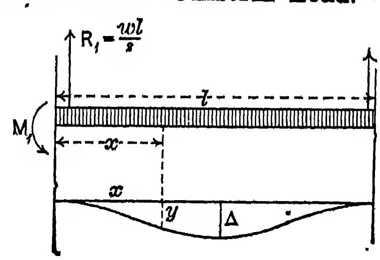
$$x = \frac{1}{\sqrt{5}}l = 0.66l.$$

For the point of inflection, if we put $\frac{d^2y}{dx^2} = 0$, we have the distance of the point of inflection from the supported end

$$x = \frac{2l}{3 - k^2}.$$

If the load is at the centre of the span this becomes $\frac{8}{11}l$.

Beam Fixed Horizontally at Both Ends—Uniform Cross-section—Uniform Load.—For a load w per unit of length the reaction is $\frac{wl}{2}$ at each end, and the bending moment at any point of the neutral axis distant x from the left end is



$$M = M_1 - \frac{wlx}{2} + \frac{wx^2}{2}.$$

Hence, from (VIII),

$$EI\frac{d^2y}{dx^2} = -M_1 + \frac{wlx}{2} - \frac{wx^2}{2}.$$

Integrating, since $\frac{dy}{dx} = 0$ for $x = 0$, we have

$$EI\frac{dy}{dx} = -M_1x + \frac{wlx^2}{4} - \frac{wx^3}{6}.$$

Integrating again, since for $x = 0$, $y = 0$, we have

$$EIy = -\frac{M_1x^2}{2} + \frac{wlx^3}{12} + \frac{wx^4}{24}.$$

Since for $x = \frac{l}{2}$ we have also $\frac{dy}{dx} = 0$, we have

$$-\frac{M_1 l}{2} + \frac{wl^3}{16} - \frac{wl^3}{48} = 0, \quad \text{or} \quad M_1 = \frac{wl^2}{12}.$$

Inserting this value of M_1, we have

$$EI\frac{dy}{dx} = -\frac{wl^2x}{12} + \frac{wlx^2}{4} - \frac{wx^3}{6},$$

$$EIy = -\frac{wl^2x^2}{24} + \frac{wlx^3}{12} - \frac{wx^4}{24}.$$

If we put the first of these equations equal to zero, we find for the point at which the deflection is a maximum $x = \frac{l}{2}$. The maximum deflection is then at the centre and given by

$$\Delta = -\frac{wl^4}{384EI},$$

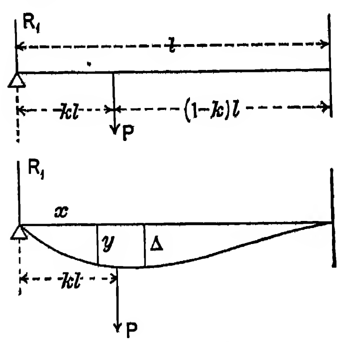
or only one fifth as much as for beam supported at the ends.

If we put $\frac{d^2y}{dx^2} = 0$, we have for the points of inflection

$$x = \frac{l}{2} - \frac{l}{2\sqrt{3}} \quad \text{and} \quad x = \frac{l}{2} + \frac{l}{2\sqrt{3}},$$

or $x = 0.2113l$ and $x = 0.7887l$.

Beam Fixed Horizontally at Both Ends—Uniform Cross-section—Concentrated Load.—Let the load P be distant from the left end a distance kl, where k is any given fraction. Then the distance from the right end is $(1 - k)l$.



Take the origin at the left end. Then for any value of x less than kl we have

$$\text{for } x < kl \qquad M = M_1 - R_1x,$$

and for x greater than kl we have

$$\text{for } x > kl \qquad M = M_1 - R_1x + P(x - kl).$$

Hence, from (VIII),

$$\text{for } x < kl \quad EI\frac{d^2y}{dx^2} = -M_1 + R_1x; \qquad \text{for } x > kl \quad EI\frac{d^2y}{dx^2} = -M_1 + R_1x - P(x - kl).$$

Integrating, we have, since $\frac{dy}{dx} = 0$ for $x = 0$,

$$EI\frac{dy}{dx} = -M_1x + \frac{R_1x^2}{2} \quad \text{and} \quad EI\frac{dy}{dx} = -M_1x + \frac{R_1x^2}{2} - \frac{Px^2}{2} + Pklx + C_2.$$

For $x = kl$ these two values of $\frac{dy}{dx}$ are equal, and hence

$$C_2 = -\frac{Pk^2l^2}{2}.$$

Inserting this value of C_2 and integrating again, we have, since $y = 0$ for $x = 0$,

$$EIy = -\frac{M_1x^2}{2} + \frac{R_1x^3}{6} \quad \text{and} \quad EIy = -\frac{M_1x^2}{2} + \frac{R_1x^3}{6} - \frac{Px^3}{6} + \frac{Pklx^2}{2} - \frac{Pk^2l^2x}{2} + C_4.$$

For $x = kl$ these two equations are equal, hence

$$C_4 = +\frac{Pk^3l^3}{6}.$$

For $x = l$, y in the second equation is zero, hence

$$-3M_1 + R_1l - Pl(1 - k)^3 = 0.$$

Also for $x = l$, $\frac{dy}{dx} = 0$. Hence

$$-2M_1 + R_1l - Pl(1 - k)^2 = 0.$$

From these two equations we obtain

$$M_1 = Plk(1 - k)^2, \quad \text{and} \quad R_1 = P(1 - k)^2(1 + 2k).$$

This is the same value for R_1 that we have already found (page 526) by the principle of least work.

If we put the values of $\frac{dy}{dx} = 0$, we have for the value of x which makes the deflection a maximum

when $x < kl$ $\qquad x = \frac{2lk}{1 + 2k}$;

when $x > kl$ $\qquad x = \frac{l}{3 - 2k}$,

When $x = kl$ in these equations the maximum deflection will be at the load and will be tne greatest possible. Putting, therefore, $x = kl$, we obtain from both these equations the condition

$$k = \frac{1}{2}.$$

That is, the greatest maximum deflection is at the load when the load is at the centre of the span. For any other position of the load the maximum deflection is between the load and the farthest end from the load.

Inserting, then, these values of x in the values of y, we have for the maximum deflection in general

when $k > \frac{1}{2}$ $\qquad \Delta = -\frac{2Pl^3k^3(1-k)^2}{3EI(1+2k)^2};$

when $k < \frac{1}{2}$ $\qquad \Delta = -\frac{2Pl^3k^2(1-k)^3}{3EI(3-2k)^2}.$

Both of these are equal and have their greatest value when $k = \frac{1}{2}$. Inserting this value of k, we have for the greatest maximum deflection at the load

when $k = \frac{1}{2}$ $\qquad \Delta = -\frac{Pl^3}{192EI},$

or only one fourth as much as for beam supported at the ends.

For the load at the centre

$$M_1 = +\frac{Pl}{8}, \quad \text{and} \quad R_1 = \frac{P}{2}.$$

For the points of inflection, if we put $\frac{d^2y}{dx^2} = 0$, we have for the distance of the points of inflection from the left end

$$x = \frac{kl}{1+2k} \quad \text{and} \quad x = \frac{(2-k)l}{3-2k}.$$

If the load is at the centre of the span, $k = \frac{1}{2}$ and these values of x become $\frac{1}{4}l$ and $\frac{3}{4}l$.

Examples.—(1) *A rectangular beam of wrought iron 5 feet long, 3 inches wide and 3 inches deep is deflected $\frac{1}{10}$ of an inch by a load of 3000 pounds applied at the centre. Find E.*

ANS. We have (page 543)

$$\Delta = \frac{Pl^3}{48EI}, \quad \text{hence} \quad E = \frac{Pl^3}{48I\Delta} = \frac{Pl^3}{4bd^3\Delta}.$$

Inserting numerical values

$$E = \frac{3000 \times 60 \times 60 \times 60 \times 10}{4 \times 3 \times 27} = 20\,000\,000 \text{ pounds per square inch,}$$

provided the elastic limit has not been exceeded.

In order to find whether this is the case we have (page 497) for the unit stress in the outer fibre

$$S_f = \frac{Mv}{I} = \frac{Plv}{4I} = \frac{3000 \times 60 \times 3 \times 12}{4 \times 2 \times 3 \times 27} = 10000 \text{ pounds per square inch.}$$

Since the elastic limit (page 476) is 25000, the elastic limit is not exceeded.

(2) *An iron rectangular beam whose length is 12 ft., breadth $1\frac{1}{2}$ inches, coefficient of elasticity 24 000 000, has a load of 10000 pounds at the centre. Find the depth in order that the deflection may be $\frac{1}{480}$ of the length.*

ANS. We have

$$d^3 = \frac{Pl^3}{4bE\Delta} = \frac{10000 \times 144 \times 144 \times 144 \times 2 \times 480}{4 \times 3 \times 24\,000\,000 \times 144} = 691.2 \text{ inches, hence } d = 8.8 \text{ inches,}$$

provided the limit of elasticity is not exceeded.

In order to find whether this is the case, we have as before

$$S_f = \frac{Plv}{4I} = \frac{10000 \times 144 \times 4.4 \times 12 \times 2}{4 \times 3 \times 8.8 \times 8.8 \times 8.8} = 18595 \text{ pounds per square inch.}$$

Since the elastic limit is 25000 pounds per square inch, the elastic limit is not exceeded.

(3) *Find the depth of a rectangular beam so loaded at the centre that the elongation of the lowest fibre shall equal $\frac{1}{1400}$ of its original length.*

ANS. We have $\lambda = \frac{l}{1400}$, and (page 477) $\lambda = \frac{S_f l}{E}$. We also have (page 497) $S_f = \frac{Mv}{I}$. Hence

$$I = \frac{1400 Mv}{E}.$$

Since $v = \frac{d}{2}$, $I = \frac{1}{12}bd^3$ and $M = \frac{Pl}{4}$, we have

$$d = \sqrt{\frac{2100 Pl}{Eb}},$$

provided that S_f is less than the elastic limit.

(4) *Find the radius of curvature at the middle point of a wooden beam when the load at middle is 3000 pounds, the length 10 feet, breadth 4 inches, depth 8 inches and E is 1 000 000 pounds per square inch.*

ANS. We have (page 532)

$$\rho = \frac{EI}{M} = \frac{4EI}{Pl} = \frac{4 \times 1\,000\,000 \times 4 \times 8 \times 8 \times 8}{12 \times 3000 \times 120} = 1896 \text{ inches,}$$

provided the elastic limit is not exceeded. To find whether this is the case we have

$$S_f = \frac{Plv}{4I} = \frac{3000 \times 120 \times 4 \times 12}{4 \times 4 \times 8 \times 8 \times 8} = 2109 \text{ pounds per square inch.}$$

Since the elastic limit is 3000 pounds per square inch (page 476), the elastic limit is not exceeded.

(5) *A wrought-iron 15 inch I beam, whose moment of inertia is 691 in inches, has a length of 30 feet and E = 24 000 000 pounds per square inch.*

If supported at the ends with a uniform load of 75 pounds per inch of length over the first 10 feet, find the deflection at the end of the load.

ANS. Deflection = 0.23444 inch.

Find the deflection at the centre.

ANS. Deflection = 0.24421 inch.

Find the deflection 10 feet from the unloaded end.

ANS. Deflection = 0.19537 inch.

Where is the point of greatest deflection and what is the greatest deflection?

ANS. At 13.1676 feet. Greatest deflection = 0.24847 inch.

If the weight of the beam itself is 5.573 pounds per inch of length, find the deflection at the centre.

ANS. Deflection = 0.07349 inch.

If the same 10-foot load is moved to the centre, find the deflection at the centre.

ANS. Deflection = 0.50063 inch.

If the uniform load of 75 pounds per inch covers the whole span, find the deflection at the centre.

ANS. Deflection = 0.98905 inch.

If the same beam is half loaded with 75 pounds per inch, find the deflection at the centre, the maximum deflection and the point at which the deflection is a maximum.

ANS. Deflection = 0.494525 inch. Max. deflection = 0.49855 inch within the loaded portion at 14.48 inches from centre.

If the same beam has three weights of 4500 pounds each placed at intervals of 60 inches beginning at one end, find the deflection at centre.

ANS. Deflection = 0.6154 inch.

If the beam is fixed horizontally at both ends and loaded uniformly with 75 pounds per inch, find the deflection at 10 feet from either end and at the centre.

ANS. Deflection = 0.1563 inch; at centre = 0.19781 inch.

If only one end is fixed, the other supported, find the deflection at 10 feet; at centre; at 20 feet. Find the maximum deflection. Where is it?

ANS. At 10 feet = 0.39074 inch; at centre = 0.39563 inch; at 20 feet = 0.27352 inch. Maximum deflection = 0.41018 inch at 151.7524 inches from supported end.

If the beam is fixed horizontally at both ends, with a load of 27000 pounds at the centre, find the deflection at the quarter points and at the centre. Where are the points of inflection?

ANS. At quarter points = 0.19781 inch; at centre = 0.39562 inch. Point of inflection at 90 inches from each end.

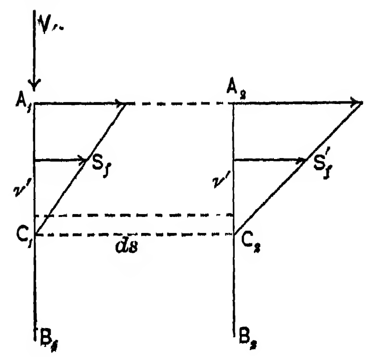
If only the right end is fixed and the other supported, and the load of 27000 pounds is at the centre, find the deflections d at the quarter points and at the centre, and the maximum deflection.

ANS. At the quarter points 0.5316 and 0.3091 inch; at centre 0.69234 inch; maximum deflection = 0.70732 inch at $l\sqrt{\frac{1}{5}}$ from supported end.

CHAPTER VI.

SHEARING STRESS.

Shearing Stress in Beams.—Let A_1B_1 and A_2B_2 be two consecutive cross-sections, so that the distance C_1C_2, between them is ds, and let M_1 be the bending moment at C_1 and M_2 the bending moment at C_2.



Then by equation (II), page 497, the unit stress S_f' in any fibre at a distance v' is

$$S_f' = \frac{M_2 v'}{I},$$

and the unit stress S_f in the same fibre at the other cross-section is

$$S_f = \frac{M_1 v'}{I}.$$

The difference

$$S_f' - S_f = \frac{(M_2 - M_1)v'}{I}$$

is the horizontal unit shear at this fibre. But $M_2 - M_1$ is dM, and if V is the shear, $dM = Vds$. Hence the horizontal unit shear at this fibre is

$$\frac{Vv'ds}{I},$$

and if the area of cross-section of the fibre is a, then the horizontal shear for this fibre is

$$\frac{Vds \,.\, av'}{I}.$$

For the total horizontal shear at the neutral axis we have then

$$H = \frac{Vds \,.\, \Sigma av'}{I},$$

the summation extending to all the fibres above C.

Let b_0 be the breadth of cross-section at the neutral axis. Then $b_0 ds$ is the area sheared, and the *horizontal unit shear at the neutral axis* is

$$S_h = \frac{V\Sigma av'}{b_0 I}.$$

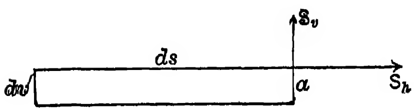
Thus for a rectangular beam of breadth b and depth d we have $I = \frac{bd^3}{12}$, $\Sigma av' = \frac{bd}{2} \times \frac{d}{4} = \frac{bd^2}{8}$ and hence

$$S_h = \frac{3V}{2bd}.$$

For a cylindrical beam of radius r, $I = \frac{\pi r^4}{4}$ and $\Sigma av = \frac{\pi r^2}{2} \times \frac{4r}{3\pi} = \frac{4r^3}{6}$, $b_0 = 2r$, and hence

$$S_h = \frac{4V}{3\pi r^2}.$$

Take the elementary fibre at the neutral axis of length ds and depth dv' and area of cross-section $a = b_0 dv'$. Let S_v be the vertical unit shear at the neutral axis. Then the vertical shearing force is $S_v a = S_v b_0 dv'$ and the horizontal shearing force is $S_h b_0 ds$. We have then for equilibrium



$$S_h b_0 ds \times dv' = S_v b_0 dv' \times ds.$$

Hence

$$S_h = S_v = \frac{V \Sigma av'}{b_0 I}.$$

That is, *the horizontal and vertical unit shear at the neutral axis are equal.*

Work of the Shearing Force in Beams.—We have from equation (I), page 477, for the strain due to shear

$$\lambda = \frac{S_v ds}{E_s} = \frac{V ds \Sigma av'}{b_0 E_s I}$$

where E_s is the coefficient of elasticity for shear.

If then V is the shear, we have the work of the shear between two consecutive sections

$$\frac{V\lambda}{2} = \frac{V^2 ds \Sigma av'}{2b_0 E_s I}.$$

The total work of shear is then

$$\text{work} = \int_0^s \frac{V^2 ds \Sigma av'}{2b_0 E_s I} \quad \ldots\ldots\ldots\ldots \quad \text{(X)}$$

For straight beams we have $s = l$, and dx in place of ds.

Influence of the Shearing Force upon Deflection.— If we divide (X) by $\frac{V}{2}$, we have for the deflection due to the shear, for straight beam,

$$\text{deflection} = \int_0^l \frac{V dx \Sigma av'}{b_0 E_s I},$$

and this deflection should be added to the deflection due to bending, as already found in the preceding pages.

Thus for a beam of constant cross-section, fixed at one end, with a load P at the free end, we have already found the deflection due to bending (page 533)

$$\Delta = \frac{Pl^3}{3EI},$$

where E is the coefficient of elasticity for tension or compression.

For the deflection due to shear we have, since $V = P$

$$\Delta = \int_0^l \frac{Pdx\Sigma av}{b_0 E_s I} = \frac{Pl\Sigma av}{b_0 E_s I},$$

where E_s is the coefficient of elasticity for shear.

The total deflection is then

$$\Delta = \frac{Pl}{I}\left[\frac{l^2}{3E} + \frac{\Sigma av}{b_0 E_s}\right]. \quad . \quad . \quad . \quad (1)$$

For rectangular cross-section $b_0 = b$, $I = \frac{1}{12}bd^3$, $\Sigma av = \frac{bd^2}{8}$, and hence

$$\Delta = \frac{4Pl^3}{Ebd^3}\left[1 + \frac{3Ed^2}{8E_s l^2}\right],$$

while for bending only we should have $\Delta = \frac{4Pl^3}{Ebd^3}$.

If we assume the ratio $\frac{E}{E_s} = 3$, we have

$$\Delta = \frac{4Pl^3}{Ebd^3}\left[1 + \frac{9d^2}{8l^2}\right]. \quad . \quad . \quad . \quad (2)$$

We have then for $\frac{l}{d} = 10$, $\Delta = 1.011 \cdot \frac{4Pl^3}{Ebd^3}$. We see at once that *for depth small compared to the length, the effect of the shear can be disregarded.* In all practical cases where l is greater than $10d$, the deflection due to bending only is then sufficiently accurate, and the results already obtained can be taken as practically correct.

Determination of Coefficient of Elasticity for Shear.—From equation (1) we have at once

$$E_s = \frac{Pl}{b_0} \cdot \frac{3E\Sigma av}{3EI\Delta - Pl^3}. \quad . \quad . \quad . \quad (3)$$

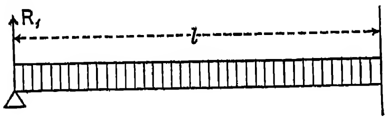
For rectangular cross-section $I = \frac{1}{12}bd^3$ and $\Sigma av = \frac{bd^2}{8}$, and (3) becomes

$$E_s = \frac{3E}{8\left[\frac{Ebd\Delta}{4Pl} - \frac{l^2}{d^2}\right]}. \quad . \quad . \quad . \quad (4)$$

From (3) or (4), then, we can determine experimentally the coefficient of elasticity for shear E_s, when the coefficient of elasticity E for tension or compression is known, by measuring the deflection Δ for a given load P at the end of a beam of known length l and known cross-section.

Influence of the Shearing Force upon Reaction.—For all cases of beams fixed at one end and free at the other, or simply supported at both ends, the reactions are entirely determined by statical conditions and are therefore independent of the shearing force. But for beams fixed at both ends or fixed at one end and supported at the other the shearing force has an influence on the reaction.

Let us take, for instance, a beam of uniform cross-section fixed horizontally at the right end, supported at the left end and uniformly loaded.



The work due to bending is from equation (IV′), page 522,

$$\int_0^l \frac{M^2 dx}{2EI},$$

and the work due to shear is, from equation (X), page 554,

$$\int_0^l \frac{V^2 dx \Sigma av'}{2b_0 E_s I}.$$

The total work is then

$$\text{work} = \int_0^l \frac{M^2 dx}{2EI} + \int_0^l \frac{V^2 dx \Sigma av'}{2b_0 E_s I}.$$

In the present case, if w is the load per unit of length, we have

$$M = -R_1 x + \frac{wx^2}{2}, \quad \text{and} \quad V = R_1 - wx,$$

and the value of R_1 must make the work a minimum. Hence

$$\text{work} = \int_0^l \left[-R_1 x + \frac{wx^2}{2}\right]^2 \frac{dx}{2EI} + \int_0^l [R_1 - wx]^2 \frac{dx \Sigma av'}{2b_0 E_s I}$$

and

$$\frac{d(\text{work})}{dR_1} = 0 = \int_0^l \left[R_1 x - \frac{wx^2}{2}\right] \frac{x dx}{EI} + \int_0^l [R_1 - wx] \frac{dx \Sigma av'}{E_s I b_0}.$$

If we omit the last term and perform the integration, we have $R_1 = \frac{3}{8}wl$, just as already found on page 524 for bending only.

If, however, we perform the integrations as given, we have

$$R_1 = \frac{3wl}{8} \cdot \frac{1 + \dfrac{4E\Sigma av}{E_s l^2 b_0}}{1 + \dfrac{3E\Sigma av}{E_s l^2 b_0}}.$$

For rectangular cross-section $b_0 = b$, $\Sigma av' = \frac{bd^2}{8}$ and

$$R_1 = \frac{3wl}{8} \cdot \frac{1 + \dfrac{Ed^2}{2E_s l^2}}{1 + \dfrac{3Ed^2}{8E_s l^2}}.$$

We see at once that for $\frac{l}{d}$ large, or length great compared to the depth, this becomes $R_1 = \frac{3wl}{8}$. Hence in all practical cases where l is large compared to d, the effect of the shear upon the reaction can be neglected.

Maximum Internal Stresses in a Beam.—From page 484 we have already proved that if s is the direct unit shear and t the direct unit tension, the maximum combined unit shear is

$$S_s = \sqrt{s^2 + \frac{t^2}{4}},$$

and the angle which it makes with t is given by

$$\tan 2\alpha = \frac{t}{2s}.$$

Also, the maximum combined unit tension is

$$S_t = \frac{t}{2} + \sqrt{s^2 + \frac{t^2}{4}},$$

and the angle which it makes with t, $\beta = 90 - \alpha$, where α is given by

$$\tan 2\alpha = -\frac{2s}{t}.$$

Now the direct unit shear in a beam is, from page 553,

$$s = \frac{V\Sigma av'}{bI},$$

and the direct unit tension is, from page 497,

$$t = \frac{Mv'}{I}.$$

We have then for a beam in general

$$S_s = \sqrt{\left(\frac{V\Sigma av'}{bI}\right)^2 + \left(\frac{Mv'}{2I}\right)^2}, \quad \text{and} \quad \tan 2\alpha = \frac{bMv'}{2V\Sigma av'};$$

$$S = \frac{Mv'}{2I} + \sqrt{\left(\frac{V\Sigma av'}{bI}\right)^2 + \left(\frac{Mv'}{2I}\right)^2}, \text{ and } \beta = 90 - \alpha,$$

where α is given by

$$\tan 2\alpha = -\frac{2V\Sigma av'}{bMv'}.$$

Let us take a beam of uniform cross-section, fixed horizontally at one end, with a load P at the free end. Then $M = Px$, $I = \frac{bd^3}{12}$, $V = -P$, and, taking $v' = y$,

$$\Sigma av' = b\left(\frac{d}{2} - y\right)\left(y + \frac{\frac{d}{2} - y}{2}\right) = \frac{b\left(\frac{d^2}{4} - y^2\right)}{2}.$$

Hence we have in this case, for the maximum unit shear at any point given by x and y, the origin being at the free end of the neutral axis,

$$S_s = \sqrt{\left[\frac{6P\left(\frac{d^2}{4} - y^2\right)}{bd^3}\right]^2 + \left[\frac{6Pxy}{bd^3}\right]^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and its angle α with the neutral axis is given by

$$\tan 2\alpha = \frac{xy}{\frac{d^2}{4} - y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the maximum unit tension at any point we have

$$S_t = \frac{6Pxy}{bd^3} + \sqrt{\left[\frac{6P\left(\frac{d^2}{4} - y^2\right)}{bd^3}\right]^2 + \left[\frac{6Pxy}{bd^3}\right]^2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and its angle β with the neutral axis is given by $\beta = 90 - \alpha$ where α is given by

$$\tan 2\alpha = -\frac{2\left(\frac{d^2}{4} - y^2\right)}{xy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

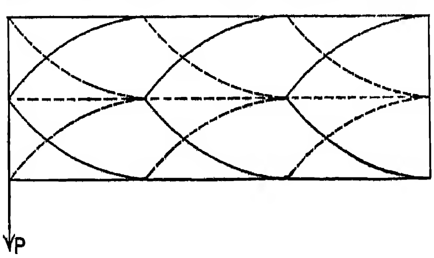
At the neutral axis $y = 0$, and, from (1), the maximum unit shear at the neutral axis is $S_s = \frac{3P}{2bd}$, and its direction, from (2), is $\alpha = 0$. At the upper surface $y = \frac{d}{2}$ and $S_s = \frac{3Px}{bd^2}$, and its direction, from (2), is $\alpha = 45°$.

The maximum unit tension at the neutral axis is, from (3), when $y = 0$, $S_t = \frac{3P}{2bd}$, and its direction, from (4), is $\beta = 45°$. At the upper surface $y = \frac{d}{2}$ and $S_t = \frac{6Px}{bd^2}$, and its direction, from (4), is $\beta = 0$.

We see, then, that equation (II), page 497,

$$S_f = \frac{Mv}{I} = \frac{6Px}{bd^2},$$

gives the maximum unit stress S_t in the outer fibre, but for other fibres this maximum unit stress S_t is greater than that given by equation (II) unless the shear is zero or is disregarded. We have just seen that the shear can be disregarded in practical cases (pages 555 and 557).



In the figure the full lines above the neutral axis represent the direction of maximum tension S_t, and those below the neutral axis, the direction of maximum compression S_c.

The dotted lines represent the direction of maximum shear S_s.

CHAPTER VII.

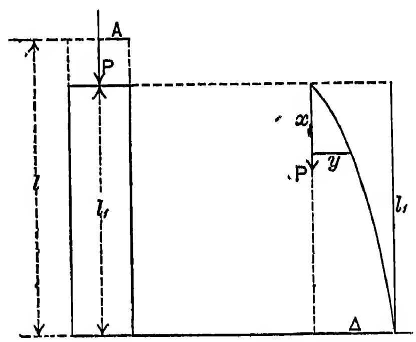
STRENGTH OF LONG COLUMNS.

The Ideal Column.—The ideal column is supposed to be perfectly homogeneous so that the coefficient of elasticity E is constant for every portion, to have a uniform cross-section A, a perfectly straight axis, and to have a load P applied exactly in that axis.

Such an ideal column, ideally loaded, has no tendency to bend in any direction. It is simply compressed by the loading.

Theory of the Ideal Column.—Suppose, then, an ideal column whose original length is l to be compressed by the load P in its axis. The new length l_1 is, from equation (I), page 477,

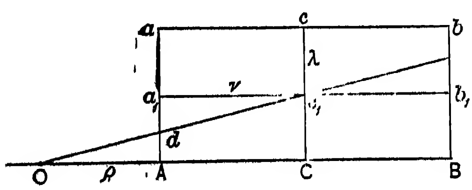
$$l - l_1 = \frac{Pl}{AE}, \quad \text{or} \quad l_1 = l\left(1 - \frac{P}{AE}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$



Now suppose this compressed column of length l_1 to be bent very slightly by a horizontal force, and suppose the column does not spring back completely when the horizontal force is removed, but takes a certain position of equilibrium as shown in the figure.

Let x and y be the co-ordinates of any point of the elastic curve of the axis, taking the free end of the axis as origin. Then the bending moment at any point is $M = Py$.

Let dx be the distance cC between two consecutive cross-sections ab and AB before the load P is applied. Then when the load P is applied the unit stress is $\frac{P}{A}$, and the shortening of the axis $\lambda = cc_1$ is, by equation (I), page 477,



$$\lambda = \frac{Pdx}{AE}.$$

If now the column deflects towards the left, the unit stress on the inner compressed fibre at a distance v from the axis is, from equation (II), page 497,

$$\frac{P}{A} + \frac{Mv}{I},$$

and hence the compression $ad = \lambda'$ of that fibre is

$$\lambda' = \frac{Pdx}{AE} + \frac{Mvdx}{IE}.$$

The distance a_1d is then

$$a_1d = \lambda' - \lambda = \frac{Mvdx}{IE}.$$

If now $\rho = OC$ is the radius of curvature of the axis, we have at once, from the figure,

$$\rho : dx - \lambda :: v : a_1d,$$

or

$$\rho : dx\left(1 - \frac{P}{AE}\right) :: v : \frac{Mvdx}{EI}.$$

But, from (1), $1 - \frac{P}{AE} = \frac{l_1}{l}$. Hence

$$\frac{EI}{\rho} = \frac{l}{l_1}M.$$

Now, from Calculus, $\frac{1}{\rho} = -\frac{d^2y}{dx^2}$. Hence

$$EI\frac{d^2y}{dx^2} = -\frac{l}{l_1}M. \qquad (2)$$

Comparing with equation (VIII), page 531, we see that when we take $l_1 = l$ we have bending only without compression.

If we multiply both sides of (2) by $2dy$ and integrate, we obtain, since $M = Py$,

$$EI\frac{dy^2}{dx^2} = -\frac{lPy^2}{l_1} + C.$$

When $y = \varDelta =$ the maximum deflection, $\frac{dy}{dx} = 0$. Hence $C = \frac{lP\varDelta^2}{l_1}$, and we have

$$dx = \sqrt{\frac{l_1EI}{lP}} \cdot \frac{dy}{\sqrt{\varDelta^2 - y^2}}.$$

Integrating again, we have

$$x = \sqrt{\frac{l_1EI}{lP}} \text{ arc sin} \frac{y}{\varDelta} + C'.$$

When $y = 0$, $x = 0$. Hence $C' = 0$, and we have

$$y = \varDelta \sin x\sqrt{\frac{lP}{l_1EI}}. \qquad (3)$$

(*a*) COLUMN FIXED AT ONE END, FREE AT THE OTHER.—For a column fixed at one end and free at the other we have from (3), when $x = l_1$, $y = \Delta$, and hence

$$l_1\sqrt{\frac{lP}{l_1EI}} = \frac{\pi}{2},$$

or, since $Ia = A\kappa^2$, where κ is the radius gyration of the cross-section,

$$\frac{P}{A} = \frac{\pi^2E\kappa^2}{4ll_1}.$$

(*b*) COLUMN WITH TWO PIN ENDS (Fig. 1).—In this case we have only to make, in (3), $y = \Delta$ when $x = \frac{l_1}{2}$. We thus obtain

$$\frac{P}{A} = \frac{\pi^2E\kappa^2}{ll_1}.$$

FIG. 1.

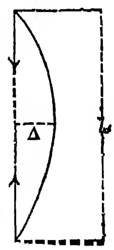


FIG. 2.

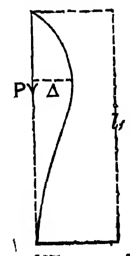
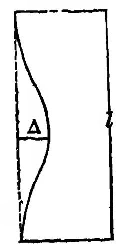


FIG. 3.



(*c*) COLUMN FIXED AT ONE END, PIN AT THE OTHER (Fig. 2).—In this case we have, in (3), $y = \Delta$ when $x = \frac{1}{3}l_1$. We thus have

$$\frac{P}{A} = \frac{9\pi^2E\kappa^2}{4ll_1}.$$

(*d*) COLUMN FIXED AT BOTH ENDS (Fig. 3).—In this case we have, in (3), $y = \Delta$ when $x = \frac{1}{4}l_1$, and hence

$$\frac{P}{A} = \frac{4\pi^2E\kappa^2}{ll_1}.$$

GENERAL EQUATION.—All these equations are of the form

$$\frac{P}{A} = \frac{n^2E\kappa^2}{ll_1}, \quad \ldots \ldots \ldots \quad (4)$$

where we have for n the values $\frac{1}{2}\pi$, $\frac{2}{2}\pi$, $\frac{3}{2}\pi$, $\frac{4}{2}\pi$ for one fixed and one free end, two pin ends, one fixed and one pin end, and two fixed ends.

EQUATION OF THE ELASTIC CURVE.—Substituting the value of $\frac{P}{A}$ from (4) in (3), we have for the equation of the elastic curve

$$y = \Delta \sin \frac{nx}{l_1}, \quad \ldots \ldots \ldots \quad (5)$$

and hence

$$\frac{dy}{dx} = \frac{n\Delta}{l_1} \cos \frac{nx}{l_1}. \quad \ldots \ldots \ldots \quad (6)$$

WORK OF P DURING BENDING.—During the direct compression of the column from the length l to l_1, the work done is $\frac{P}{2}(l - l_1)$, or, from (1), $\frac{P^2 l}{2AE}$.

If now, when the column is pushed slightly to one side, it continues to bend and assumes a deflection Δ, we have, from equation (6), for the moment M at any point of the neutral axis

$$M = Py = P\Delta \sin \frac{nx}{l_1}.$$

From equation (IV′), page 522, we have then for the work of P during this bending

$$\textit{work of } P \textit{ during bending} = \int_0^{l_1} \frac{M^2 dx}{2EI} = \int_0^{l_1} \frac{P^2\Delta^2}{2EI} \sin^2 \frac{nx}{l_1} \cdot dx = \frac{P^2\Delta^2 l_1}{4EI}.$$

WORK OF P NECESSARY TO PRODUCE THE DEFLECTION Δ.—The horizontal component of P at any point of the neutral axis is $P\frac{dy}{dx}$. Its work is $\frac{Pdy}{dx} \cdot \frac{dy}{2}$, and hence, from (6), we have

$$\textit{work of } P \textit{ necessary to produce the deflection} = \int_0^{l_1} \frac{Pdy^2}{2dx} = \int_0^{l_1} \frac{Pn^2\Delta^2}{2l_1^2} \cos^2 \frac{nx}{l_1} \cdot dx = \frac{Pn^2\Delta^2}{4l_1}.$$

Deportment of the Ideal Column.—If the work of P during bending is greater than the work of P necessary to produce the deflection Δ, that is, if

$$\frac{P^2\Delta^2 l_1}{4EI} > \frac{Pn^2\Delta^2}{4l_1},$$

the compressed vertical column when pushed slightly to one side will continue to bend until the deflection Δ is reached. The compressed vertical column is then in unstable equilibrium, and if pushed slightly to one side by a horizontal force H, will not return to its original vertical position when H is removed.

If the work of P during bending is just equal to the work of P necessary to produce the deflection Δ, that is, if

$$\frac{P^2\Delta^2 l_1}{4EI} = \frac{Pn^2\Delta^2}{4l_1},$$

the compressed vertical column when pushed slightly to one side will remain in its new position. The compressed vertical column is then in indifferent equilibrium.

If the work of P during bending is less than the work of P necessary to produce the deflection Δ, that is, if

$$\frac{P^2\Delta^2 l_1}{4EI} < \frac{Pn^2\Delta^2}{4l_1},$$

the compressed vertical column is in stable equilibrium. If pushed slightly to one side by a horizontal force H, it will return to its original position when H is removed.

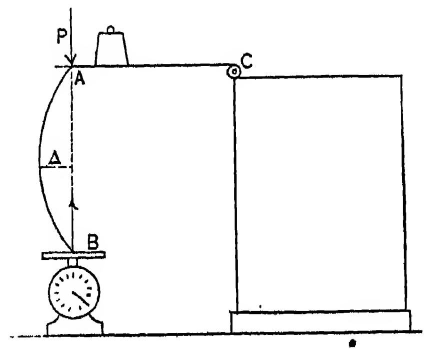
If we put for I its value $A\kappa^2$, where κ is the radius of gyration, we have in these three cases

$$\frac{l_1}{\kappa} \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \sqrt{\frac{n^2 EA}{P}}.$$

If then the ratio $\frac{l_1}{\kappa}$ of the length l_1 to the radius of gyration κ is greater than $\sqrt{\frac{n^2 EA}{P}}$,

the column is in unstable equilibrium, and if pushed slightly to one side will not return. If $\frac{l_1}{\kappa}$ is equal to $\sqrt{\frac{\pi^2 EA}{P}}$, the column is in indifferent equilibrium, and if pushed slightly to one side will remain where placed. If $\frac{l_1}{\kappa}$ is less than $\sqrt{\frac{\pi^2 EA}{P}}$, the column is in stable equilibrium, and if pushed slightly to one side will return to its original position.

EXPERIMENTAL VERIFICATION.—These theoretic conclusions as to the deportment of the ideal column can be verified by the following experiment.* Let a thin bar of wrought iron AB be placed with one end on a spring balance, and apply a load P in the axis by means of a hinged lever AC upon which weights can be placed. After a little adjusting it will be found that for $\frac{P}{A}$ less than $\frac{\pi^2 E\kappa^2}{l_1^2}$ the column will not deflect, and if we deflect it by applying a lateral force at the middle, the column will straighten when this force is removed.



If, however, $\frac{P}{A}$ is equal to $\frac{\pi^2 E\kappa^2}{l_1^2}$ the column will not straighten, and if $\frac{P}{A}$ is greater than this by a very small amount, for the least disturbance it will be bent by the load to a very great extent, and even bent almost double or broken.

CRIPPLING LOAD—EULER'S FORMULA.—We see, then, that $\frac{P}{A} = \frac{\pi^2 E\kappa^2}{l_1^2}$ gives the crippling load for the ideal column. For a load less than this for slight disturbance the column recovers.

Now from equation (1), page 559, we have

$$l_1 = l\left(1 - \frac{P}{AE}\right).$$

But $\frac{P}{A}$ can never exceed the elastic limit S_e, and hence $\frac{P}{AE}$ is always a small fraction which can be disregarded with respect to unity. Thus (pages 476 and 478) for wrought iron it is $\frac{1}{1000}$, for steel $\frac{1}{750}$, for cast iron $\frac{1}{2500}$, and for timber $\frac{1}{500}$.

We have then practically for the crippling unit stress

$$\frac{P}{A} = \frac{\pi^2 E\kappa^2}{l^2}. \qquad \text{(E)}$$

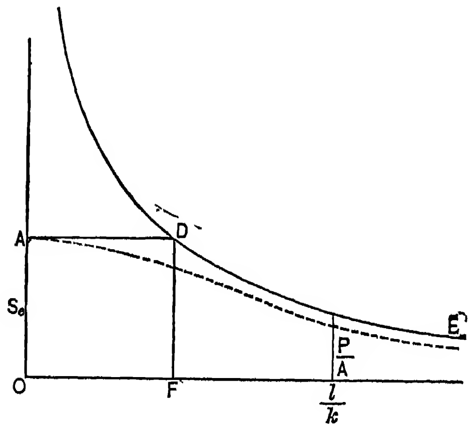
Formula (E) is known as *Euler's* formula for long struts. As we see, it neglects the change of length due to direct compression. It is usually deduced directly by writing in place of equation (2), page 560, the equation

$$EI\frac{d^2y}{dx^2} = -M = -Py,$$

and then integrating twice.

* Given by T. Claxton Fidler in "A Practical Treatise on Bridge Construction" (London, Charles Griffin & Co.), page 158.

Diagram for Ideal Column.—If we lay off $\frac{l}{\kappa}$ as abscissa, and the corresponding value of $\frac{P}{A}$ as given by (E) as ordinate, we have the curve DE of Euler's formula. The ordinate to this curve to scale gives the crippling unit stress $\frac{P}{A}$ for any given $\frac{l}{\kappa}$, and, as we have seen for a less load, the column will recover if slightly disturbed.



But when $\frac{P}{A}$ becomes equal to the elastic limit S_e, the corresponding ratio is

$$\frac{l}{\kappa} = OF = \sqrt{\frac{n^2 E}{S_e}}.$$

For less values of $\frac{l}{\kappa}$ than this the column can be loaded up to the elastic limit S_e. The ordinates, then, to ADE give the crippling unit stress for the ideal column for any value of $\frac{l}{\kappa}$.

Actual Column.—The preceding discussion and the conclusions and deportment of the ideal column do not hold good for actual columns, because in such columns the ideal conditions are not realized. Thus no actual column is perfectly homogeneous, has a perfectly straight axis or has the load exactly centred.

Lack of ideality in any of these conditions will cause a column of any length to deflect when loaded, and this is in accord with common experience.

We can, then, only load the actual column up to the elastic limit S_e when it is very short—theoretically only when the length is zero. For any finite length $\frac{P}{A}$ must always be less than given by the preceding diagram.

The actual curve, then, for any actual column will be some curve such as represented by the broken curve in the preceding diagram, which is tangent at A to the line AD, and at an infinite distance to the curve DE.

We see at once that any such curve which should give the actual values of the crippling unit stress $\frac{P}{A}$ for any one actual column must depend upon the actual eccentricity of the load and upon all other actual deviations from ideal conditions.

As all such deviations can never be identical for any two actual columns, the actual curve *must be a different one for each column.*

It is therefore obvious that any one curve which gives the *average* experimental values of $\frac{P}{A}$ for any number of actual columns must rest at bottom upon the *average* deviations from ideal conditions. A single curve must, then, be based upon average experimental results, and to try to deduce any single theoretic curve which shall give actual results for all columns is to attempt the impossible. Any practical formula must be directly based upon *average* experimental results.

Practical Values for n.—The theoretic values of n given on page 561 disregard friction. Also, the ends in practice cannot be perfectly "fixed." We have to do practically with two pin

ends with friction, or one pin end with friction and one flat end, or two flat ends. Hence the practical values of n should be different from the theoretic values given on page 561.

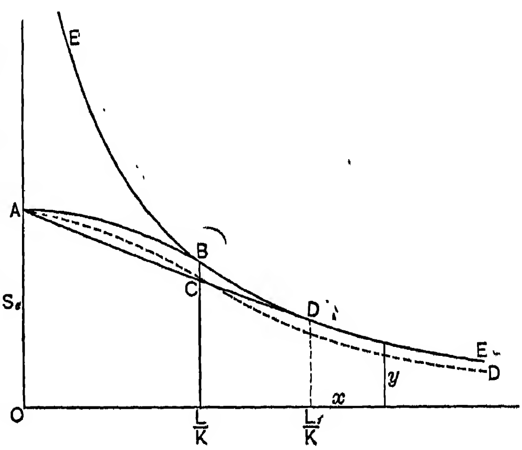
Experiments show that Euler's formula (E) gives the average crippling unit stress $\frac{P}{A}$ with very good accuracy *for very long columns*, i.e. when $\frac{l}{\kappa}$ is very great, provided we take for n the following values:

Two pin ends.	One pin end and one flat end.	Two flat ends.
$n = \pi\sqrt{\frac{5}{3}} = 4.053,$	$\frac{5\pi}{2\sqrt{3}} = 4.524,$	$\pi\sqrt{\frac{5}{2}} = 4.964.$

These values of n should be taken when Euler's formula is used.

Practical Formulas for Long Columns.—Experiment also shows that actual values of $\frac{P}{A}$ for short columns, instead of following the ideal diagram ADE, page 564, are so scattered, that almost any curve through A and tangent to Euler's curve DE gives very satisfactory average results. On this fact all our practical column formulas are based.

STRAIGHT-LINE FORMULA.—Of these practical formulas one of the best known is the "straight-line formula," first given by Thomas H. Johnson, C.E. (*Trans. Am. Soc. C. E.*, July, 1886). It consists in drawing a straight line through A tangent to Euler's curve EE. The point of tangency D is at a distance from O (see figure) given by $\frac{L}{\kappa}$. Both curve EE and straight line AD have then a common ordinate at the point D.



The equation of a straight line through A is

$$y = S_e + bx. \quad . \quad . \quad . \quad . \quad (1)$$

The equation of Euler's curve EE is

$$y = \frac{n^2E}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we differentiate (1) and (2), and equate the values of $\frac{dy}{dx}$, making $x = \frac{L}{\kappa}$, we have for the condition of a common tangent at D

$$b = -\frac{2n^2E\kappa^3}{L^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we equate (1) and (2), making $x = \frac{L}{\kappa}$, we have for the condition of a common ordinate at D

$$S_e + \frac{bL}{\kappa} = \frac{n^2E\kappa^2}{L^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4) we find for the limiting value of $\frac{L}{\kappa}$

$$\frac{L}{\kappa} = n\sqrt{\frac{3E}{S_e}}, \text{ and hence } b = -\frac{2S_e\sqrt{S_e}}{3n\sqrt{3E}}.$$

Inserting this value of b in (1), and putting $y = \frac{P}{A}$ and $x = \frac{l}{\kappa}$, we have for the straight-line formula,

$$\text{when } \frac{l}{\kappa} < n\sqrt{\frac{3E}{S_e}}, \qquad \frac{P}{A} = S_e\left[1 - \frac{2\sqrt{S_e}}{3n\sqrt{3E}} \cdot \frac{l}{\kappa}\right], \quad . \quad . \quad . \quad . \quad (S)$$

where k is the *least radius of gyration of the cross-section.*

This formula holds for any value of $\frac{l}{\kappa}$ so long as

$$\frac{l}{\kappa} < n\sqrt{\frac{3E}{S_e}}.$$

Beyond this limit we use Euler's formula, and have,

$$\text{when } \frac{l}{\kappa} > n\sqrt{\frac{3E}{S_e}}, \qquad \frac{P}{A} = \frac{n^2E\kappa^2}{l^2}.$$

The straight-line formula is simple and easily applied, and contains no experimental constants except S_e, E, and n.

It gives values for $\frac{P}{A}$ for small values of $\frac{l}{\kappa}$ considerably less than the average of experiments, owing to the fact that the tangent at A is not horizontal.

PARABOLA FORMULA.—This formula is given by Prof. J. B. Johnson (Theory and Practice of Modern Framed Structures—Wiley & Sons). The curve AB (figure, page 565) is assumed as a parabola tangent to Euler's curve at B. We have then

$$y = S_e + bx^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where b must be determined by the condition of tangency.

This equation gives $y = S_e$ for $x = 0$, and the tangent at A is horizontal.

From Euler's formula we have

$$y = \frac{n^2E}{x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating (1) and (2), and proceeding as before, we have the parabola formula,

$$\text{when } \frac{l}{\kappa} < n\sqrt{\frac{2E}{S_e}}, \qquad \frac{P}{A} = S_e\left[1 - \frac{S_e}{4n^2E} \cdot \frac{l^2}{\kappa^2}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (P)$$

where, as always, κ is the least radius of gyration of the cross-section.

This formula holds for any value of $\frac{l}{\kappa}$ so long as

$$\frac{l}{\kappa} < n\sqrt{\frac{2E}{S_e}}.$$

Beyond this limit we use Euler's formula, and have,

$$\text{when } \frac{l}{\kappa} > n\sqrt{\frac{2E}{S_e}}, \qquad \frac{P}{A} = \frac{n^2 E \kappa^2}{l^2}.$$

The parabola formula is as simple and easily applied as the straight-line formula. It also contains no experimental constants except S_e, E and n. It gives on the whole better average values for $\frac{P}{A}$, owing to the fact that the tangent at A is horizontal.

Remarks on these Formulas.—Formula (P) gives greater values for $\frac{P}{A}$ than formula (S), and is to be preferred, therefore, for very perfect columns with load very accurately centered.

Both the straight-line and the parabola formulas are lines tangent to Euler's curve EE at points D and B (figure, page 565). This means, in the light of our remarks, page 564, that *both assume ideal conditions for all columns at and beyond a certain length L, which is a different length for each formula.*

Such an assumption is of course incorrect. There is no one length, to say nothing of two different lengths, at which ideal conditions can be considered as existing. Experiments, however, show that average values of $\frac{P}{A}$ approach at and beyond these lengths very closely to Euler's curve, being always, however, slightly below, and hence the assumption is practically justified.

The actual average curve, however, as we have seen (page 564), should run through A as shown by the broken curve in the figure page 564, should have a horizontal tangent at A, and should then run, as shown, somewhat below Euler's curve, and be tangent to it at an infinite distance.

Rankine's Formula.—Such a curve is *Rankine's* formula. Let Δ be the deflection. Then the maximum moment is $P\Delta$.

From equation (II), page 497, we have for the unit stress S_f due to bending in the most compressed fibre at a distance v from the axis

$$S_f = \frac{P\Delta v}{I} = \frac{P\Delta v}{A\kappa^2}.$$

We have in addition a direct compressive unit stress $\frac{P}{A}$.

If then S_e is the elastic limit unit stress, we have for the crippling unit stress

$$\frac{P}{A} + \frac{P\Delta v}{A\kappa^2} = S_e, \quad \text{or} \quad \frac{P}{A} = \frac{S_e}{1 + \frac{\Delta v}{\kappa^2}}. \qquad (1)$$

Equation (1) is rational in form, and if we knew Δ it would give accurately the crippling unit stress.

If we suppose for small deflections the curve of deflection to be practically a circle of radius of curvature ρ, we should have

$$\Delta : l :: l : \rho - \Delta,$$

or, since Δ is small compared to ρ,

$$\Delta = \frac{l^2}{\rho}.$$

In general whatever the curve of deflection, we can assume Δ to be some function of l^2 and to vary inversely as v, since ρ increases with v. We can then write

$$\Delta = \frac{cl^2}{v}, \quad \text{or} \quad \frac{\Delta v}{\kappa^2} = \frac{cl^2}{\kappa^2}.$$

Inserting this value of $\frac{\Delta v}{r^2}$ in (1), we have

$$\frac{P}{A} = \frac{S_e}{1 + \frac{cl^2}{\kappa^2}}. \quad \ldots\ldots\ldots\ldots \quad \text{(R)}$$

where, as always, κ is the least radius of gyration of the cross-section, and c is a constant to be determined by experiment, depending upon the material and the end conditions. Since the column bends easiest in the direction of its least dimension, *we take for κ the least radius of gyration.*

Equation (R) is Rankine's formula for long columns. It holds for all values of $\frac{l}{\kappa}$.

We see that Rankine's formula gives $\frac{P}{A} = S_e$ for $\frac{l}{\kappa} = 0$. The tangent at A (figure, page 556) is horizontal, and we have $\frac{P}{A} = 0$ for $\frac{l}{\kappa} = \infty$. It therefore complies with the conditions for the average actual curve given on page 564.

It is not so simple or easily applied as the straight-line or the parabola formula, and the experimental constant c must be determined before it can be used in any case.

Gordon's Formula.—Since κ is a function of the *least dimension d* of the cross-section, we may also write for the crippling unit stress

$$\frac{P}{A} = \frac{S_e}{1 + c\frac{l^2}{d^2}}, \quad \ldots\ldots\ldots\ldots \quad \text{(G)}$$

where c is again a constant, to be determined by experiment. Equation (G) is known as Gordon's formula for long struts. It also holds for all values of $\frac{l}{\kappa}$, and the same remarks apply as for Rankine's formula.

Merriman's Formula.—The equation of the curve AB (figure, page 564) has been assumed by Prof. Merriman (*Engineering News*, July 19, 1894) as identical in form with Rankine's formula. We have, then,

$$y = \frac{S_e}{1 + bx^2}.$$

Instead, however, of regarding b as an experimental constant, Prof. Merriman determines b precisely as in the case of the straight-line and parabola formulas, by the condition of tangency.

We thus obtain

$$\frac{P}{A} = \frac{S_e}{1 + \frac{S_e l^2}{n^2 E \kappa^2}}, \quad \ldots \ldots \ldots \ldots \quad \text{(M)}$$

where, as always, κ is the *least radius of gyration of the cross-section.*

Equation (M) is Merriman's formula for long columns. Like Rankine's formula, it complies with the conditions of the average actual curve given on page 564. It is preferable to Rankine's in that it contains no experimental constant. It is therefore probably nearer the true curve for an average actual column than any of the formulas thus far given.

Allowable Unit Stress—Factor of Safety.—The preceding formulas will enable us to find the crippling unit stress.

In practice only a portion of this is taken as the allowable working unit stress. This portion is called the factor of safety (page 481). For *quiescent* loads (buildings, etc.) this factor is taken at $f = 4$ for wrought iron and steel, and $f = 6$ for cast iron and wood.

For variable loads a variable factor of safety is used equal to

$$f = 4 + \frac{l}{20d} \qquad \text{for wrought iron and steel,}$$

$$f = 7 + \frac{l}{20d} \qquad \text{for cast iron,}$$

$$f = 6 + \frac{l}{20d} \qquad \text{for wood,}$$

where l is the length in inches and d the least dimension in inches of the rectangle which encloses the given cross-section.

We have then in general for the working unit stress

$$S_w = \frac{P}{fA}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where $\frac{P}{A}$ is found by any one of the preceding formulas, and the value of f is taken as just given.

Examples.—(1) *Let the ratio of the length of a steel column to the least radius of gyration of its cross-section A be* $\frac{l}{\kappa} = 100$, *and to the least dimension of the enclosing rectangle be* $\frac{l}{d} = 25$. *Let* $S_e = 40000$ *and* $E = 30\,000\,000$ *pounds per square inch. Find the crippling and working unit stress by the straight-line formula.*

ANS. We have, using the practical values of n on page 565, $\frac{l}{\kappa} = 100$ less than $n\sqrt{\frac{3E}{S_e}}$ in all cases. Hence by the straight-line formula the crippling unit stress is

$$\frac{P}{A} = S_e\left[1 - \frac{100}{71n}\right].$$

Hence for

Two pin ends.................... $\frac{P}{A} = 0.65S_e = 26000$ pounds per square inch;

One pin end and one flat end...... $\frac{P}{A} = 0.69S_e = 27600$ " " " "

Two flat ends. .... $\frac{P}{A} = 0.72S_e = 28800$ " " " "

The factor of safety is $f = 4 + \frac{l}{20d} = 5.25$. Hence the working stress in these three cases is

$$S_w = 4952, \quad 5257, \quad 5486 \text{ pounds per square inch.}$$

(2) *Find the crippling and working unit stress by the parabola formula.*

ANS. We have by this formula

$$\frac{P}{A} = S_e\left[1 - \frac{10}{3n^2}\right].$$

Hence for

Two pin ends.................... $\frac{P}{A} = 0.80S_e = 32000$ pounds per square inch

One pin end and one flat end....... $\frac{P}{A} = 0.84S_e = 33600$ " " " "

Two flat ends .... $\frac{P}{A} = 0.86S_e = 34400$ " " " "

The factor of safety is as before $f = 5.25$. Hence the working stress in these three cases is

$$S_w = 6100, \quad 6400, \quad 6550 \quad \text{pounds per square inch.}$$

It will be seen that the values given by the parabola formula are greater than those given by the straight-line formula. For very perfect columns and load very accurately centred the parabola formula results are preferable; for poorer columns, the straight-line.

(3) *Find the crippling and working unit stress by the Merriman formula.*

ANS. We have by this formula

$$\frac{P}{A} = \frac{S_e}{1 + \frac{40}{3n^2}}.$$

Hence for

Two pin ends.................... $\frac{P}{A} = 0.55S_e = 22000$ pounds per square inch;

One pin end and one flat end....... $\frac{P}{A} = 0.60S_e = 24000$ " " " "

Two flat ends.................... $\frac{P}{A} = 0.65S_e = 26000$ " " " "

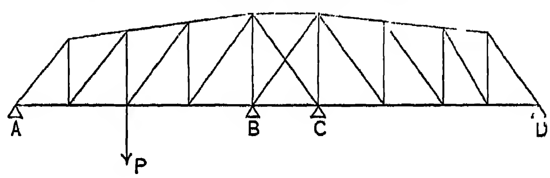
The factor of safety is as before $f = 5.25$. Hence the working stresses in these three cases are

$$S_w = 4190, \quad 4570, \quad 4950 \quad \text{pounds per square inch.}$$

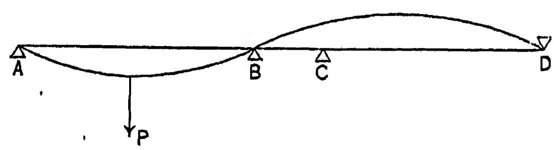
CHAPTER VIII.

THE PIVOT OR SWING BRIDGE.

Pivot or Swing Bridge.—The pivot or swing bridge is a girder continuous or partially continuous over three or four supports. If over three supports, it is a pivot span. If over four supports, the length of the short centre span is the width of the turn-table, and loads in this span come directly on the turn-table and hence cause no stresses in the truss members.



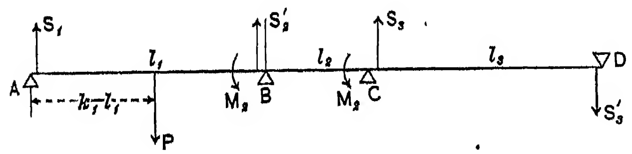
If the bracing is carried through the centre span as shown in the figure, it is evident that a load in one end span, as AB, tends to lift the span from the support C. It will be found in general difficult to hold the span down at C.



For this reason *the bracing in the centre span is omitted.* The continuity in such case is partial, but the span can be held down at C.

Centre Span without Bracing. — By means of the principle of least work we can find the reactions at the supports for a load P placed anywhere, and then the stresses in the truss members for this load are easily found.

1st. LOAD P IN SPAN $AB = l_1$.—Let the span be $AB = l_1$, $BC = l_2$, $CD = l_3$. Let the load P be at the distance k_1l_1 from the left end, where k_1 is any given fraction. Let M_2 be the moment on the left of the second support. Let the pressure on the right of A be S_1, on the left of B be S_2', on the right of C, S_3, and on the left of D, S_3'.



There are no braces in the span BC and hence no pressure on right of B or left of C, and the moment on left of C will also be M_2, the same as on left of B.

Taking moments about B, we have

$$- S_1l_1 + P(l_1 - k_1l_1) = M_2.$$

Taking moments about D we have

$$M_2 - S_3l_3 = 0.$$

From these equations we have

$$S_1 = -\frac{M_2}{l_1} + P(1 - k_1), \qquad S_2' = P - S_1 = \frac{M_2}{l_1} + Pk_1, \qquad S_3 = \frac{M_2}{l_3} = - S_3'. \quad . \quad (1)$$

Equations (1) give the pressures at the supports in terms of M_2.
If then we can determine M_2, we can find these pressures.
For any point distant x from A we have the moment,

between A and P, $M = -S_1 x = \dfrac{M_2 x}{l_1} - P(1 - k_1)x$;

between P and B, $M = -S_1 x + P(x - k_1 l_1) = \dfrac{M_2 x}{l_1} - P(1 - k_1)x + P(x - k_1 l_1)$.

For any point between B and C we have

$$M = M_2.$$

For any point distant x from D we have,

between C and D, $M = S_3' x = \dfrac{M_2 x}{l_3}$.

Let v be the lever-arm for any truss member, and M the moment at the centre of moments for that member. Then the stress in that member is $\dfrac{M}{v}$. Let a be the cross-section of the member, and s its length. Then, from equation (III), page 515, the work of straining that member is

$$\frac{M^2 s}{2Eav^2};$$

the total work of straining all the members is then

$$\text{work} = \sum \frac{M^2 s}{2Eav^2},$$

and this work must be a minimum by the principle of least work, page 517.

We have then, in the present case, inserting the value of M just found,

$$\text{work} = \sum_0^{\kappa_1 l_1} \left[\frac{M_2 x}{l_1} - P(1 - k_1)x\right]^2 \frac{s}{2Eav^2} + \sum_{\kappa_1 l_1}^{l_1} \left[\frac{M_2 x}{l_1} - P(1 - k_1)x + P(x - k_1 l_1)\right]^2 \frac{s}{2Eav^2}$$
$$+ \sum_0^{l_2} \frac{M_2^2 s}{2Eav^2} + \sum_0^{l_3} \frac{M_2^2 x^2 s}{2l_3^2 Eav^2}.$$

Suppose the span AB to be fixed horizontally at B and the support at A removed, and let u be the stress in any member due to a unit load at A, and p the stress in any member due to a unit load at P. Then we have

$$uv = 1 \times x, \quad \text{or} \quad \frac{x^2}{v^2} = u^2;$$

$$pv = 1(x - k_1 l_1), \quad \text{or} \quad \frac{(x - k_1 l_1)xs}{v^2} = pus.$$

If, then, we insert these values, and put $\dfrac{d(\text{work})}{dM_2} = 0$, we have for the value of M_2 which makes the work a minimum

$$\sum_0^{\kappa_1 l_1} \frac{M_2 u^2 s}{a l_1^2} - \sum_0^{\kappa_1 l_1} \frac{P(1-k_1)u^2 s}{a l_1} + \sum_{\kappa_1 l_1}^{l_1} \frac{M_2 u^2 s}{a l_1^2} - \sum_{\kappa_1 l_1}^{l_1} \frac{P(1-k_1)u^2 s}{a l_1} + \sum_{\kappa_1 l_1}^{l_1} \frac{Ppus}{a l_1}$$

$$+ \sum_0^{l_2} \frac{M_2 s}{a v^2} + \sum_0^{l_3} \frac{M_2 u^2 s}{a l_3^2} = 0,$$

or, solving for M_2,

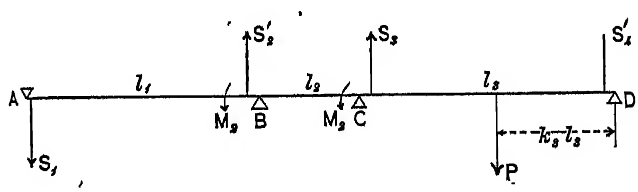
$$M_2 = P l_1 \frac{(1-k_1)\sum_0^{l_1} \frac{u^2 s}{a} - \sum_{\kappa_1 l_1}^{l_1} \frac{pus}{a}}{\sum_0^{l_1} \frac{u^2 s}{a} + l_1^2 \sum_0^{l_2} \frac{s}{a v^2} + \frac{l_1^2}{l_3^2} \sum_0^{l_3} \frac{u^2 s}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) gives the value of M_2, or the moment at the support B, for a load P at any distance $k_1 l_1$ from the left end A in the first span. If M_2 is known, then equations (1) give the pressures at the supports. When these are known the stresses can be easily calculated. Note that in equation (2) u is the stress in any member due to a unit load at A, and p the stress in any member due to a unit load at P, *considering the span AB as fixed horizontally at B and without any support at A.*

2d. LOAD P IN SPAN $CD = l_3$.—Let the load P be in the third span l_3, at the distance $k_3 l_3$ from support D, where k_3 is any given fraction.

Then we have

$$\left.\begin{aligned} S_1 &= -\frac{M_2}{l_1} = -S_2', \\ S_3 &= \frac{M_2}{l_3} + Pk_3, \\ S_4' &= -\frac{M_2}{l_3} + P(1-k_3), \end{aligned}\right\} \quad . \quad (3)$$

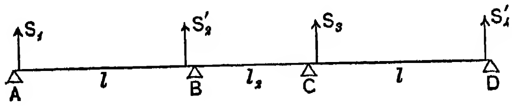


and proceeding now precisely as before, we find in the same way

$$M_2 = P l_3 \frac{(1-k_3)\sum_0^{l_3} \frac{u^2 s}{a} - \sum_{\kappa_3 l_3}^{l_3} \frac{pus}{a}}{\frac{l_3^2}{l_1^2}\sum_0^{l_1} \frac{u^2 s}{a} + l_3^2 \sum_0^{l_2} \frac{s}{a v^2} + \sum_0^{l_3} \frac{u^2 s}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) gives the value of M_2, of the moment at the support B, for a load P at any distance $k_3 l_3$ from the right end D in the third span. If M_2 is known, then equations (3) give the pressures at the supports. When these are known the stresses can be easily calculated. Note that in equation (4) u is the stress in any member due to a unit load at D, and p the stress in any member due to a unit load at P, *considering the span CD as fixed horizontally at C and without any support at D.*

Special Cases.—The preceding equations are general and include all cases. Thus if the two end spans are equal, we have only to make $l_1 = l_3 = l$ and $k_1 = k_3 = k$. If we have only two spans, we make $l_2 = 0$. Hence we have



THREE SPANS, END SPANS EQUAL.—Making $l_1 = l_3 = l$ and $k_1 = k_3 = k$, we have the following equations:

1st. *Load in span AB:*

$$S_1 = -\frac{M_2}{l} + P(1 - k), \qquad S_2' = \frac{M_2}{l} + Pk, \qquad S_3 = \frac{M_2}{l} = -S_3'. \quad . \quad . \quad . \quad (5)$$

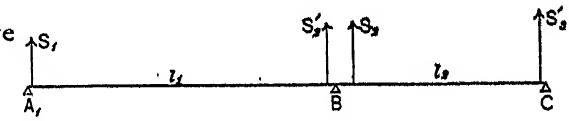
2d. *Load in span CD:*

$$S_1 = -S_2' = -\frac{M_2}{l}, \qquad S_3 = \frac{M_2}{l} + Pk \qquad S_4' = -\frac{M_2}{l} + P(1 - k). \quad . \quad (6)$$

And in both cases

$$M_2 = Pl\frac{(1 - k)\sum_0^l \frac{u^2 s}{a} - \sum_{kl}^l \frac{pus}{a}}{2\sum_0^l \frac{u^2 s}{a} + l\sum_0^{l_2} \frac{s}{av^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

TWO SPANS ONLY.—Making $l_2 = 0$, we have the following equations:



1st. *Load in span AB:*

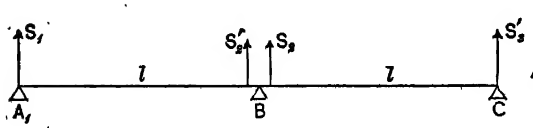
$$S_1 = -\frac{M_2}{l_1} + P(1 - k_1), \qquad S_2' = \frac{M_2}{l_1} + Pk_1, \qquad S_2 = -S_3' = \frac{M_2}{l_2}, \quad . \quad . \quad (8)$$

$$M_2 = Pl_1\frac{(1 - k_1)\sum_0^{l_1} \frac{u^2 s}{a} - \sum_{k_1 l_1}^{l_1} \frac{pus}{a}}{\sum_0^{l_1} \frac{u^2 s}{a} + \frac{l_1^2}{l_2^2}\sum_0^{l_2} \frac{u^2 s}{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

2d. *Load in span BC:*

$$S_1 = -S_2' = -\frac{M_2}{l_1}, \qquad S_2 = \frac{M_2}{l_2} + Pk_2, \qquad S_3' = -\frac{M_2}{l_2} + P(1 - k_2), \quad . \quad . \quad (10)$$

$$M_2 = Pl_2\frac{(1 - k_2)\sum_0^{l_2} \frac{u^2 s}{a} - \sum_{k_2 l_2}^{l_2} \frac{pus}{a}}{\frac{l_2^2}{l_1^2}\sum_0^{l_1} \frac{u^2 s}{a} + \sum_0^{l_2} \frac{u^2 s}{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$



TWO EQUAL SPANS.—Making $l_2 = 0$ and $l_1 = l_3 = l$, we have the following equations:

1st. *Load in span AB:*

$$S_1 = -\frac{M_2}{l} + P(1 - k), \quad S_2' = \frac{M_2}{l} + Pk, \qquad S_2 = -S_3' = \frac{M_2}{l}. \quad . \quad . \quad . \quad (12)$$

2d. *Load in span BC:*

$$S_1 = -S_2' = -\frac{M_2}{l}, \qquad S_2 = \frac{M_2}{l} + Pk, \qquad S_3' = -\frac{M_2}{l} + P(1 - k). \quad . \quad . \quad (13)$$

And in both cases

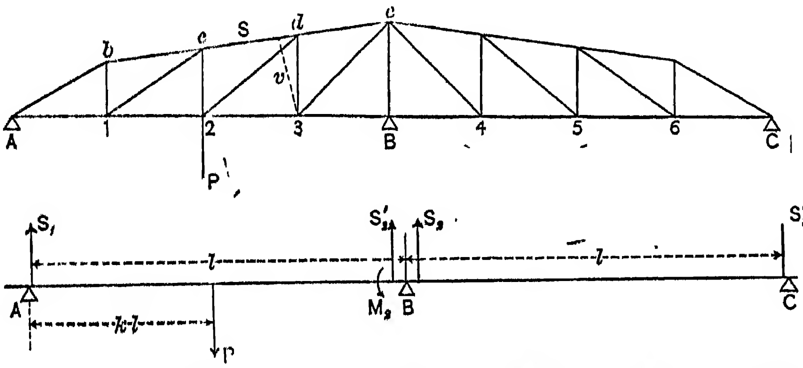
$$M_2 = Pl\frac{(1-k)\sum_0^l \frac{u^2 s}{a} - \sum_{kl}^l \frac{pus}{a}}{2\sum_0^l \frac{u^2 s}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Values of a Indeterminate.—It will be noted that our equations for M_2 require that the area of cross-section a for each member shall be known, while it is our object to determine these areas by first finding the stress in each member and then dividing this stress by the allowable unit stress.

It is necessary, then, to make a first approximation by supposing a the same for each member. It will then cancel out of our equations for M_2. We can then find M_2 approximately, and determine the stresses and corresponding areas a. With these values of a we can again determine M_2.

A short example will thoroughly illustrate the application of our equations.

Example.—*Find the stresses for a swing bridge of two equal spans $l = 80$ ft., each span divided into four panels of 20 ft. Let the spans be symmetrical, the centre height $Be = 10$ ft. and the height at ends $b1 = 7$ ft.*



For these dimensions the lever-arm v for any upper-chord member, as de, will not differ appreciably from the height $d3$, and the length s for any upper-chord member, as cd, will not differ appreciably from the panel length.

We have then the following lever-arms

	$A1$	1–2	2–3	3–B	bc	cd	de	Ab	$b1$	$c1$	$c2$	$d2$	$d3$	$e3$
lever-arms	7	8	9	10	7	8	9	39.64	140	52	160	65.66	180	80.5

We can now calculate the stresses u in every member due to a unit load at A, and the stresses p in every member due to a unit load at P, considering the span AB as fixed horizontally at B and without support at A. We can then form the following table:

(1)	(2) s	(3) u	(4) p_1	(5) p_2	(6) p_3	(7) u^2s	(8) p_1us	(9) p_2us	(10) p_3us
$A1$	20	− 2.857	....	....	....	163.25	....	....	....
1–2	20	− 5.000	− 2.500	....	....	500.00	250.00	....	....
2–3	20	− 6.666	− 4.444	− 2.221	....	888.88	592.59	296.30	....
3–B	20	− 8.000	− 6.000	− 4.000	− 2.000	1280.00	960.00	640.00	320.00
bc	20	+ 2.857	....	....	....	163.25	....	....	....
cd	20	+ 5.000	+ 2.500	....	....	500.00	250.00	....	....
de	20	+ 6.666	+ 4.444	+ 2.222	....	888.88	592.59	296.30	....
Ab	21.2	+ 3.027	....	....	....	194.25	....	....	....
$b1$	7	− 0.857	....	....	....	5.14	....	....	....
1–c	21.54	+ 2.307	+ 2.692	....	....	114.64	133.77	....	....
c–2	8	− 0.750	− 0.875	....	....	4.50	5.25	....	....
2–d	21.93	+ 1.827	+ 2.132	+ 2.436	....	72.87	85.42	97 60	....
$d3$	9	− 0.666	− 0.777	− 0.888	....	4.00	4.66	5.33	....
3–e	22.36	+ 1.490	+ 1.739	+ 1.987	+ 2.236	49.64	57.94	66.20	74.49
eB	10	− 0.600	− 0.700	− 0.800	− 0.900	3.60	4.20	4.80	5.40
						4833.90	2936.42	1406.53	399 89

In the first column we place the designation of each member; in column (2) the length s of each member; in column (3) the stress u in each member for a load of unity at the end A, considering the span AB as a semi-girder fixed horizontally at B; in columns (4), (5), (6) the stress p in each member for a load of unity at apex 1, 2 and 3, considering the span AB as a semi-girder fixed horizontally at B. In column (7) we have the values of u^2s for each member; in columns (8), (9), (10), the values of pus for each member. At the bottom of columns (7), (8), (9), (10) we give the summation of the values in each column.

We have then, from equation (14),

$$M_2 = Pl\frac{4833.9(1-k) - \Sigma_{kl}^{l} pus}{9667.8},$$

where k has the values $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ at apex 1, 2, 3, and the summation Σpus is given by columns (8), (9), (10).

We have then for load P at apex 1, from equations (12),

$$M_2 = +0.0712\,Pl, \qquad S_1 = +0.6788P.$$

For P at apex 2,

$$M_2 = +0.1045\,Pl, \qquad S_1 = +0.3955P.$$

For P at apex 3,

$$M_2 = +0.0836\,Pl, \qquad S_1 = +0.1664P.$$

For P at apex 4, S_1 will be the same as S_3' for P at 3, or

$$S_1 = -0.0836\,P.$$

For P at apex 5, S_1 will be the same as S_3' for P at 2, or

$$S_1 = -0.1045P.$$

For P at apex 6, S_1 will be the same as S_3' for P at 1, or

$$S_1 = -0.0712P.$$

Negative values denote that S_1 acts downwards.

We can now find the stresses in each member for each load, and then by tabulation can find the loading which gives the maximum stress and the maximum stress itself in each member.

Solid Beam—Uniform Cross-section.—We can easily find, from the equations already deduced, the value of M_2 for a solid beam of uniform cross-section. Thus, let x be the distance of the point of moments for any member from the left end A, and v its lever-arm. Then $u = \frac{x}{v}$, $p = \frac{x - \kappa_1 l_1}{v}$ Insert these values in equation (2) and we have

$$M_2 = Pl_1\frac{(1-\kappa_1)\sum_0^{l_1}\frac{x^2s}{av^2} - \sum_{\kappa_1 l_1}^{l_1}\frac{(x-\kappa_1 l_1)xs}{av^2}}{\sum_0^{l_1}\frac{x^2s}{av^2} + l_1^2\sum_0^{l_2}\frac{s}{av^2} + \frac{l_1^2}{l_3^2}\sum_0^{l_3}\frac{x^2s}{av^2}}.$$

Now if the girder is a solid beam of uniform cross-section, we have $s = dx$, and av^2 constant. Hence we have

$$M_2 = Pl_1\frac{1-\kappa_1)\int_0^{l_1}(x^2dx - \int_{\kappa_1 l_1}^{l_1}(x-\kappa_1 l_1)xdx}{\int_0^{l_1}x^2dx + l_1^2\int_0^{l_2}dx + \frac{l_1^2}{l_3^2}\int_0^{l_3}x^2dx}.$$

If we perform the integrations, we obtain

for load in span AB $\quad M_2 = \frac{Pl_1^2k_1(1-k_1^2)}{2(l_1 + 3l_2 + l_3}$. . . . . (15)

The pressures of the supports are then given by equations (1).

In the same way we have, from equation (4),

for load in span CD $$M_2 = \frac{Pl_3^2 k_3(1 - k_3^2)}{2(l_1 + 3l_2 + l_3)}. \qquad (16)$$

The pressures of the supports are then given by equations (3).

For three spans, end spans equal, the pressures on the supports are given by equations (5) for load in span AB, and by equations (6) for load in span CD, and in both cases

$$M_2 = \frac{Pl^2k(1 - k^2)}{4l + 6l_2}. \qquad (17)$$

For two spans only we have

for load in span AB $$M_2 = \frac{Pl_1^2 k_1(1 - k_1^2)}{2(l_1 + l_2)}. \qquad (18)$$

The pressures of the supports are given by equations (8).

for load in span BC $$M_2 = \frac{Pl_2^2 k_2(1 - k_2^2)}{2(l_1 + l_2)}. \qquad (19)$$

The pressures of the supports are given by equations (10).

For two equal spans the pressures on the supports are given by equations (12) for load in span AB, and by equations (13) for load in span BC, and in both cases

$$M_2 = \frac{Plk(1 - k^2)}{4}. \qquad (20)$$

Example.—*In the example given on page 575, consider the girder as a solid beam of uniform cross-section.* In this case we have, from equation (20),

$$M_2 = 20Pk(1 - k^2),$$

where k has the values $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$.

We have then for load P at $\frac{1}{4}l$ from left end, from equations (12),

$$M_2 = +4.6875P, \qquad S_1 = +0.6914P.$$

For load P at $\frac{2}{4}l$ from left end,

$$M_2 = +7.5P, \qquad S_1 = +0.4062P.$$

For load P at $\frac{3}{4}l$ from left end,

$$M_2 = +6.5625P, \qquad S_1 = +0.1680P.$$

For P at $\frac{3l}{4}$ from right end, from equations (13),

$$M_2 = +6.5625P, \qquad S_1 = -0.0820P.$$

For P at $\frac{2}{4}l$ from right end,

$$M_2 = +7.5P, \qquad S_1 = -0.0938P.$$

For P at $\frac{1}{4}l$ from right end,

$$M_2 = +4.6875P, \qquad S_1 = -0.0586P.$$

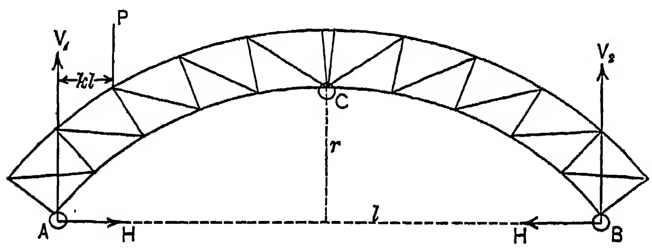
CHAPTER IX.

THE METAL ARCH.

Three Kinds of Metal Arch.—We may distinguish three kinds of metal arch, viz., arch hinged at crown and ends; arch hinged at ends only; arch without hinges.

If the arch is a framed structure, the stresses in the members can be found in any case, if for a given load we can find the horizontal thrust and vertical reactions at the ends and the moments, if any, which exist at the ends.

Framed Arch Hinged at Crown and Ends.—This form of construction is an arch only in form, but in principle is simply two braced rafters the thrust of which is taken by the abutments instead of by a tie-rod. It is therefore a very simple matter to find the end reactions for a given load.



Let the span AB be l, and P the load at the distance kl from the left end, where k is any given fraction. Let the rise, measured always from the chord AB to the hinge C at the crown, be denoted by r.

Then taking moments about the right-hand hinge at B, we have for the reaction V_1 at the left end for any position of P

$$-V_1 l + Pl(1-k) = 0, \quad \text{or} \quad V_1 = P(1-k). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking moments about the hinge C at the crown, we have for the horizontal thrust H at the left end, when kl is less than $\frac{l}{2}$, that is when P is on the left of the centre,

$$\frac{-V_1 l}{2} + Hr + Pl\left(\frac{1}{2} - k\right) = 0, \quad \text{or} \quad H = \frac{V_1 l}{2r} - \frac{Pl}{2r}(1-2k) = \frac{Pkl}{2r}. \quad . \quad . \quad (2)$$

When kl is greater than $\frac{l}{2}$, that is when P is on the right of the centre,

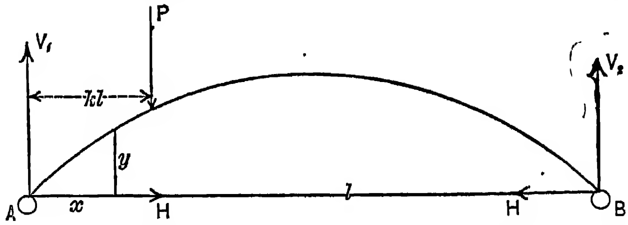
$$\frac{-V_1 l}{2} + Hr = 0, \quad \text{or} \quad H = \frac{V_1 l}{2r} = \frac{Pl(1-k)}{2r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These values of V_1 and H are independent of the shape of the arch.

Change of temperature causes no stresses in the arch hinged at crown and ends. Each half is free to turn about the hinges and accommodate itself to any change of shape due to change of temperature.

Equations (1), (2) and (3) hold for a solid as well as for a braced arch. We can then determine the stress in each member for each load, and then by tabulation can find the loads which give the maximum stress and the maximum stress itself in each member.

Framed Arch Hinged at Ends Only.—In this case we have just as before, taking moments about B, for any position of P



$$\left.\begin{array}{l} -V_1l + Pl(1-k) = 0, \\ \text{or} \quad V_1 = P(1-k). \end{array}\right\} \quad . \quad . \quad (1)$$

It remains to find the horizontal thrust H.

Let v be the lever-arm for any member, as determined by the method of sections, page 401, and M the moment at the centre of moments for that member. Then the stress in that member is $\frac{M}{v}$. Let a be the cross-section of the member, and s its length. Then from equation (III), page 515, the work of straining that member is

$$\frac{M^2 s}{2Eav^2}.$$

The total work of straining all the members is then

$$\text{work} = \sum \frac{M^2 s}{2Eav^2},$$

and by the principle of least work, page 517, this work must be a minimum.

Let x and y be the co-ordinates of the point of moments for any member, as determined by the method of sections, page 401.

Then for any member on the left of P we have the moment

$$M = Hy - V_1 x = Hy - P(1-k)x,$$

and for any member on the right of P we have

$$M = Hy - V_1 x + P(x - kl) = Hy - P(1-k)x + P(x - kl).$$

We have then for the work of straining all the members

$$\text{work} = \sum_0^{kl} [Hy - P(1-k)x]^2 \frac{s}{2Eav^2} + \sum_{kl}^{l} [Hy - P(1-k)x + P(x - kl)]^2 \frac{s}{2Eav^2}.$$

If we differentiate with reference to H and put the differential coefficient equal to zero, we have for least work

$$\frac{d(\text{work})}{dH} = 0 = \sum_0^{kl} [Hy^2 - Py(1-k)x] \frac{s}{Eav^2} + \sum_{kl}^{l} [Hy^2 - P(1-k)xy + P(x - kl)y] \frac{s}{Eav^2}.$$

Hence, since E is constant,

$$H = P \frac{(1-k)\sum_0^{kl} \frac{xys}{av^2} + \sum_{kl}^{l} \left[(1-k)\frac{xys}{av^2} - (x - kl)\frac{ys}{av^2}\right]}{\sum_0^{l} \frac{y^2 s}{av^2}}. \quad . \quad . \quad . \quad . \quad (2)$$

Let h be the stress in any member due to a *negative* unit horizontal force at the left end A. Let p be the stress in any member due to a unit load at P, considering the arch as simply *supported at the ends*. Then we have for any member on the left of P

$$\pm hv = 1 \times y, \qquad \pm pv = 1 \times (1 - k)x.$$

Multiplying these two equations, we have for any member on the left of P

$$ph = (1 - k)\frac{xy}{v^2}.$$

For any member on the right of P we have

$$\pm hv = 1 \times y, \qquad \pm pv = 1 \times (1 - k)x - 1 \times (x - kl);$$

hence

$$ph = (1 - k)\frac{xy}{v^2} - (x - kl)\frac{y}{v^2}.$$

We see, then, that equation (2) can be written

$$H = P\frac{\sum_0^l \frac{phs}{a}}{\sum_0^l \frac{h^2s}{a}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where p is the stress in any member with its proper sign, (+) for tension and (−) for compression for a unit load at P, *considering the arch as simply supported at the ends*, and h is the stress in any member for a unit *negative* thrust at the left end A, also taken with its proper sign, (+) for tension and (−) for compression.

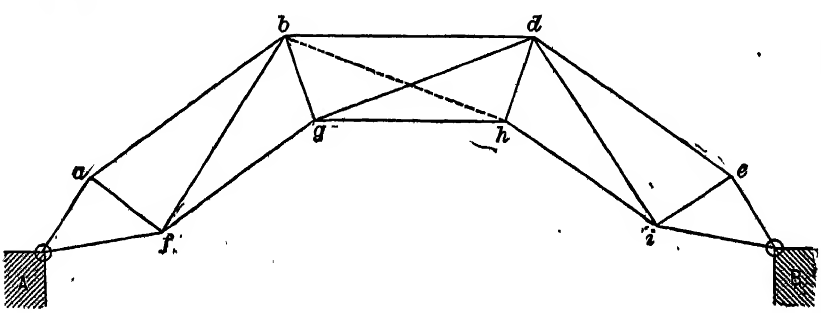
From equations (1) and (3) we can find V_1 and H for a load P placed anywhere, and can then determine the stresses for this loading.

We can then easily determine the stress in each member for each load, and then by tabulation can find the loads which give the maximum stress, and the maximum stress itself, in each member.

It will be noted that equation (3) requires that the area of cross-section a for each member shall be known in advance, while it is our object to determine these areas by first finding the stress and then dividing by the allowable unit stress. It will in general, then, be necessary to first find values for H, assuming a to be constant. It then cancels out. Using these values of H, we can find the areas a, and then with these areas can find new values for H and new areas.

Example.—A short example will illustrate the use of equations (1) and (3).

Let us take a circular arch, the radius of the outer chord being 44 feet, and of the inner cord 36 feet. The apices A, a, b, d, e, B are on the outer circle and f, g, h, i on the inner circle, the hinges being at A and B, and bracing as shown,



The members af, bg, dh, ei, are radial, and the chords ab, bd, de subtend an angle of 30° at the centre, while Aa, eB subtend an angle of 15°.

We can now find the stress h in each member for a negative thrust of unity at the end A, and also the stress p in each member for a unit load at a, b d, e, considering the arch as simply supported at the ends. We can then draw up the following table. In the first column we have the members; in column two the

lengths s of the members; in the third column the stresses h; in the fourth column the quantities h^2s, and in the following columns the quantities p_1hs, p_2hs, p_3hs, p_4hs.

The minus sign for a stress denotes compression, and the plus sign tension.

	s	h	h^2s	p_1hs	p_2hs	p_3hs	p_4hs
Aa	11.4866	− 0.4357	2.1805	+ 4.0073	+ 2.8652	+ 2.7966	+ 0.7302
ab	22.7762	− 0.4472	4.5550	+ 7 6816	+ 10.8253	+ 5.1662	+ 1.5255
bd	22.7762	− 1.6530	62.2339	+ 21.1919	+ 85.9151	+ 85.9151	+ 21.1919
de	22.7762	− 0.4472	4.5550	+ 1.5255	+ 5.1662	+ 10.8253	+ 7.6816
eB	11.4866	− 0.4357	2.1805	+ 0.7302	+ 2.7966	+ 2.8652	+ 4.0073
Af	13.1133	+ 1.3116	22.5587	+ 15.7202	+ 11.2415	+ 6.0628	+ 1.5789
fg	18.6350	+ 2.6530	131.1615	+ 39.0452	+ 111.0000	+ 59.9147	+ 15.6671
gh	18.6350	+ 2.6530	131.1615	+ 15.6671	+ 59.9147	+ 59 9147	+ 15 6671
hi	18.6350	+ 2.6530	131.1615	+ 15.6671	+ 59.9147	+ 111.0000	+ 39.0452
iB	13.1133	+ 1.3116	22.5587	+ 1.5789	+ 6.0628	+ 11.2415	+ 15.7202
af	8	+ 0.1726	0.2383	− 0.4457	+ 0.5665	+ 0.2719	+ 0.0802
fb	22.1005	− 1.4300	45.1933	− 7.1709	+ 50.9041	+ 29.9284	+ 7.1835
bg	8	+ 1.3733	15.0876	+ 2.2972	+ 8.6188	+ 6.8917	+ 1.8017
gd	22.1005	0	0	0	0	0	0
dh	8	+ 1.3733	15.0876	+ 1.8017	+ 6.8917	+ 8.6188	+ 2.2972
di	22.1005	− 1.4300	45.1933	+ 7.1835	+ 29.9284	+ 50.9041	− 7.1709
ei	8	+ 0.1726	0.2383	+ 0.0802	+ 0.2719	+ 0.5665	− 0.4457
			635.3252	+ 126.561	+ 452.8835	+ 452.8835	+ 126.561

From equation (3) we have then for the horizontal thrust, assuming the cross-sections a constant,

$$H = P\frac{\sum_0^l phs}{635.3252},$$

where $\sum_0^l phs$ is 126.561, 452.8835, 452.8835 and 126.561 for the load P at a, b, d and e.

We have then for the loads P_1, P_2, P_3, P_4 at a, b, d, e:

	P_1	P_2	P_3	P_4
$H =$	$0.2P$,	$0.7128P$,	$0.7128P$,	$0.2P$.

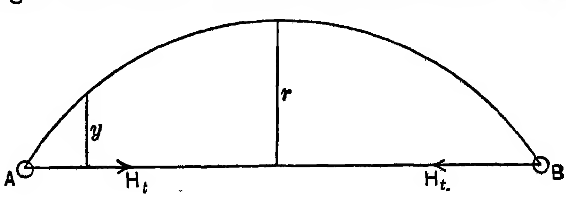
For the vertical reaction we have, from equation (1),

$$V_1 = P(1 - k),$$

	P_1	P_2	P_3	P_4
$V_1 =$	$0.9082P$,	$0.6494P$,	$0.3506P$,	$0.0918P$.

With these values of H and V_1 we can easily find the stresses in every member for each load P, either by computation or diagram. Then by tabulation we can find the loading which gives the maximum stress and the maximum stress itself in each member.

Temperature Stress—Framed Arch Hinged at Ends.—While in the arch with three hinges there are no temperature stresses, in the arch hinged at the ends only the stresses due to change of temperature may be considerable.



The effect of a change of temperature is to cause a horizontal thrust at the ends. For a rise of temperature we have a positive thrust H_t. For a fall of temperature we have a negative thrust H_t. If H_t is known, the resulting stresses are easily found either by computation or diagram.

Let ϵ be the coefficient of expansion, and t the number of degrees rise or fall of temperature. Then the change of length of the span is ϵlt. The work is then $\frac{H_t\epsilon lt}{2}$.

The moment at any point is $M = H_t y$. If v is the lever-arm for any member, and M the moment at the centre of moments for that member, the stress in that member is $\frac{M}{v}$. Let a be the cross-section of the member, and s its length. Then, from equation (III), page 515, the work of straining that member is

$$\frac{M^2 s}{2Eav^2} = \frac{H_t^2 y^2 s}{2Eav^2}.$$

The total work for all the members is then

$$\frac{H_t \epsilon l t}{2} = \sum_0^l \frac{H_t^2 y^2 s}{2Eav^2}.$$

Hence we have

$$H_t = \frac{E\epsilon l t}{\sum_0^l \frac{y^2 s}{av^2}}.$$

Suppose the arch to be rigidly fixed at the right end and free at the left end, and let h be the stress in any member due to a unit horizontal force at the left end. Then we have $hv = 1 \times y$, or $h = \frac{y}{v}$, and hence we can write

$$H_t = \pm \frac{E\epsilon l t}{\sum_0^l \frac{h^2 s}{a}}, \quad \ldots \ldots \ldots \ldots \quad (3)$$

where the summation $\sum_0^l \frac{h^2 s}{a}$ is made as in the preceding example, t is the number of degrees rise or fall of temperature above or below the mean temperature of erection, ϵ is the coefficient of expansion or the change of length per unit of length for one degree, and E is the coefficient of elasticity (page 478). The plus (+) sign is taken for rise of temperature, and the minus (−) sign for fall of temperature.

We have for one degree Fahrenheit:

For cast-iron.... $\epsilon = 0.00000617$,
wrought-iron.. $\epsilon = 0.00000686$,
steel......... $\epsilon = 0.00000599$.

Values of E are given on page 478.

Equation (3) requires that the areas of the members should be known in advance, whereas these are what we wish to find. We must in general, then, first assume a constant value for a in (3) equal, say, to the section at the crown. Denote this assumed constant section by a_0. We have then for our first calculation

$$H_t = \pm \frac{E\epsilon a_0 l t}{\sum_0^l h^2 s}. \quad \ldots \ldots \ldots \ldots \quad (4)$$

Example.—In the preceding example, page 580, let the cross-section at crown be $a_0 = 2$ square inches Let the arch be of steel, and let us take $E = 30\,000\,000$ pounds per square inch. Let the change of temperature be $t = 40°$.

Then from the table page 581 we have $\sum_0^l h^2 s = 635.297$; and since $l = 76.21024$ ft., we have for $\epsilon = 0.0000065$99

$$H_t = \pm 1728 \text{ pounds.}$$

The stresses due to this thrust can now be found.

Solid Arch Hinged at Ends Only.—For any load P we have, just as for the framed arch page 579, the left vertical reaction

$$V_1 = P(1 - k). \quad \ldots \ldots \ldots \quad (1)$$

We have found on page 579, equation (2), for the horizontal thrust of a framed arch

$$H = P \frac{(1-k)\sum_0^l \frac{xys}{av^2} - \sum_{kl}^l \frac{y(x-kl)s}{av^2}}{\sum_0^l \frac{y^2 s}{av^2}}.$$

If the arch is a solid beam, we can put ds for s, and $av^2 = I =$ moment of inertia of the cross-section. Hence if x and y are the co-ordinates of any point of the neutral axis, we have

$$H = P \frac{(1-k)\int_0^l \frac{xyds}{I} - \int_{kl}^l \frac{y(x-kl)ds}{I}}{\int_0^l \frac{y^2 ds}{I}}. \quad \ldots \ldots \quad (2)$$

This equation is general. If the moment of inertia I of the cross-section is constant, I cancels out.

Instead of performing the integrations indicated in (2) we can in any case divide the neutral axis into a number of equal arcs of length s. We have then, since s cancels out,

$$H = P \frac{(1-k)\sum_0^l \frac{xy}{I} - \sum_{kl}^l \frac{y(x-kl)}{I}}{\sum_0^l \frac{y^2}{I}}. \quad \ldots \ldots \quad (3)$$

If I is constant, it cancels out.

From (1) and (3), then, we can find V_1 and H for any given load, and can then find the moment M at any point of the neutral axis for a load anywhere. Then by tabulation we can find the loading which gives the maximum moment at any point of the neutral axis, and this maximum moment $M_{max.}$ itself.

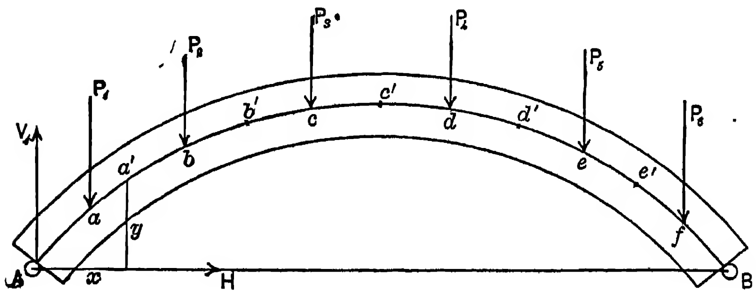
We have then, from equation (II), page 497,

$$I = \frac{144 M_{max.} v}{S_f} \quad \ldots \ldots \ldots \quad (4)$$

where $M_{max.}$ is *in pound-feet*, S_f is the allowable unit stress *in pounds per square inch* in the most remote fibre at a distance v *in feet*, and I is given for dimensions *in inches*.

Example.—Let us take a circular arch of radius of neutral axis 40 ft. and central angle 120°, so that $l = 69.282$ ft. and rise $r = 20$ ft. We first suppose I constant so that it cancels out.

Divide the neutral axis into a sufficiently large number of equal segments, say six, so that the length of a segment is $s = 13.963$ ft., and let the end segments Aa and Bf *be each one half of s*.



Let the ordinates to the middle points a', b', c', etc., of each segment for origin at A be x, y, and take the loads P_1, P_2, etc., acting half way between A and a', a' and b', b' and c', etc.

In order to apply equation (3), page 583, we can then draw up the following table.

	x	y	xy	y^2
Aa	1.875	2.943	5.523	8.661
ab	8.929	10.642	95.022	113.252
bc	20.960	17.588	368.644	309.338
cd	34.641	20	692.820	400
de	48.322	17.588	849.887	309.338
ef	60.352	10.642	642.266	113.252
fB	67.407	2.943	198.379	8.661
			2750.590	1253.841

In the first column we have the segments Aa, ab, bc, etc.; in the second and third columns the values of x and y for the middle points of these segments; in the last two columns the values of xy and y^2.

We have then $\Sigma_0^l xy = 2750.590$ and $\Sigma_0^l y^2 = 1253.841$. Note that in finding these summations, since the end segments Aa and fB are only half length, we take in the summations *one half the values for Aa and fB.*

We have now for the values of kl and $1 - k$ for each load:

	P_1	P_2	P_3	P_4	P_5	P_6
$kl =$	4.465	14.945	27.8	41.481	54.337	64.817
$1 - k =$	0.935	0.784	0.599	0.401	0.216	0.065

Hence from equation (1), page 583, the values of V_1 for each load are

$$V_1 = 0.935P_1, \quad 0.784P_2, \quad 0.599P_3, \quad 0.401P_4, \quad 0.216P_5, \quad 0.065P_6.$$

In order to find the summations $\Sigma_{kl}^l y(x - kl)$ for each load we can draw up the following table.

	P_1		P_2		P_3		P_4		P_5		P_6	
	$x - kl.$	$y(x - kl).$	$x - kl.$	$y(x - kl).$	$x - kl.$	$y(x - kl).$	$x - kl.$	$y(x - kl).$	$x - kl.$	$y(x - kl).$	$x - kl.$	$y(x - kl).$
ab	4.464	47.506										
bc	16.495	290.114	6.015	105.792								
cd	30.176	603.520	19.696	393.920	6.841	136.820						
de	43.857	771.357	33.377	587.035	20.522	360.941	6.841	120.309				
ef	55.887	594.749	45.407	483.221	32.552	346.418	18.871	200.825	6.015	64.911		
fB	63.042	179.533	52.462	154.396	39.607	116.563	25.926	76.300	13.070	38.465	2.59	7.622
		2397.012		1653.366		902.460		359.284		84.14[illegible]		3.811

We have then, from equation (3), page 583, for I constant

$$H = P\frac{2750.59(1 - k) - \Sigma_{kl}^l y(x - kl)}{1253.841},$$

where the summation $\Sigma^l_{kl} y(x - kl)$ for each load is given by the preceding table. Note that in finding these summations, since the end segments Aa and fB are only half length, we take in the summations *one half the values* for fB.

We have then for each load

$$H = 0.139P_1, \quad 0.401P_2, \quad 0.6P_3, \quad 0.6P_4, \quad 0.401P_5, \quad 0.139P_6.$$

Since we now know H and V_1 for each load, we can find the moment M at the centre of each segment for each load. By tabulation, then, we can find the maximum moment $M_{max.}$ at each of these points. Then from equation (4), page 583, we can find I at each of these points.

Solid Semicircle, Hinged at Ends Only—Constant I.—The preceding method applies to any solid arch of any shape and any loading. In the case of a semicircle of constant I the integrations of equation (2), page 583, are easily made.

Thus let R be the radius of the neutral axis. Then we have $\frac{ds}{dx} = \frac{R}{y}$, or $ds = \frac{Rdx}{y}$. Inserting this value of ds in (2), we obtain

$$H\int_0^l ydx = P(1 - k)\int_0^l xdx - P\int_{kl}^l (x - kl)dx.$$

Now $\int_0^l ydx$ is the area $A = \frac{\pi R^2}{2}$ of the semicircle. Performing the other integrations, we have, since $l = 2R$,

$$H = \frac{4Pk(1 - k)}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) gives H directly for a load P anywhere.

Temperature Thrust—Solid Arch Hinged at Ends.—We have, just as on page 582, for a framed arch

$$\frac{H_t \epsilon lt}{2} = \sum_0^l \frac{H_t^2 y^2 s}{2Eav^2}.$$

For a solid arch we can put ds for s, and I for av^2; hence

$$H_t = \pm \frac{E\epsilon lt}{144\int_0^l \frac{y^2 ds}{I}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where l, y, ds are *in feet*, E in pounds per *square inch*, and I is given for dimensions *in inches*.

Instead of performing the integration, we can, as before, divide the neutral axis into a number of equal arcs of length s. We have then

$$H_t = \pm \frac{E\epsilon lt}{144\sum_0^l \frac{y^2 s}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These equations are general.

If the arch is a semicircle, we have $ds = \frac{Rdx}{y}$, and (5) becomes

$$H_t = \pm \frac{E\epsilon lt}{144R\int_0^l \frac{ydx}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let I_0 be the moment of inertia at the crown. Then for our first calculation we have in general, from (6),

$$H_t = \pm \frac{E\epsilon I_0 lt}{144\sum_0^l y^2 s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and for a semicircle from (7), since $\int_0^l ydx = A = \frac{\pi R^2}{2}$,

$$H_t = \pm \frac{2E\epsilon I_0 lt}{144\pi R^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In all equations, distances are *in feet*, E in pounds per *square inch*, and I is given for dimensions *in inches*.

Example.—In the example page 584, let the moment of inertia at the crown be $I_0 = 2000$ in.^{4}. Let the arch be of steel, and let us take $E = 30\,000\,000$ pounds per square inch. Let the change of temperature $t = 40°$.

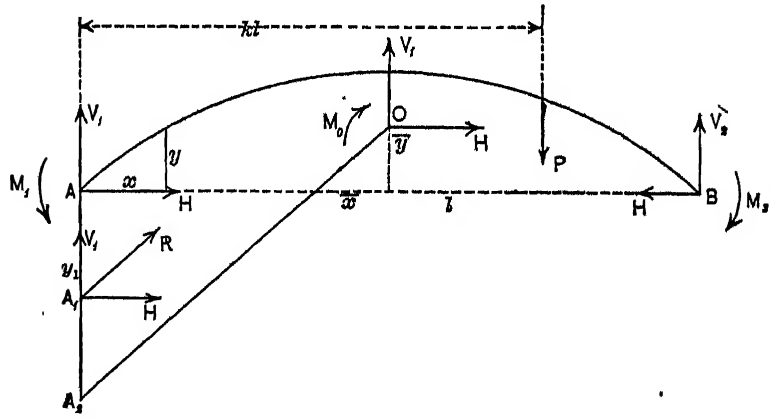
Then, from the table page 584, we have $\sum_0^l y^2 s = 17307.06$ ft.^{3}, and since $l = 69.28$ ft., we have, from (8), for $\epsilon = 0.00000599$

$$H_t = \pm 400 \text{ pounds.}$$

For a semicircle, since $R = 40$ ft., we have, from (9),

$$H_t = \pm 72 \text{ pounds.}$$

Framed Arch Fixed at Ends.—Let the arch be fixed at the ends, and a load P act at the distance kl from the left end A of the upper or lower chord. In this case we have at A not only a vertical reaction V_1 and a horizontal thrust H, but also a moment M_1.



That is, the resultant R of V_1 and H, instead of passing through A as in the case of end hinges, page 579, must now pass through some point A_1 at a distance y_1 vertically below A, so that $-Hy_1 = M_1$.

Let v be the lever-arm for any member as determined by the method of sections, page 401, and a the cross-section of any member, and s its length. Let x, y be the co-ordinates for origin at A of the point of moments for any member as determined by the method of sections, page 401.

Let O be a point whose co-ordinates for origin at A are given by

$$\bar{x} = \frac{\sum_0^l \frac{xs}{av^2}}{\sum_0^l \frac{s}{av^2}}, \qquad \bar{y} = \frac{\sum_0^l \frac{ys}{av^2}}{\sum_0^l \frac{s}{av^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

so that we have

$$\sum_0^l (x-\bar{x})\frac{s}{av^2} = 0 \quad \text{and} \quad \sum_0^l (y-\bar{y})\frac{s}{av^2} = 0.$$

For arch symmetrical with respect to the centre we have

$$\bar{x} = \frac{l}{2}, \quad \text{and} \quad \sum_0^l (y-\bar{y})(x-\bar{x})\frac{s}{av^2} = 0.$$

Draw through this point O a parallel OA_2 to the resultant R of V_1 and H.

If now we consider O as a fixed point rigidly connected to the arch at A_2 by members OA_2 and A_2A, we can remove the abutment at A and the equilibrium of the arch will not be affected. We shall then have at O the reactions V_1 and H and a moment M_0.

For any member on the right of P we have the moment

$$M = H(y-\bar{y}) - V_1\left(x-\frac{l}{2}\right) + P(x-kl) + M_0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For any member on the left of P we have

$$M = H(y-\bar{y}) - V_1\left(x-\frac{l}{2}\right) + M_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have then, just as on page 579, for the work of straining all the members

$$\text{work} = \sum_0^{kl}\left[H(y-\bar{y}) - V_1\left(x-\frac{l}{2}\right) + M_0\right]^2\frac{s}{2Eav^2}$$

$$+ \sum_{kl}^{l}\left[H(y-\bar{y}) - V_1\left(x-\frac{l}{2}\right) + P(x-kl) + M_0\right]^2\frac{s}{2Eav^2}.$$

Since H, V_1 and M_0 must have such values as to make the work a minimum, we place the first differential coefficients of the work with reference to H, V_1 and M_0 equal to zero. Hence

$$\frac{d(\text{work})}{dM_0} = 0 = \sum_0^l\left[H(y-\bar{y}) - V_1\left(x-\frac{l}{2}\right) + M_0\right]\frac{s}{Eav^2} + \sum_{kl}^{l} P(x-kl)\frac{s}{Eav^2},$$

$$\frac{d(\text{work})}{dV_1} = 0 = \sum_0^l\left[V_1\left(x-\frac{l}{2}\right) - H(y-\bar{y}) - M_0\right]\frac{\left(x-\frac{l}{2}\right)s}{Eav^2} - \sum_{kl}^{l} P\left(x-\frac{l}{2}\right)(x-kl)\frac{s}{Eav^2},$$

$$\frac{d(\text{work})}{dH} = 0 = \sum_0^l\left[H(y-\bar{y}) - V_1\left(x-\frac{l}{2}\right) + M_0\right]\frac{(y-\bar{y})s}{Eav^2} + \sum_{kl}^{l} P(y-y)(x-kl)\frac{s}{Eav^2}.$$

But since for symmetrical arch

$$\sum_0 (y-\bar{y})\frac{s}{av^2} = 0, \quad \sum_0^l\left(x-\frac{l}{2}\right)\frac{s}{av^2} = 0, \quad \sum_0^l (y-\bar{y})\left(x-\frac{l}{2}\right)\frac{s}{av^2} = 0,$$

these equations reduce to

$$\left.\begin{aligned} M_0\sum_0^l \frac{s}{av^2} &= -P\sum_{kl}^l \frac{(x-kl)s}{av^2},\\ V_1\sum_0^l \frac{\left(x-\frac{l}{2}\right)^2 s}{av^2} &= P\sum_{kl}^l \frac{\left(x-\frac{l}{2}\right)(x-kl)s}{av^2},\\ H\sum_0^l \frac{(y-\bar{y})^2 s}{av^2} &= -P\sum_{kl}^l \frac{(y-\bar{y})(x-kl)s}{av^2}. \end{aligned}\right\} \quad (4)$$

These equations give M_0, V_1 and H for a load P anywhere at a distance kl from the left end A, as shown in the figure, page 586.

Let the arch be supposed to be fixed at the right end and free at the left end, the left support being removed, and in this condition let u be the stress in any member for a unit load at the left end A, h the stress in any member for a *negative* unit horizontal force at A, p the stress in any member due to a unit load at P, m the stress in any member due to a *negative* unit moment at the point of moments for that member.

Then we have for any member

$$\pm m = \frac{1}{v}, \quad \pm h = \frac{y}{v}, \quad \pm u = -\frac{x}{v}, \quad \pm p = -\frac{x-kl}{v}, \quad \pm\frac{l}{2}m = \frac{l}{2v}, \quad \pm\bar{y}m = \frac{\bar{y}}{v}.$$

Hence $m^2 = \dfrac{1}{v^2}$, $mh = \dfrac{y}{v^2}$, $pm = -\dfrac{x-kl}{v^2}$, $p\left(u+\dfrac{l}{2}m\right) = \dfrac{\left(x-\frac{l}{2}\right)(x-kl)}{v^2}$,

$$p(h-\bar{y}m) = -\frac{(y-\bar{y})(x-kl)}{v^2}.$$

We have then, from equations (1) and (4), for a load P anywhere on a symmetrical arch at a distance kl from the left end

$$\left.\begin{aligned} \bar{x} = \frac{l}{2}, \qquad \bar{y} &= \frac{\sum_0^l \frac{mhs}{a}}{\sum_0^l \frac{m^2 s}{a}},\\ M_0 &= P\frac{\sum_{kl}^l \frac{pms}{a}}{\sum_0^l \frac{m^2 s}{a}},\\ V_1 &= P\frac{\sum_{kl}^l p\left(u+\frac{lm}{2}\right)\frac{s}{a}}{\sum_0^l \left(u+\frac{lm}{2}\right)^2\frac{s}{a}},\\ H &= P\frac{\sum_{kl}^l p(h+\bar{y}m)\frac{s}{a}}{\sum_0^l (h-\bar{y}m)^2\frac{s}{a}}. \end{aligned}\right\} \quad (5)$$

Equations (5) give M_0 and V_1 and H at the left end A for a load P anywhere on a symmetrical framed arch at a distance kl from the left end A.

If we take moments about the left end A (figure, page 586), we have

for load P on right of the centre $$M_1 = \frac{V_1 l}{2} - H\bar{y} + M_0, \qquad (6)$$

where M_1 is the moment at the left end A for a load anywhere *on the right-hand half*, and H, V_1, M_0 are given by (5).

If we take moments about the right end B, we have

for load P on right of the centre $$M_2 = M_1 - V_1 l + Pl(1 - k), \qquad (7)$$

where M_1 is given by (6). This value of M_2 is the same as the moment M_1' *at the left end A* for a similarly placed load *on the left of the centre.*

If V_1 as given by (5) is the reaction at the left end for a load P on the right-hand half, we have for the reaction V_1' for a similarly placed load on the left-hand half

$$V_1' = P - V_1. \qquad (8)$$

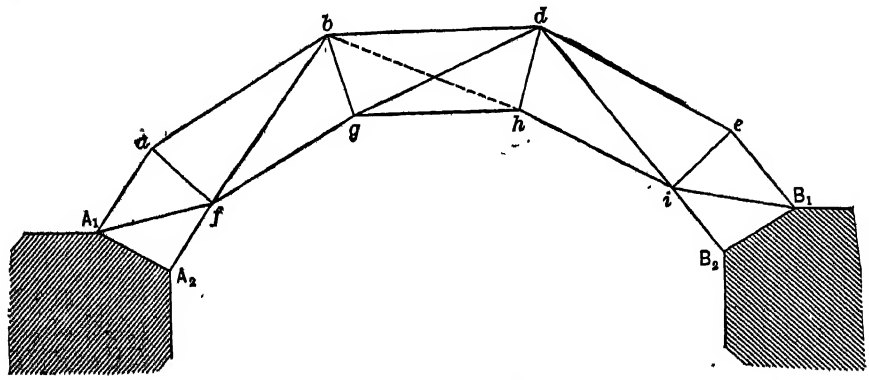
The value of H is the same in both cases whether the similarly placed load is on the right or left half.

From equations (5) and (6) we can find M_1, V_1 and H for a load anywhere on the *right-hand half* of a symmetrical arch. From equations (5), (7) and (8) we can then find V_1', H and M_1' at the left end A for a *similarly placed load anywhere on the left-hand half.*

We can then determine the stresses in the members for a load P on either half. For a first approximation we can take the cross-section a constant for all members, so that it cancels out in equations (5).

We can thus find the stress in each member for each load, and then by tabulation the maximum stress in each member for all the loads.

Example.—Let us take a circular arch, the radius of the outer chord being 44 ft. and of the inner chord 36 ft. The apices A_1, a, b, d, e, B_1 are on the outer circle, and A_2, f, g, h, i, B_2 on the inner circle. The bracing as shown.



The members af, bg, dh, ei are radial, and the chords ab, bd, de subtend an angle of 30° at the centre, while A_1a, eB_1 subtend an angle of 15°.

Considering the arch fixed at the right end and free at the left end, we can now find the stress u in every member due to a unit load at A_1; the stress m in every member due to a negative unit moment; the stress h in every member due to a negative unit horizontal force at A_1; and the stresses p_3, p_4 due to a unit load

at d and e on the right-hand half of the arch. We can then fill up the following table. The plus (+) sign for a stress denotes tension, and the minus (−) sign compression.

	s	u	h	m	p_2	p_4	mhs	m^2s
A_1a	11.4863	+ 1.5947	− 0.4357	− 0.1260	....	....	0.6306	0.1929
ab	22.7761	+ 1.6369	− 0.4472	− 0.1294	....	....	1.3164	0.3804
bd	22.7761	+ 3.7254	− 1.6930	− 0.1294	....	....	4.8718	0.3804
de	22.7761	+ 8.2254	− 0.4472	− 0.1294	+ 1.8205	....	1.3164	0.3804
eB_1	11.4863	+ 8.0149	− 0.4357	− 0.1260	+ 1.7736	− 0.7132	0.6306	0.1929
A_2f	9.3979	0	0	+ 0.1260			0	0.1579
fg	18.6349	− 3.4574	+ 2.6530	+ 0.1294	....	....	6.3974	0.3112
gh	18.6349	− 6.4048	+ 2.6530	+ 0.1294	....	....	6.3974	0.3112
hi	18.6349	− 6.4048	+ 2.6530	+ 0.1294	....	....	6.3974	0.3112
iB_2	9.3979	− 9.6085	+ 0.5043	+ 0.1260	− 3.3684	− 0.8815	0.5967	0.1579
A_3f	13.1128	− 1.0064	+ 1.3116	0	....	....	0	0
af	8	− 0.6317	+ 0.1726	+ 0.0499	....	....	0.0690	0.0200
fb	22.1004	+ 2.4768	− 1.4300	0	....	....	0	0
bg	8	− 2.8250	+ 1.3733	+ 0.0669	....	....	0.7349	0.0359
gd	22.1004	+ 2.8600	0	0	....	....	0	0
dh	8	− 3.3152	+ 1.3733	+ 0.0669	....	....	0.7349	0.0359
di	22.1004	− 2.4768	− 1.4300	0	− 2.4768	....	0	0
ei	8	− 3.1746	+ 0.1726	+ 0.0499	− 0.7026	....	0.0690	0.0200
iB_1	13.1128	+ 1.0064	+ 1.3116	0	+ 1.0064	+ 1.0064	0	0
							30.1625	2.8882

In the first column we have the members; in column two the lengths s of the members; in the third column the stresses u; in the fourth and fifth columns the stresses h and m; in the next two columns the stresses p_2 and p_4; finally, in the last two columns, the products mhs and m^2s. We see from the table that

$$\Sigma_0^l\, mhs = 30.1625 \quad \text{and} \quad \Sigma_0^l\, m^2s = 2.8882.$$

We have, then, from the first of equations (5), page 588,

$$\bar{x} = \frac{l}{2} = +\ 38.105 \text{ ft.}, \qquad \bar{y} = \frac{30.1625}{2.8882} = +\ 10.4433 \text{ ft.}$$

For the quantities $(u + \frac{l}{2}m)$, $(h - \bar{y}m)$, $(u + \frac{l}{2}m)^2s$, $(h - \bar{y}m)^2s$, pms, $p(u + \frac{l}{2}m)s$, $p(h - \bar{y}m)s$, we can now fill up the following table:

	$u + \frac{l}{2}m$	$h - \bar{y}m$	$(u + \frac{l}{2}m)^2s$	$(h - \bar{y}m)^2s$	p_2ms	p_4ms	$p_2(u + \frac{l}{2}m)s$	$p_4(u + \frac{l}{2}m)s$	$p_2(h - \bar{y}m)s$	$p_4(h - \bar{y}m)s$
A_1a	− 3.2065	+ 0.8801	118.0680	8.8968						
ab	− 3.2938	+ 0.9041	247.0949	18.6170						
bd	− 1.2053	− 0.3417	33.0878	2.6593						
de	+ 3.2947	+ 0.9041	247.2350	18.6170	− 5.3654	....	+136.6110	....	+ 37.4874	....
eB_1	+ 3.2947	+ 0.8801	124.6811	8.8968	− 2.5669	+ 1.0322	+ 67.1200	−26.9902	+ 17.9295	− 7.2098
A_2f	+ 4.8012	− 1.3158	216.6358	16.2710						
fg	+ 1.4733	+ 1.3017	40.4494	31.5760						
gh	− 1.4741	+ 1.3017	40.4930	31.5760						
hi	− 1.4741	+ 1.3017	40.4930	31.5760						
iB_2	− 4.8073	− 0.8115	217.6870	6.1888	− 3.9885	− 1.0437	+152.1792	+30.8248	+ 25.6887	+ 6.7227
A_3f	− 1.0064	+ 1.3116	13.2183	22.5580						
af	+ 1.2735	− 0.3495	12.9744	0.9772						
fb	+ 2.4768	− 1.4300	135.5730	45.1920						
bg	− 0.2758	+ 0.6747	0.6085	3.6418						
gd	+ 2.8600	0	180.7691	0						
dh	− 0.7660	+ 0.6747	4.6940	3.6418						
di	− 2.4768	− 1.4300	135.5730	45.1920	0	....	+135.5755	....	+ 78.2755	....
ei	− 1.2694	− 0.3495	12.8910	0.9772	− 0.3085	....	+ 7.1350	....	+ 0.1964	....
iB_1	+ 1.0064	+ 1.3116	13.2183	22.5580	0	0	+ 13.2811	+13.2811	+ 17.3087	+17.3087
			1835.5346	319.6127	−12.2293	− 0.0115	+511.9018	+26.1157	+176.8602	+16.8216

We have then, from equations (5), page 588 and the tables,

$$M_0 = P\frac{\Sigma_{kl}^l\, pms}{2.8882},$$

and hence for the apex loads P_3, P_4 at apices d, e

	apex d	apex e
$M_0 =$	$-4.2342\,P_3$	$-0.094\,P_4$

We have also

$$V_1 = P\frac{\Sigma_{kl}^{l}\left(u + \frac{l}{2}m\right)ps}{1835.5346},$$

and hence for the apex loads P_3 and P_4 at apices d, e

	apex d	apex e
$V_1 =$	$+0.2788P_3$	$+0.0142P_4$

We have also

$$H = P\frac{\Sigma_{kl}^{l}\ (h - \bar{y}m)ps}{319.6127}$$

and hence for the apex loads P_3 and P_4 at apices d, e

	apex d	apex e
$H =$	$+0.5534P_3$	$+0.0526P_4$

We have the same values of H for similarly placed loads at apices a, b. Hence

	apex a	apex b	apex d	apex e
$H =$	$+0.0526P_1$	$+0.5534P_2$	$+0.5534P_3$	$+0.0526P_4$

Also, from (8) we have

	apex a	apex b	apex d	apex e
$V_1 =$	$+0.9858P_1$	$+0.7212P_2$	$+0.2788P_3$	$+0.0142P_4$

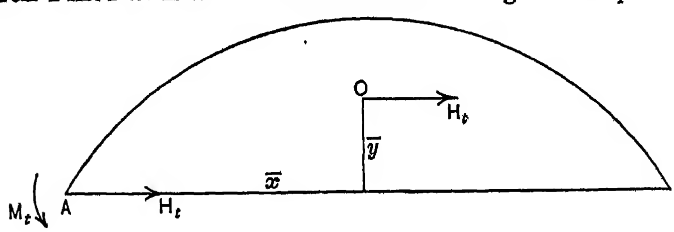
From (6) and (7) we now have

	apex a	apex b	apex d	apex e
$(l - kl) =$	6.9925	26.7172	....	....
$M_1 =$	$+5.8982P_1$	$+6.0801P_2$	$+0.6101P_3$	$-0.0122P_4$

We can now find the stresses in each member for each apex load, and then by tabulation find the maximum stress in each member.

Temperature Stress—Framed Arch Fixed at Ends.—The effect of a change of temperature is to cause a horizontal thrust H_t at the point O, or a horizontal thrust H_t and moment M_t at the left end A, where

$$M_t = -H_t\bar{y}. \quad . \quad . \quad (1)$$



For the moment at the point of moments for any member we have

$$M = H_t(y - \bar{y}).$$

We have then for the work of straining all the members

$$\text{work} = \sum_0^l \frac{H_t^2(y - \bar{y})^2 s}{2Eav^2}.$$

If ϵ is the coefficient of expansion and t the number of degrees rise or fall of temperature, the change of length of the span is ϵlt. The work is then

$$\text{work} = \frac{H_t\epsilon lt}{2}.$$

Hence

$$\frac{H_t\epsilon lt}{2} = \sum_0^l \frac{H_t^2(y - \bar{y})^2 s}{2Eav^2},$$

or

$$H_t = \frac{E\epsilon lt}{\sum_0^l \frac{(y-\bar{y})^2 s}{av^2}},$$

or, just as on page 588,

$$H_t = \pm \frac{E\epsilon lt}{\sum_0^l (h - \bar{y}m)^2 \frac{s}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Where considering the arch fixed at the right end and free at the left, h is the stress in any member for a negative unit horizontal force at the left end A, and m the stress in any member due to a negative unit moment at the point of moments for that member.

The summation $\sum_0^l (h - \bar{y}m)^2 \frac{s}{a}$ is made as in the preceding example. The plus $(+)$ sign is taken for rise of temperature, and the minus $(-)$ sign for fall of temperature.

The values of ϵ are given on page 582, and of E on page 478. The value of M_t at the left end A is given by (1) when H_t is known.

Equation (2) requires that the areas a of the members should be known in advance. We must then in general assume a constant value for a in (2) equal, say, to the section at the crown. Denote this assumed constant cross-section by a_0. We have then for our first calculation

$$H_t = \pm \frac{E\epsilon a_0 lt}{\sum_0^l (h - \bar{y}m)^2 s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Example.—In the preceding example, page 589, let the cross-section at crown be $a = 2$ square inches. Let the arch be of steel, and let us take $E = 30\,000\,000$ pounds per square inch. Let the change of temperature be $t = 40°$.

Then from the table page 590 we have $\sum_0^l (h - \bar{y}m)^2 s = 319.6127$, and since $l = 76.21$ ft., we have for $\epsilon = 0.00\,000\,599$, from (3),

$$H_t = \pm\ 3428 \text{ pounds},$$

and from (2), since $\bar{y} = 10.4433$ feet,

$$M_t = \mp\ 35800 \text{ pound-feet}.$$

Solid Arch—Fixed at Ends.—If the arch is a solid beam, we can put in equations (4), page 588, ds for s and I for av^2, where I is the moment of inertia of the cross-section. Hence if x and y are the co-ordinates of the neutral axis, we have for a load P anywhere on a symmetrical arch

$$\left.\begin{aligned}
M_0\int_0^l \frac{ds}{I} &= -P\int_{kl}^l \frac{(x-kl)ds}{I}, \\
V_1\int_0^l \left(x-\frac{l}{2}\right)^2\frac{ds}{I} &= P\int_{kl}^l \left(x-\frac{l}{2}\right)(x-kl)\frac{ds}{I}, \\
H\int_0^l (y-\bar{y})^2\frac{ds}{I} &= -P\int_{kl}^l (y-\bar{y})(x-kl)\frac{ds}{I}, \\
\bar{y} &= \frac{\int_0^l \frac{yds}{I}}{\int_0^l \frac{ds}{I}}.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

These equations are general. If the moment of inertia I is constant, it cancels out.

Instead of performing the integrations indicated in equations (1), we can in any case divide the neutral axis into a number n of equal segments of length s. We have then, since s is constant,

$$M_0 = -P\frac{\sum_{kl}^{l}\frac{(x-kl)}{I}}{\sum_0^l\frac{1}{I}},$$

or, if I is constant,

$$M_0 = -P\frac{\sum_{kl}^{l}(x-kl)}{n},$$

where n is the number of segments;

$$V_1 = P\frac{\sum_{kl}^{l}\frac{\left(x-\frac{l}{2}\right)(x-kl)}{I}}{\sum_0^l\frac{\left(x-\frac{l}{2}\right)^2}{I}},$$

$$H = -P\frac{\sum_{kl}^{l}\frac{(y-\bar{y})(x-kl)}{I}}{\sum_0^l\frac{(y-\bar{y})^2}{I}}.$$

If I is constant, it cancels out in these two equations.

$$\bar{y} = \frac{\sum_0^l\frac{y}{I}}{\sum_0^l\frac{1}{I}},$$

or, if I is constant,

$$\bar{y} = \frac{\Sigma_0^l y}{n},$$

where n is the number of segments. . . . . (2)

From (1) or (2), then, we can find M_0 and V_1 and H at the left end A of the neutral axis for a load P anywhere on the right-hand half of a symmetrical arch. We have then for the moment M_1 at the left end for a load on the right-hand half, just as on page 589,

$$M_1 = -H\bar{y} + \frac{V_1 l}{2} + M_0, \quad\quad (3)$$

and for the moment M_2 at the right end, or for a similarly placed load on the left-hand half, the moment M_1' at the left end,

$$M_1' = M_1 - V_1 l + Pl(1-k). \quad\quad (4)$$

The reaction V_1' at the left end for a similarly placed load on the left-hand half is

$$V_1' = P - V_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The value of H is the same in both cases, whether the similarly placed load is on the right- or left-hand half.

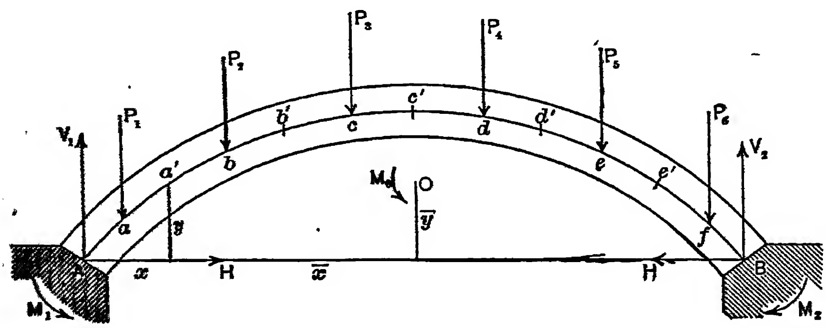
From equations (1) or (2) and (3) we can find, then, M_1, V_1 and H at the left end for a load anywhere on the right-hand half of a symmetrical solid arch. From (1) or (2) and (4) and (5) we can find M_1', V_1' and H at the left end for a load anywhere on the left-hand half.

We can now find the moment M at any point of the neutral axis for a load anywhere. Then by tabulation we can find the loading which gives the maximum moment at any point of the neutral axis, and this maximum moment $M_{max.}$ itself. We have then, from equation (II), page 497,

$$I = \frac{144 M_{max.} v}{S_f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $M_{max.}$ is in *pound-feet*, S_f is the allowable unit stress in *pounds per square inch* in the most remote fibre at a distance v *in feet*, and I is given for dimensions in inches.

Example.—Let us take a circular arch of radius of neutral axis 40 ft. and central angle 120°, so that the span $l = 69.282$ ft. and rise $r = 20$ ft. We first suppose I constant, so that it cancels out.



Divide the neutral axis into a sufficiently large number of equal segments, say six, so that the length of a segment is $s = 13.963$ ft., and let the end segments Aa and fB be each *one half of s*.

Let the ordinates to the middle points a', b', c', etc., of each segment for origin at A be x, y, and take the loads P_1, P_2, etc., acting half way between A and a', a' and b', etc.

We can now draw up the following table.

	x	y	$x-\frac{l}{2}$	$\left(x-\frac{l}{2}\right)^2$	$y-\bar{y}$	$(y-\bar{y})^2$
Aa	1.875	2.943	−32.766	1073.611	−10.291	105.905
ab	8.929	10.642	−25.711	661.081	−2.592	6.718
bc	20.960	17.588	−13.681	187.170	+4.354	18.957
cd	34.641	20	0	0	+6.766	45.779
de	48.322	17.588	+13.681	187.170	+4.354	18.957
ef	60.3525	10.642	+25.711	661.081	−2.592	6.718
fB	67.407	2.943	+32.766	1073.611	−10.291	105.905
		79.403		2770.113		203.034

We have then $\Sigma_0^l y = 79.403$. Note that in taking this summation and the other summations of the table, since the end segments Aa and fB are only half length, we take in the summations *one half the values for Aa and fB*.

We have then

$$\bar{y} = \frac{79.403}{6} = 13.234 \text{ ft.},$$

and can fill out the last two columns.

For the values of kl, $(1 - k)$ and $l(1 - k)$ for each load we have now

	P_1	P_2	P_3	P_4	P_5	P_6
$kl =$	4.465	14.945	27.8	41.481	54.337	64.817
$1 - k =$	0.935	0.784	0.599	0.401	0.216	0.065
$l(1 - k) =$	64.817	54.337	41.481	27.8	14.945	4.465

We can now draw up the following table for loads P_4, P_5, P_6.

	P_4			P_5			P_6		
	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$
de	+ 6.841	+ 93.592	+ 29.786						
ef	+18.872	+485.237	− 48.916	+ 6.016	+154.683	− 15.593			
fB	+25.926	+849.491	+266.802	+13.070	+428.252	−134.503	+ 2.590	+ 84.864	− 26.654
	+38.670	+1003.574	−152.531	+12.551	+368 809	− 82.845	+ 1.295	+ 42.432	− 13.327

Note that in taking the summations, since fB is of half length, *we take one half* the values for fB in summing up.

We have then from these tables and equations (2), page 593, and (3), (4), (5), page 594, for loads P_4, P_5, P_6:

$$\begin{aligned}
M_0 &= -6.446P_4, & &-2.092P_5, & &-0.216P_6;\\
V_1 &= +0.3623P_4, & &+0.1331P_5, & &+0.0153P_6;\\
H &= +0.7513P_4, & &+0.4081P_5, & &+0.0658P_6;\\
M_1 &= -3.838P_4, & &-2.882P_5, & &-0.558P_6;\\
M_1' &= -1.138P_4, & &+2.842P_5, & &+2.846P_6;\\
V_1' &= 0.6377P_4, & &+0.8669P_5, & &+0.9847P_6.
\end{aligned}$$

Hence we have at the left end A for each load:

$$\begin{aligned}
M_1 &= +2.846P_1, & &+2.842P_2, & &-1.138P_3, & &-3.838P_4, & &-2.882P_5, & &-0.558P_6;\\
V_1 &= +0.9847P_1, & &+0.8669P_2, & &+0.6377P_3, & &+0.3623P_4, & &+0.1331P_5, & &+0.0153P_6;\\
H &= +0.0658P_1, & &+0.4081P_2, & &+0.7513P_3, & &+0.7513P_4, & &+0.4081P_5, & &+0.0658P_6.
\end{aligned}$$

We can now find the moment M at the centre of each segment for each load. Then by tabulation we can find the maximum moment $M_{max.}$ at each of these points. Finally, from equation (5), page 594, we can find I at each of these points.

Temperature Thrust—Solid Arch Fixed at Ends.—We have, as on page 591, for a framed arch

$$\frac{H_t \epsilon l t}{2} = \sum_0^l \frac{H_t^2 (y - \bar{y})^2 s}{2Eav^2}.$$

For a solid arch we can put ds for s, and the moment of inertia I for the cross-section in place of av^2. Hence

$$H_t = \pm \frac{E\epsilon l t}{144 \int_0^l \frac{(y-\bar{y})^2 ds}{I}}, \qquad (1)$$

where l, y, ds are in feet, E in pounds per *square inch* and I is given for dimensions *in inches*.

Instead of performing the integration we can, as in the preceding example, divide the neutral axis into a number of equal parts of length s. We have then

$$H_t = \pm \frac{E\epsilon lt}{144 \sum_0^l \frac{(y-\bar{y})^2 s}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the moment at the left end we have, as on page 591,

$$M_t = -H_t \bar{y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let I_0 be the moment of inertia at the crown, then for our first calculation we have

$$H_t = \pm \frac{\frac{E\epsilon I_0 lt}{I}}{144 \sum_0 (y-\bar{y})^2 s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Example.—In the example page 594 let the moment of inertia at the crown be 2000 in.^{4}. Let the arch be of steel, and let us take $E = 30\,000\,000$ pounds per square inch. Let the change of temperature be $t = 40°$.

Then from the preceding example we have $\sum_0^l (y-\bar{y})^2 s = 4714.18$, and since $l = 69.28$ ft., we have, from (4), for $\epsilon = 0.00000599$

$$H_t = \pm 1467 \text{ pounds}.$$

Since $\bar{y} = 10.38$ ft., we have, from (3),

$$M_t = \mp 15227 \text{ pound-ft.},$$

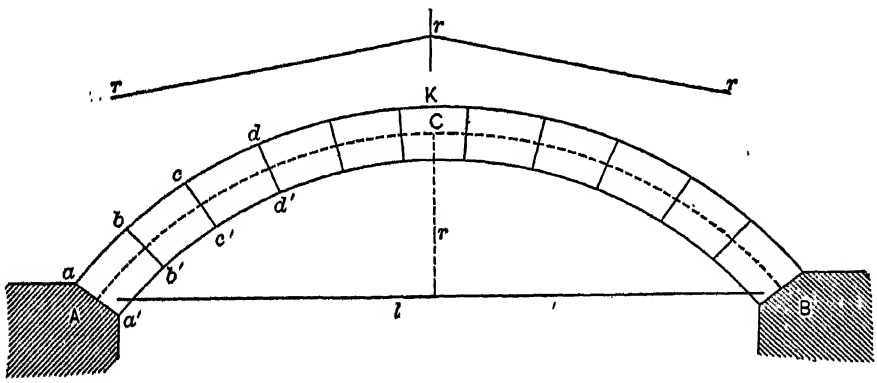
the top signs being taken for expansion and the bottom signs for contraction.

CHAPTER X.

THE STONE ARCH.

Definitions.—The stone arch consists of a number of *arch-stones* or *voussoirs* which press upon each other. The central one of these is the keystone. The *extrados* is the exterior outline of the arch proper. The *intrados* is the interior line, and the corresponding surface of the arch is the *soffit*. The sides of the arch are the *haunches*, and the spaces above are the *spandrels*. The ends of the arch or the area between intrados and extrados are the *faces*. The inclined surfaces or joints upon which the arch rests at the ends are the *skewbacks*. The permanent load supported by the arch in addition to its own weight is the *surcharge*. The masonry or other material which supports two successive arches is the pier; at the extreme ends this is the abutment.

Thus in the figure $cdd'c'$ is a voussoir or arch-stone, and k is the keystone. The line $abcd$, etc., is the extrados, and $a'b'c'd'$, etc., the intrados, and the interior surface corresponding the soffit. The line rrr marks the upper limit of the surcharge. Between this



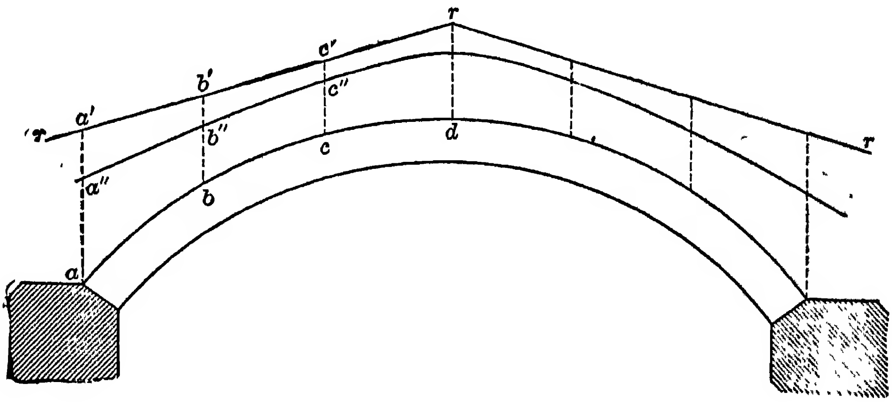
and the haunches on either side is the spandrel space, and the material with which this space is filled is the spandrel filling or surcharge. The area between the extrados $abcd$, etc., and the intrados $a'b'c'd'$, etc., is the face, and aa' is the skewback. At A and B we have the abutments or piers. Arch and abutments or piers are stone. The surcharge may be stone or filled in with rubble or lighter material.

The upper limit of surcharge may be level or on any desired grade. The extrados and intrados may be circular, elliptic or any desired curve, and may or may not be parallel. Often the depth at key is less than at ends.

In all investigations and calculations we suppose the width to be one foot and the effect of mortar between the joints is disregarded. The neutral axis ACB or centre line is the curve passing through the centre of the voussoirs. The rise r of this axis is the rise of the arch, and the span l of this axis is the span of the arch.

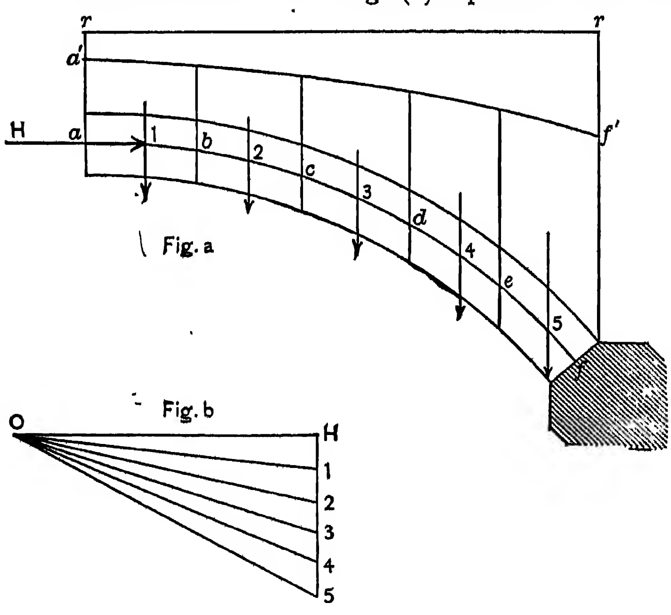
Reduced Surcharge.—The arch proper is constructed of cut stone. The material above it may also be of stone or of some lighter material. The density or weight per cubic foot of the surcharge is then in general less than the density of the arch proper.

Thus in the figure let, *abcd*, etc., be the extrados, and *rrr* the roadway. Between the roadway and the extrados the surcharge may have a less density than for the arch proper. Say, for instance, that the density of the surcharge is $\frac{2}{3}$ of that of the arch. Then if we



draw verticals aa', bb', cc', etc., and lay off aa'' equal to $\frac{2}{3}$ of aa', bb'' equal to $\frac{2}{3}$ of bb' and so on, we obtain the line $a''b''c''$, etc., which marks the limit of the *reduced surcharge*. We can then treat and discuss all the area below this line as if it were homogeneous and of the same density as the arch itself.

Pressure Curve.—Let Fig. (*a*) represent one half an arch with its surcharge *rr*, and $a'f'$ the line of reduced surcharge, so that all the area below can be considered as homogeneous and of the same density as the arch.



Let us take the width as one foot, and divide the area into a suitable number of slices by vertical lines. In this case we have five. The weight and centre of mass of each slice can be found. Let 1, 2, 3, 4, 5 represent the weights acting at the centres of mass. Let *H* be the horizontal thrust at the crown due to the pressure of the other half of the arch. Let the magnitude and point of action *a* of *H* be known. In Fig. (*b*) lay off the weights 1, 2, 3, 4, 5 to scale, let *OH* be the known thrust to scale, and draw *O*1, *O*2, *O*3, *O*4, *O*5. Then *O*1 is the resultant of *H* and weight 1; *O*2 is the resultant of *O*1 and weight 2; *O*3 is the resultant of *O*2 and weight 3, and so on.

In Fig. (*a*) produce *H* acting at *a* till it meets weight 1. From 1 draw 1–2 parallel to *O*1 till it meets weight 2; from 2 draw 2–3 parallel to *O*2 till it meets weight 3, and so on

We thus have a polygon $a\ 1\ 2\ 3\ 4\ 5\ f$, each segment of which is in the direction of the resultant of the forces acting at its end. Thus the resultant of H and weight 1 is in the direction 1–2, the resultant of $O1$ and weight 2 is in the direction 2–3 and so on.

As we increase the number of divisions this polygon approaches a curve tangent at the points of division a, b, c, d, e, f. This curve is the *curve of pressures* in the arch.

Conditions of Stability of the Arch.—The conditions for stability of the arch are the same as for walls, page 425. Thus:

1st. The joints of the voussoirs should be so arranged that the tangent to the curve of pressure at each joint shall make an angle less than the angle of friction with the normal to the joint, otherwise there is danger of sliding.

2d. The curve of pressures must lie entirely within the arch, otherwise there is danger of rotation.

3d. The curve of pressure must not approach too near the edge of a joint, otherwise there is danger of crushing.

Let d be the depth of any joint, N the normal pressure on the joint, and e the *least* edge distance of N. Then for a width of one foot we have, as already shown page 426, for the maximum unit pressure p,

$$\left.\begin{array}{ll} \text{when } e \text{ is greater than } \frac{1}{3}d, & p = \frac{2N}{d}\left(2 - \frac{3e}{d}\right); \\ \text{when } e = \frac{1}{3}d, & p = \frac{2N}{d} \\ \text{when } e \text{ is less than } \frac{1}{3}d, & p = \frac{2N}{3e}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In any case the value of p must not exceed the allowable compressive unit stress C, which for stone may be taken at the average value of 25 tons per square foot or 50000 pounds per square foot. (See table page 424.)

VALUE OF N.—If we make the joints nearly at right angles to the curve of pressure, the first condition of stability is complied with, and we have with sufficient accuracy, if we denote by P_n the sum of all the loads between the crown and any joint on the right,

$$N = \sqrt{H^2 + P_n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

VALUE OF e.—If M is the moment at any point of the neutral axis, then $\frac{M}{N}$ is the distance of N from that point. If we subtract this from $\frac{d}{2}$, we have for the edge distance e' *from the intrados*, with sufficient accuracy,

$$e' = \frac{d}{2} - \frac{M}{N}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If e' at any joint is negative or is positive and greater than d, the equilibrium curve runs outside of the arch.

If M is negative, e' is greater than $\frac{d}{2}$ and the *least* edge distance is

$$e = d - e'.$$

If M is positive, e' is less than $\frac{d}{2}$ and the *least* edge distance is

$$e = e'.$$

We see, then, that if in any case we can find the values of H and the moment M at any point of the axis, we can find the normal pressure N at any joint, the distance of the normal pressure from the intrados and the maximum unit pressure p on the joint.

If the normal pressure at every joint is within the arch ring, and p does not exceed the allowable unit stress, and the joints are taken perpendicular or nearly so to the pressure line, the arch is stable at every point.

Determination of H and M.—From equations (2), page 593, we have already for a solid arch fixed at the ends

$$\bar{y} = \frac{\sum_0^l \frac{y}{I}}{\sum_0^l \frac{1}{I}},$$

or, if I is constant,

$$\bar{y} = \frac{\sum_0^l y}{n},$$

where n is the number of segments;

$$M_0 = -P\frac{\sum_{kl}^l \frac{(x-kl)}{I}}{\sum_0^l \frac{1}{I}},$$

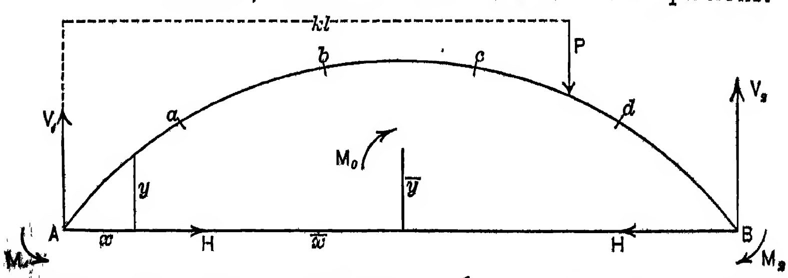
or, if I is constant,

$$M_0 = -P\frac{\sum_{kl}^l (x-kl)}{n}, \qquad \ldots \ldots \ldots (4)$$

$$V_1 = P\frac{\sum_{kl}^l \frac{\left(x-\frac{l}{2}\right)(x-kl)}{I}}{\sum_0^l \frac{\left(x-\frac{l}{2}\right)^2}{I}},$$

$$H = -P\frac{\sum_{kl}^l \frac{(y-\bar{y})(x-kl)}{I}}{\sum_0^l \frac{(y-\bar{y})^2}{I}}.$$

If I is constant, it cancels out in the last two equations. In these equations I is the moment of inertia of the cross-section, H the horizontal thrust, V_1 the reaction at the left end A of the neutral axis for a load P at a distance kl from A. The neutral axis is divided into a number n of equal segments, Aa, ab, etc., of length s, and x, y are the co-ordinates of the middle point of a segment for origin at A. The load P acts half way between the ends of a segment.



We have then for the moment M_1 at the left end A for a load P on the right-hand half, just as on page 593,

$$M_1 = -H\bar{y} + \frac{V_1 l}{2} + M_0, \quad \ldots \quad (5)$$

and for the moment M_1' at the left end for a similarly placed load on the left-hand half

$$M_1' = M_1 - V_1 l + Pl(1 - k). \quad \ldots \quad (6)$$

The reaction V_1' at the left end for a similarly placed load on the left-hand half is

$$V_1' = P - V_1. \quad \ldots \quad (7)$$

The value of H is the same in both cases, whether the similarly placed load is on the right- or the left-hand half.

From equations (4) we can find M_0 and V_1 and H at the left end for a load on the right-hand half, and then from (5), (6) and (7) can find H and M_1 and V_1 at the left end for each load. By summation, then, we can find ΣM_1, ΣV_1 and ΣH at the left end for all the loads.

For the moment at any point of the neutral axis we have then

$$M = \Sigma M_1 - x\Sigma V_1 + y\Sigma H + \Sigma_0^x P(x-)kl. \quad \ldots \quad (8)$$

This is the value of M to be used in equation (3), page 599, and ΣH is the value of H to be used in equation (2), page 599. We thus finally have N and e, and then from equations (1), page 599, can see if the maximum unit pressure p is less than the allowable unit stress at any joint.

It will generally be necessary to first assume some constant depth, so that I is constant and cancels out of equations (4). We can then determine for this assumed case H, e and p at the crown and N, e and p at the skewback. If p at either point is too great, we can increase the depth assumed; if very small, we can decrease it. If H or N takes effect outside of the middle third, then, as we have seen page 426, the entire joint is not effective and we can assume a depth of $3e$ at crown and skewback. We thus obtain proper depths at crown and skewback and can then make a second and final calculation.

In many cases much computation can be avoided by drawing the arch and reduced surcharge to scale and taking off many of the required quantities directly from the drawing. Having found H and e at the crown, we can then construct the curve of pressures as in the figure page 598.

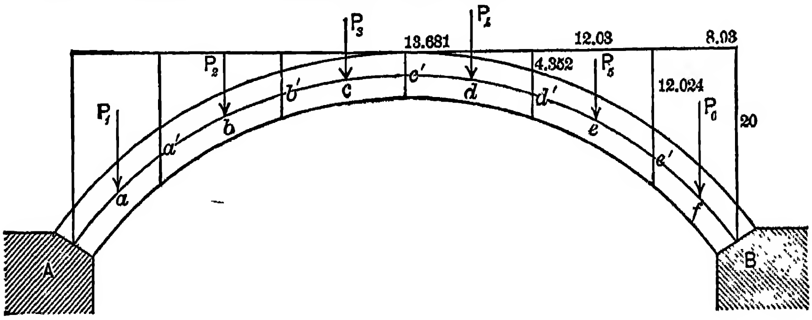
The method of calculation thus outlined adapts itself to any shape of arch with any surcharge. Its detailed application will be perfectly illustrated by the following examples.

Examples.—(1) *Required to design a circular arch, radius of neutral axis 40 ft. and central angle 120°, so that $l = 69.282$ ft. The allowable unit stress to be 25 tons per square foot, and the surcharge level with the crown of extrados and of the same material as the arch, density of both 165 lbs. per cubic foot.*

Since we first assume a constant depth, I cancels out in equations (4), page 600, and we proceed precisely as for the solid arch of the same dimensions given in the example page 594. It will not be necessary, therefore, to repeat here this portion of the calculation. Referring to the results there obtained, we have for the values of M_1, V_1, H for loads P_1, P_2, etc.,

$M_1 =$	$+2.846P_1$	$+2.842P_2$	$-1.138P_3$	$-3.838P_4$	$-2.882P_5$	$-0.558P_6$
$V_1 =$	$+0.9847P_1$	$+0.8669P_2$	$+0.6377P_3$	$+0.3623P_4$	$+0.1331P_5$	$+0.0153P_6$
$H =$	$+0.0658P_1$	$+0.4081P_2$	$+0.7513P_3$	$+0.7513P_4$	$+0.4081P_5$	$+0.0658P_6$

Let us first assume the constant depth at 2 feet. If we take one foot width, we have for the loads



$P_1 = P_6 = 25966,$

$P_2 = P_5 = 16253,$

$P_3 = P_4 = \ \ 7169$ pounds.

Hence

$\Sigma M_1 = +\ 23087$ pound-feet,

$\Sigma V_1 = +\ 49388$ pounds,

$\Sigma H = +\ 27455$ pounds.

For the values of $\sum_0^x P(x - kl)$ we have for the points of division A, a', b', c', d', e', B

	A	a'	b'	c'
$\sum_0^x P(x-kl)$	$= 0$	$4.464P_1$	$16.495P_1 + 6.015P_2$	$30.176P_1 + 19.696P_2 + 6.841P_3$
	$= 0$	115912	526071	1152712

	d'	e'
$\sum_0^x P(x-kl)$	$= 43.857P_1 + 33.377P_2 + 27.363P_3$	$55.881P_1 + 51.422(P_2 + P_3)$
	$=$ 1877432	2655412

	B
$\sum_0^x P(x-kl)$	$= 69.282(P_1 + P_2 + P_3)$
	$=$ 3421700

We have then from equation (8), page 601, for the moment at any point of the axis

$$M = +\ 23087 - 49388x + 27455y + \sum_0^x P(x - kl).$$

For the points c', d', e', B we have then

	c'	d'	e'	B
$x =$	34.641	48.322	60.3525	69.282 ft.
$y =$	20	17.588	10.642	0 ft.
$M =$	+ 14050	− 3130	− 10014	+ 23087 pound-feet.

From equation (2), page 599, we have then

	c'	d'	e'	B
$N =$	27455	28375	36088	56506 pounds.

For the edge distance of N *from the intrados* we have now, from equation (3), page 599,

	c'	d'	e'	B
$e' =$	0.488	1.11	1.27	0.592 ft.,

and hence the *least edge distance* of N at each point is

	c'	d'	e'	B
$e =$	0.488	0.99	0.73	0.592 ft.

We have then, from equations (1), page 599, for the maximum unit pressure p

	c'	d'	e'	B
p =	37507	14755	32840	63633 pounds per square foot.

Taking the allowable unit stress at 25 tons or 50000 pounds per square ft., we see that this is exceeded only at B. Also, we see that the least edge distance at crown and springing is less than one third the depth. We ought, then, to have the depth at springing greater than 2 ft. At the crown we can have a less depth than 2 ft. if we wish. As we have seen, page 426, if N is outside of the middle third, the entire joint is not brought into action. If then we take the depth at springing B equal to $\frac{2N}{p} = \frac{2 \times 56506}{50000} = 2.26$ ft., the whole joint there will act and the allowable unit stress not be exceeded. We may take this constant depth, or take the depth at crown equal to $\frac{2N}{p} = \frac{2 \times 27455}{50000} = 1.1$ ft. The loads P_1, P_2 etc., will now be somewhat changed. To allow for this let us take, say, a uniform depth of 2.5 ft. Or, if preferred, we may take 2.5 ft. at springing and, say, 1.5 ft. at crown. In the latter case I will vary. The calculation can now be repeated to be sure that the allowable unit stress is not exceeded at any joint, and that the curve of pressures does not pass outside of the middle third.

(2) *In the preceding example suppose, in addition to the surcharge, a moving load of 8000 pounds per lineal foot moves over the arch, the width of arch being 20 feet.*

We still have, just as before, for the values of M_1, V_1, H at the left end for loads P_1, P_2, etc.,

$$M_1 = +2.846P_1 + 2.842P_2 - 1.138P_3 - 3.838P_4 - 2.882P_5 - 0.558P_6$$
$$V_1 = +0.9847P_1 + 0.8669P_2 + 0.6377P_3 + 0.3623P_4 + 0.1331P_5 + 0.0153P_6$$
$$H = +0.0658P_1 + 0.4081P_2 + 0.7513P_3 + 0.7513P_4 + 0.4081P_5 + 0.0658P_6$$

Since the moving load is 8000 pounds per lineal ft. and width 20 ft., we have $\frac{8000}{20} = 400$ pounds per lineal ft. for width of one foot. Hence the live loads are

$$P_1 = P_6 = 8.93 \times 400 = 3572 \text{ pounds},$$
$$P_2 = P_5 = 12.03 \times 400 = 4812 \text{ pounds},$$
$$P_3 = P_4 = 13.681 \times 400 = 5472 \text{ pounds}.$$

Let us find the moment at c' and B due to each of these loads.

For the moment at c' we have for any load on left of c', if r is the rise of the neutral axis,

$$M = M_1 - V_1\frac{l}{2} + Hr + P\left(\frac{l}{2} - kl\right),$$

and for any load on right of c'

$$M = M_1 - \frac{V_1 l}{2} + Hr.$$

For the moment at B we have for any load

$$M = M_1 - V_1 l + P(l - kl).$$

We have then

at the crown c'

$$M = +2.846P_1 - 0.9847P_1 \times 34.641 + 0.0658P_1 \times 20 + P_1 \times 30.176 = +810 \text{ pound-feet},$$
$$M = +2.842P_2 - 0.8669P_2 \times 34.641 + 0.4081P_2 \times 20 + P_2 \times 19.696 = +3220 \quad ''$$
$$M = -1.138P_3 - 0.6377P_3 \times 34.641 + 0.7513P_3 \times 20 + P_3 \times 6.841 = -7450 \quad ''$$
$$M = -3.838P_4 - 0.3623P_4 \times 34.641 + 0.7513P_4 \times 20 = -7450 \quad ''$$
$$M = -2.882P_5 - 0.1331P_5 \times 34.641 + 0.4081P_5 \times 20 = +3220 \quad ''$$
$$M = -0.558P_6 - 0.0153P_6 \times 34.641 + 0.0658P_6 \times 20 = +810 \quad ''$$

at the springing B

$$M = +2.846P_1 - 0.9847P_1 \times 69.282 + P_1 \times 64.817 = -\ \ 1996 \text{ pound-feet,}$$
$$M = +2.842P_2 - 0.8669P_2 \times 69.282 + P_2 \times 54.337 = -13866 \quad ''$$
$$M = -1.138P_3 - 0.6377P_3 \times 69.282 + P_3 \times 41.482 = -20997 \quad ''$$
$$M = -3.838P_4 - 0.3623P_4 \times 69.282 + P_4 \times 27.8 \quad = -\ \ 6232 \quad ''$$
$$M = -2.882P_5 - 0.1331P_5 \times 69.282 + P_5 \times 14.945 = +13674 \quad ''$$
$$M = -0.558P_6 - 0.0153P_6 \times 69.282 + P_6 \times \ 4.465 = +10169 \quad ''$$

Since for the surcharge alone we have at c', $M = +14050$, we see that there can never be a negative moment at c', and the maximum moment at c' is when the live loads P_1, P_2, P_5, P_6 only act together with the surcharge, and is

$$M_{\text{max.}} \text{ at } c' = +14050 + 8060 = +22110 \text{ pound-feet.}$$

Also, since for the surcharge alone we have at B, $M = +23087$, we see that the maximum moment at B is when the live loads P_5 and P_6 act and is given by

$$M_{\text{max.}} \text{ at } B = 23087 + 23843 = +46930 \text{ pound-feet.}$$

When P_1, P_2, P_5, P_6 act together with the surcharge we have

$$H = +27455 + 3927 = +31382 \text{ pounds.}$$

Hence the edge distance of H from the intrados at the crown is, from equation (3), page 599, $e' = 1 - \frac{22110}{31382} = 0.7$. This is the least edge distance, and hence, from equations (1), page 599, we have at the crown

$$p = 29813 \text{ pounds per square foot.}$$

When P_5 and P_6 act together with the surcharge we have

$$H = +27455 + 2198 = +29653 \text{ pounds.}$$

Hence, from equation (2), page 599,

$$N = \sqrt{29653^2 + 57772^2} = 64937 \text{ pounds.}$$

The edge distance of N from the intrados at B is, from equation (3), page 599,

$$e' = 1 - \frac{46930}{64937} = 0.72.$$

This is the least edge distance e, and hence, from equations (1), page 599, we have at the springing B

$$p = 59742 \text{ pounds per square foot.}$$

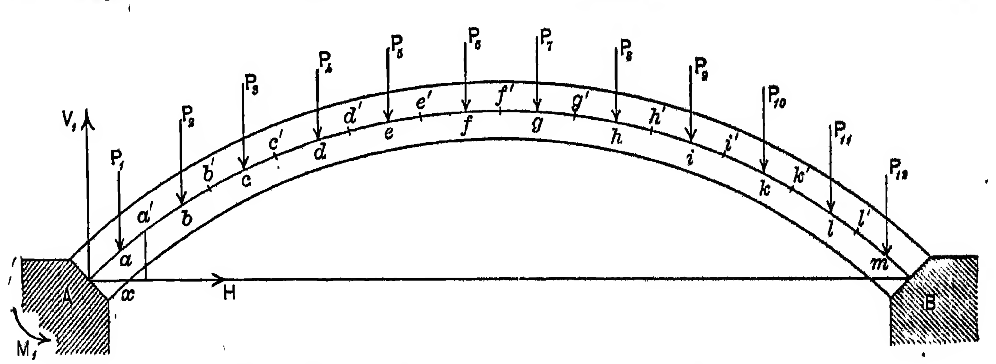
Taking the allowable unit stress at 50000 pounds per square foot, we see that this is exceeded at B.

The dimensions chosen in the preceding example will be ample for both surcharge and live load.

(3) *Investigate the conditions of stability for a circular stone arch, rise of the intrados 35 ft., span of the intrados 140 ft., uniform depth 2.5 ft., surcharge level with the crown of extrados and of the same material as the arch, density of both 160 lbs. per cubic foot.*

This arch was actually erected, and fell on the removal of the centre, the crown rising. Show that this might have been anticipated, and design the arch so as to be stable.

Since the depth is constant, I cancels out in equations (4), page 600, and we proceed precisely as for the



solid arch given in the example, page 594. Let us divide the neutral axis into twelve equal segments ab, bc, cd, etc., of length s, and let the end segments Aa and Bm be one half of s. Let the ordinates to the middle

points a', b', c', etc., of each segment for origin at A be x, y, and take the loads P_1, P_2, etc., acting half way between A and a', a' and b', etc.

We have then the following table.

	x	y	$x - \frac{l}{2}$	$\left(x - \frac{l}{2}\right)^2$	$y - \bar{y}$	$(y - \bar{y})^2$
Aa	2.11	2.70	− 68.89	4746.0321	− 20.69	428.0761
ab	9.04	10.29	− 61.96	3839.0416	− 13.10	171.6100
bc	19.56	19.07	− 51.44	2646.0736	− 4.32	18.6624
cd	31.31	26.13	− 39.69	1575.2961	+ 2.74	7.5076
de	44.00	31.29	− 27.00	729.0000	+ 7.90	62.4100
ef	57.34	34.44	− 13.66	186.5956	+ 11.05	122.1025
fg	71.00	35.50	0	0	+ 12.11	146.6521
gh	84.66	34.44	+ 13.66	186.5956	+ 11.05	122.1025
hi	98.00	31.29	+ 27.00	729.0000	+ 7.90	62.4100
ik	110.69	26.13	+ 39.69	1575.2961	+ 2.74	7.5076
kl	122.44	19.07	+ 51.44	2646.0736	− 4.32	18.6624
lm	132.96	10.29	+ 61.96	3839.0416	− 13.10	171.6100
mB .	139.89	2.70	+ 68.89	4746.0321	− 20.69	428.0761
		280.64		22698.0459		1339.3032

We have then $\sum_0^l y = 280.64$. Note that in taking this summation and the other summations of the table, since the end segments Aa and mB are only half length, we take in the summations *one half the values* for Aa and mB.

We have then

$$\bar{y} = \frac{280.64}{12} = 23.39 \text{ ft.},$$

and can now fill out the last two columns.

For the values of kl, $(1 - k)$ and $l(1 - k)$ for each load we have

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$kl =$	4.52	14.30	25.435	37.655	50.67	64.17	77.83	91.33	104.345	116.565	127.70	137.48
$l(1 - k) =$	137.48	127.70	116.565	104.345	91.33	77.83	64.17	50.67	37.655	25.435	14.30	4.52
$1 - k =$	0.968	0.899	0.821	0.734	0.643	0.548	0.452	0.357	0.266	0.179	0.101	0.032

We can now draw up the following table for the loads P_7 to P_{12}.

	P_7			P_8			P_9		
	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$
gh	6.83	93.2978	+ 75.4712						
hi	20.17	544.5900	+ 159.3430	6.67	180.0900	+ 52.6930			
ik	32.86	1304.2134	+ 90.0364	19.36	768.3984	+ 53.0464	6.35	252.0315	+ 17.3990
kl	44.61	2294.7384	− 192.7152	31.11	1600.2984	− 134.3952	18.10	931.0640	− 78.1920
lm	55.13	3415.8548	− 722.2030	41.63	2580.3948	− 545.3530	28.62	1773.2952	− 374.9220
mB	62.06	4275.3136	−1300.7776	48.54	3343.9206	−1017.3984	35.54	2449.0396	− 745.1280
	+190.63	+9790.3512	−1240.6564	+123.04	+6801.1419	−1080.7080	+ 70.84	+4180.9105	− 808.2790

	P_{10}			P_{11}			P_{12}		
	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$	$x - kl$	$\left(x-\frac{l}{2}\right)(x-kl)$	$(y-\bar{y})(x-kl)$
kl	5.87	302.0702	− 25.3584						
lm	16.39	1015.5244	− 214.7090	5.26	325.9096	− 68.9096			
mB	23.32	1606.5148	− 488.7872	12.20	839.7690	− 255.5024	2.40	166.0248	− 50.5136
	+ 33.32	+2120.8520	− 484.4610	+ 11.36	+ 745.7941	− 196.6608	+ 1.20	+ 83.0124	− 25.2568

Note that in taking the summations, since mB is of half length, we take one half the values for mB in summing up.

We have then, from these tables and equations (4), page 600,

$$
\begin{array}{llllll}
M_0 = -15.886P_7, & -10.253P_8, & -5.903P_9, & -2.827P_{10}, & -0.947P_{11}, & -0.100P_{12}; \\
V_1 = +0.431P_7, & +0.299P_8, & +0.184P_9, & +0.093P_{10}, & +0.033P_{11}, & +0.004P_{12}; \\
H = +0.926P_7, & +0.801P_8, & +0.603P_9, & +0.362P_{10}, & +0.147P_{11}, & +0.019P_{12};
\end{array}
$$

and from equations (5), (6) and (7), page 601,

$$
\begin{array}{llllll}
M_1 = -6.944P_7, & -7.759P_8, & -6.943P_9, & -4.691P_{10}, & -2.042P_{11}, & -0.260P_{12}; \\
M_1' = -10.92P_6, & -7.306P_5, & -2.35P_4, & +2.842P_3, & +5.53P_2, & +3.432P_1; \\
V_1' = +0.569P_6, & +0.701P_5, & +0.816P_4, & +0.907P_3, & +0.967P_2, & +0.996P_1.
\end{array}
$$

Since the depth is 2.5 ft. and density 160 lbs. per cubic foot, we have for one foot width the loads

$$
\begin{array}{llll}
P_1 = P_{12} = 48200, & P_2 = P_{11} = 40000, & P_3 = P_{10} = 29400, & P_4 = P_9 = 19100, \\
P_5 = P_8 = 11000, & P_6 = P_7 = 6600. & &
\end{array}
$$

Hence

$$\Sigma M_1 = +272750 \text{ pound-feet}, \qquad \Sigma V_1 = 154300 \text{ pounds}, \qquad \Sigma H = 87750 \text{ pounds}.$$

For the values of $\Sigma_0^x P\,(x - kl)$ we have for the points of division A, a', b', etc.,

	A	a'	b'	c'	d'	e'	f'	g'
$\Sigma_0^x P(x - kl) =$	0	217864	935328	2144550	3758000	5654936	7717664	9870685

	h'	i'	k'	l'	B
$=$	12090728	14392982	16809712	19338720	21910600

We have then, from equation (8), page 601, for the moment at any point of the axis

$$M = +272750 - 154300x + 87750y + \Sigma_0^x P(x - kl).$$

For the points f', g', h', etc., we have then

	f'	g'	h'	i'	k'	l'	B
$x =$	71	84.66	98	110.69	122.44	132.96	142
$y =$	35.50	34.44	31.29	26.13	19.07	10.29	0
$M =$	+ 150500	+102500	−12500	− 121000	−136650	−1300	+272750.

From equation (2), page 599, we have then

	f'	g'	h'	i'	k'	l'	B
$N =$	87750	88010	89550	95115	109860	137685	177510

For the edge distance of N *from the intrados* we have now, from equation (3), page 599.

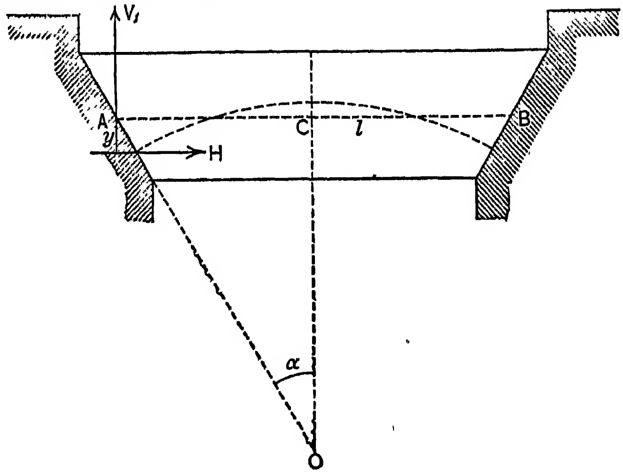
	f'	g'	h'	i'	k'	l'	B
$e' =$	− 0.47	+ 0.08	+ 1.39	+ 2.52	+ 2.50	1.26	− 0.29

We see that the curve of equilibrium passes outside of the arch and below the axis at the crown f' and springing B, and outside of the arch and above the axis at i'. The arch is then unstable and will fall, the joints at the crown and springing opening at the extrados, and at i' opening at the intrados. In other words, the haunches sinking and the crown rising.

This is precisely what happened when the arch was erected. In order to make the arch stable we should make one of two changes in the design. We can either increase the depth or make the surcharge lighter over the haunches, by building up the surcharge with hollow spaces at the haunches or lightening the surcharge there by filling in with gravel instead of stone.

The arch was actually rebuilt with hollow spaces in the surcharge over the haunches.

The Straight Arch.—The straight arch is a stone beam fixed at the ends, subjected to compression and bending. The beam is composed of voussoirs and therefore will not resist tension at any joint.



Let the loading be uniform and equal to w pounds per foot of length, and the length of the span be l.

We have at the left end the reaction $V_1 = \frac{wl}{2}$ and the thrust H acting at the distance y_1 below A.

The moment at any point of the axis is then

$$M = -Hy_1 - \frac{wx}{2}(l - x).$$

The work is

$$\text{work} = \int_0^l \frac{H^2dx}{2EA} + \int_0^l \frac{M^2dx}{2EI} = \int_0^l \frac{H^2dx}{2EA} + \int_0^l \left[-Hy_1 - \frac{wx}{2}(l-x)\right]^2 \frac{dx}{2EI}.$$

If we differentiate with respect to H and put $\frac{d(\text{work})}{dH} = 0$, we have for the value of H which makes the work a minimum

$$\int_0^l \frac{Hdx}{EA} + \int_0^l [Hy_1^2 + wxy_1(l-x)]\frac{dx}{EI} = 0.$$

If we substitute $I = A\kappa^2$, where κ^2 is the square of the radius of gyration of the cross-section A, and integrate, we have

$$\frac{Hl}{EA} + \frac{Hy_1^2l}{EA\kappa^2} + \frac{wl^3y_1}{12EA\kappa^2} = 0, \quad \text{or} \quad H = -\frac{wl^2y_1}{12(\kappa^2 + y_1^2)}. \quad . \quad . \quad . \quad . \quad (1)$$

Let the angle of the ends with the vertical be α, and let the ends be at right angles to the curve of equilibrium. Then the normal pressure on the ends is

$$N = \frac{H}{\cos\alpha},$$

and the end areas are $A_1 = \frac{A}{\cos\alpha}$. Hence $\frac{N}{A_1} = \frac{H}{A}$, and if we take the breadth unity

$$\frac{N}{d_1} = \frac{H}{d}.$$

We have then from equations (1), page 599, for the maximum unit pressure p at the end

$$p = \frac{2H}{d}\left(2 - \frac{3e}{d}\right),$$

where e is the edge distance of H, or, disregarding signs, $e = \frac{d}{2} - y_1$, $H = \frac{wl^2y_1}{12(\kappa^2 + y_1^2)}$

Hence

$$p = \frac{wl^2 y_1}{6(\kappa^2 + y_1^2)d}\left(\frac{1}{2} - \frac{3y_1}{d}\right).$$

The work will be a minimum when p is a minimum.

If we differentiate and put $\frac{dp}{dy_1} = 0$, we have for the value of y_1 which makes p a minimum, since $\kappa^2 = \frac{d^2}{12}$,

$$y_1 = \frac{d}{2} - \frac{d}{\sqrt{3}}. \quad \ldots \ldots \ldots \ldots \quad (2)$$

Substituting in equation (1), we have, since $\kappa^2 = \frac{d^2}{12}$,

$$H = \frac{wl^2}{8\sqrt{3}d}. \quad \ldots \ldots \ldots \ldots \quad (a)$$

We have then for the angle α

$$\tan \alpha = \frac{V_1}{H} = 2\sqrt{3} \cdot \frac{d}{l} = 3.464\frac{d}{l}. \quad \ldots \ldots \ldots \quad (b)$$

The maximum unit pressure is, since $\kappa^2 = \frac{d^2}{12}$,

$$p = \frac{wl^2(\sqrt{3} - 1)}{4\sqrt{3}d^2}. \quad \ldots \ldots \ldots \ldots \quad (3)$$

Let the maximum allowable unit stress be S_f, then we have

$$S_f = \frac{wl^2(\sqrt{3} - 1)}{4\sqrt{3}d^2}, \quad \text{or} \quad d = 0.325l\sqrt{\frac{w}{S_f}}. \quad \ldots \ldots \ldots \quad (c)$$

Equation (*c*) gives the depth d for the maximum allowable stress, equation (*b*) gives the angle of the ends with the vertical, equation (*a*) gives the horizontal thrust H.

Example.—*Design a straight arch of 20 feet span to sustain a load of 4000 pounds per foot of length. The allowable unit stress is 50 000 pounds per square foot.*

From (*c*) we have

$$d = 6.5\sqrt{\frac{4000}{50000}} = 1.8 \text{ ft.}$$

From (*b*) we have for the angle α of the ends with the vertical

$$\tan \alpha = 3.464\frac{1.8}{20} = 0.31176, \quad \text{or} \quad \alpha = 17° \; 19'.$$

The ends intersect, then, at a point O at a vertical distance CO below the centre C given by

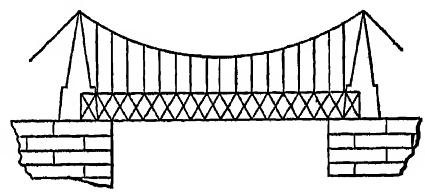
$$CO = \frac{l}{2 \tan \alpha} = \frac{10}{0.31176} = 32.07 \text{ ft.}$$

CHAPTER XI.

COMPOSITE STRUCTURES. SUSPENSION SYSTEM WITH STIFFENING TRUSS.

EACH of the structures of Chapter IX may be inverted, and constitutes in such case an inverted arch or rigid suspension system. The method of calculation is then precisely the same, the only difference being that the horizontal thrust at the end of the arch becomes a horizontal pull at the ends of the cable, and therefore members which were in compression are now in tension, and *vice versa*.

Suspension System.—A common construction for long spans, however, is that shown in the figure. Such a structure we may call a "composite" system, that is, it consists of two different systems which act together. The figure represents the most important of these, known as the "suspension system." It consists of a flexible chain or cable which is stiffened under the action of partial loads by a truss. The truss is slung from the cable by suspenders, and may be of any design, either double or single intersection, Pratt, etc. The cable carries the entire dead weight, that is, the suspenders are screwed up until the ends of the truss just bear on the abutments. The office of the truss is thus to stiffen the cable and prevent change of shape and oscillation due to partial and moving loads. It also acts to support its share of the moving load. There are usually side spans at each end. In any case the cable passes over rollers on top of the towers, and is carried on beyond and firmly fastened to large anchorages of masonry.

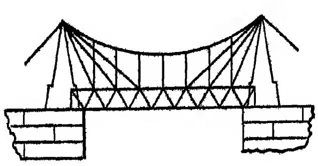


Defects of the System.—The principal defect of this system is its lack of rigidity. The cable possesses little inherent rigidity, and the stiffness is due almost entirely, therefore, to the truss.

A second disadvantage is that a rise of temperature, by increasing the deflection, throws considerable load on the truss. To obviate this objection, the truss may be hinged at the centre and placed on rollers at the ends.

Advantages of the System.—It is evident from the preceding that the system is best applied to long spans. The cable, then, carries the dead weight, and by reason of its own very considerable weight in such case resists in some degree the deforming action of partial loads. The truss can thus be very light compared to what it would have to be if there were no cable.

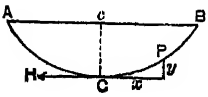
Stays Unnecessary.—The system is accordingly in practice applied only to very long spans. In such case, with cables made of steel wire it admits of great economy. But, owing to lack of rigidity, additional stiffness is sought to be obtained by the introduction of *stays* reaching from the top of the tower to various points of the truss, as shown in the accompanying figure. The use of these is not to be recommended. They render the correct determination of the stresses indeterminate. A load at any point may be carried entirely by the suspender and



stay at that point, or by the suspender and truss, or by the stay and truss. It is impossible to tell exactly the duty performed by each; and even if it were not, it would be impossible to so adjust the several systems that each shall take its proper share. If such adjustment could be made, it would not last. Variations of stress, set, and elongation of members, shocks and vibrations, rise and fall of temperature, would constantly disturb such adjustment.

The stays are also superfluous. The truss is a rigid construction. It ought to render rigid the system of which it forms a part, and should be so designed as to perform its duty without help. If such superfluous members are introduced, they can then be considered as an extra addition, contributing to strength and stiffness. But the truss should be designed without reference to their action.

Horizontal Pull of Cable.—Let the span or chord of the cable AB be c, and the rise or versed sine Cc be r. Then if w be the load per unit of horizontal, we have for uniformly distributed load, taking moments about B, if H is the horizontal pull,



$$-Hr + \frac{wc^2}{8} = 0, \quad \text{or} \quad H = \frac{wc^2}{8r}. \qquad (1)$$

Equation (1) gives the horizontal pull of the cable for uniform load, which is evidently the same at every point.

Shape of Cable.—If we take moments about any point P distant x from the centre, we have, if y is the ordinate for origin at C,

$$-Hy + \frac{wx^2}{2} = 0, \quad \text{or} \quad y = \frac{wx^2}{2H}.$$

Inserting the value of H from (1),

$$y = \frac{4rx^2}{c^2}, \qquad (2)$$

which is the equation of a parabola. Hence the curve of the cable or of a flexible string uniformly loaded along the horizontal is a parabola.

Length of Suspenders.—Let l_0 be the length of the suspender at the centre, then from (2) we have for the length of a suspender at a distance x from the centre

$$l = l_0 + \frac{4rx^2}{c^2}. \qquad (3)$$

Length of Segment of Cable.—Let n be the number of segments of the cable, the suspenders being equally spaced, so that $\frac{c}{n}$ is the distance between suspenders, or the horizontal projection of a segment. Then from (2) we have for the ordinate of the nearest end

$$y_1 = \frac{4rx^2}{c^2},$$

and for the ordinate of the farther end

$$y_2 = \frac{4r\left(x + \frac{c}{n}\right)^2}{c^2}.$$

Hence the vertical projection of a segment is

$$y_2 - y_1 = \frac{4r}{nc}\left(2x + \frac{c}{n}\right).$$

The length s_c of a segment is then

$$s_c = \sqrt{\frac{c^2}{n^2} + \frac{16r^2}{n^2c^2}\left(2x + \frac{c}{n}\right)^2}, \qquad \ldots \ldots \ldots \quad (4)$$

where x is the ordinate from the centre to the nearest end of segment.

Stress in Segment of Cable.—The secant of the angle of inclination α at any point is

$$\sec \alpha = \frac{ns_c}{c}.$$

Since the horizontal pull H is the same at every point, the stress for any segment is

$$\text{Stress in segment} = H \sec \alpha = \frac{nHs_c}{c},$$

or, from (1), for uniform load

$$\text{Stress in segment} = \frac{nwcs_c}{8r}. \qquad \ldots \ldots \ldots \quad (5)$$

Deflection of Cable Due to Temperature.—Let ϵ be the coefficient of expansion or contraction, that is, the ratio of the change of length to the original length of a member for one degree change of temperature. Let t be the change of temperature in degrees, λ the total change of length and s_c the length of a cable segment. Then $\frac{\lambda}{s_c}$ is the ratio of the change of length to original length for t degrees. For one degree we have then

$$\frac{\lambda}{s_ct} = \epsilon, \quad \text{or} \quad \lambda = \epsilon s_ct,$$

where ϵ is given by experiment, and ϵt is an abstract number or numerical value. For values of ϵ see page 582.

Suppose a load P at the centre of the cable would produce the same change of length. Since the structure is rigid so that the shape of the cable does not change, this load must be

distributed by the truss over the cable as a uniform load of $\frac{P}{c}$ per foot of horizontal projection. Inserting this for w in (5), we have

$$\text{Stress in segment} = \frac{nPs_c}{8r}.$$

From page 515 the work is one half the product of the stress and change of length, or

$$\text{Work on a segment} = \frac{nP\epsilon t s_c^2}{16r}.$$

The total work for all the segments is then

$$\text{Work} = \frac{nP\epsilon t \Sigma s_c^2}{16r}.$$

If $\varDelta$ is the deflection of cable at centre, the work is also $\frac{P\varDelta}{2}$. Hence

$$\frac{P\varDelta}{2} = \frac{nP\epsilon t \Sigma s_c^2}{16r}, \quad \text{or} \quad \varDelta = \frac{n\epsilon t \Sigma s_c^2}{8r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where n and ϵt are abstract numbers and s_c and r are lengths. If we take s_c and r in feet or inches, equation (6) will give $\varDelta$ in feet or inches.

Deflection of Truss for Uniform Load.—Let u_0 be the stress in pounds in any member of the truss for a uniformly distributed load of *one pound* per foot of length, that is, u_0 is the stress *per pound-per-foot* distributed load. Then the stress in pounds for a uniform load of w pounds per foot will be given by wu_0. The corresponding strain λ is then (page 477)

$$\lambda = \frac{wu_0 s}{aE},$$

where s is the length of truss member, a its area of cross-section in square inches and E its coefficient of elasticity in pounds per square inch. If s is taken in feet or inches, λ will be given in feet or inches.

Let p be the stress in pounds in the member due to one pound placed at the centre of the truss. That is, p is the stress in pounds *per pound of load* at centre. Then the work on the member due to this load is

$$1 \text{ pound} \times \frac{p\lambda}{2} = \frac{wpu_0 s}{2aE} \times 1 \text{ pound}.$$

The total work on all the members is then

$$\text{Work} = \frac{w}{2E}\sum\frac{pu_0 s}{a} \times 1 \text{ pound}.$$

But if Δ is the deflection at the centre the work is also 1 pound $\times \frac{\Delta}{2}$. Hence we have

$$1 \text{ pound} \times \frac{\Delta}{2} = \frac{w}{2E}\sum\frac{pu_0 s}{a} \times 1 \text{ pound}, \quad \text{or} \quad \Delta = \frac{w}{E}\sum\frac{pu_0 s}{a}, \quad . \quad . \quad . \quad . \quad (7)$$

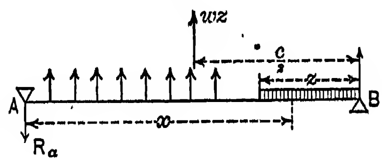
where E and a are as already specified. If s is taken in feet or inches, Δ will be given in feet or inches.

Temperature Load for Truss.—When the cable expands or contracts the centre falls or rises a distance Δ, given by (6), and the centre of the truss falls or rises with the cable a distance given by (7). Let w_t be the uniformly distributed load in pounds per foot which would cause this deflection. Then, equating (6) and (7), we have

$$w_t = \frac{Ene t \Sigma s_c^2}{8r\sum\frac{pu_0 s}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Old Theory of Suspension System.—The theory of the suspension system heretofore in use is due to Rankine, and is based upon the assumption that the cable *carries the entire load*, dead and live, the office of the truss being simply to distribute a partial loading over the cable, and thus prevent change of shape.

FIG. 1.



MAXIMUM SHEAR IN TRUSS—OLD METHOD.—Let the uniform live load w for unit of length extend over the distance z from the right end (see Fig. 1).

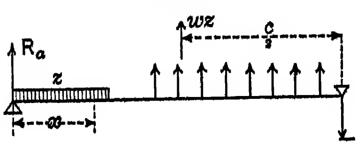
Then the load is wz, and, since by assumption the cable carries all this load, the upward load on the truss due to the cable is wz or, $\frac{wz}{c}$ for unit of length.

Let R_A be the reaction at the left end A of the truss. We have, taking moments about the right end B,

$$-R_A c - wz \cdot \frac{c}{2} + \frac{wz^2}{2} = 0, \quad \text{or} \quad R_A = -\frac{wz(c-z)}{2c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since this is negative, the truss should be tied down at the ends. If the load w extends over the distance z from the left end (Fig. 2), we have

FIG. 2.



$$-R_A c + wz\left(c - \frac{z}{2}\right) - wz \cdot \frac{c}{2} = 0,$$

or

$$R_A = +\frac{wz(c-z)}{2c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the first case (Fig. 1), when the load comes on from the right, we have for the shear at any point distant x from the left end:

when x is less than $c - z$ Shear $= R_A + \frac{wz}{c}x$;

when x is greater than $c - z$ Shear $= R_A + \frac{wz}{c}x - w[x - (c - z)]$;

or, inserting the value of R_A from (1),

$$\text{when } x < c - z \qquad \text{Shear} = \frac{wz}{2c}\Big[2x - (c - z)\Big];$$

$$\text{when } x > c - z \qquad \text{Shear} = \frac{wz}{2c}\Big[2x - (c - z)\Big] - w[x - (c - z)].$$

From the last of these equations we see that the shear is a positive maximum when the last term is zero or when $z = c - x$. That is, the shear for the unloaded portion is a positive maximum at the head of the load.

From the first of these equations we have the shear a negative maximum when $z = \frac{c}{2} - x$. That is, the shear for the unloaded portion is a negative maximum at any point when the distance covered by the load is equal to the distance of the point from the centre.

Inserting these values of $z = \frac{c}{2} - x$ and $z = c - x$, we have for any point of the unloaded portion distant x from the left end:

$$\text{unloaded portion}\left\{\begin{array}{l} \text{maximum positive Shear } = +\dfrac{w(c - x)x}{2c}, \\[2ex] \text{maximum negative Shear} = -\dfrac{w\left(\frac{c}{2} - x\right)^2}{2c}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the second case (Fig. 2), when the load comes on from the left, we have for the shear at any point distant x from the left end:

$$\text{when } x < z \qquad \text{Shear} = R_A + \frac{wz}{c}x - wx,$$

or, inserting the value of R_A from (2),

$$\text{when } x < z \qquad \text{Shear} = \frac{wz}{2c}[2x + (c - z)] - wx.$$

This is a negative maximum for $z = x$. That is, the shear for the loaded portion is negative maximum at the head of the load.

It is a positive maximum when $z = \frac{c}{2} + x$. That is, the shear for the loaded portion is a positive maximum at any point when the distance between the point and the end of the load is equal to the half span.

Inserting the values of $z = x$ and $z = \frac{c}{2} + x$, we have for any point of the loaded portion distant x from the left end:

$$\text{loaded portion}\left\{\begin{array}{l} \text{maximum positive Shear } = +\dfrac{w\left(\frac{c}{2} - x\right)^2}{2c}, \\[2ex] \text{maximum negative Shear} = -\dfrac{w(c - x)x}{2c}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We see from equations (3) and (4) that we have for the maximum shears for

$$x = 0 \qquad \frac{1}{4}c \qquad \frac{1}{2}c$$

$$\text{Shear} = \pm\frac{wc}{8} \qquad \pm\frac{3wc}{32} \qquad \pm\frac{wc}{8}$$

That is, the maximum shear is practically constant and varies but litle from $\frac{wc}{8}$.

It is therefore customary by the old method *to design every brace for the maximum shears due to live load.*

$$\text{Shear} = \pm\frac{wc}{8}. \quad \ldots \ldots \ldots \quad (1)$$

MAXIMUM MOMENT IN TRUSS—OLD METHOD.—For the moment M at any point of the unloaded portion (Fig. 1) distant x from the left end, if w is the uniform live load for unit of length, we have

$$M = -R_Ax - \frac{wzx^2}{2c},$$

or, substituting the value of R_A from (1),

$$M = -\frac{wzx}{2c}[x - (c - z)]. \quad \ldots \ldots \ldots \quad (5)$$

For any point of the loaded portion (Fig. 2) we have

$$M = -R_Ax - \frac{wzx^2}{2c} + \frac{wx^2}{2},$$

or, substituting the value of R_A from (2),

$$M = -\frac{wzx}{2c}[x + (c - z)] + \frac{wx^2}{2}. \quad \ldots \ldots \ldots \quad (6)$$

In (5) $M = 0$ for $x = c - z$, and in (6) $M = 0$ for $x = z$. That is, the moment at the head of the load is zero. Also, if x is less than $c - z$ in (5), the moment is positive, and if greater than $c - z$, the moment is negative. *The head of the load is then a point of inflection,* and the loaded and unloaded portions may be considered as simple trusses uniformly loaded. The greatest moment for each portion will then be at the centre of each portion. Making, then, $x = \frac{c - z}{2}$ in (5) and $x = \frac{z}{2}$ in (6), we have for the moment at the centre of each portion

$$+\frac{wz(c - z)^2}{8c} \qquad \text{and} \qquad -\frac{wz^2}{8c}(c - z).$$

These are a maximum, respectively, for $z = \frac{1}{3}c$ and $z = \frac{2}{3}c$.

Hence *the maximum positive moment is at the middle of the unloaded portion when the load extends over one third the span, and the maximum negative moment is at the middle of the load when it covers two thirds the span.*

We have then for the maximum positive moment at any point of the unloaded left half span, by putting $z = \frac{1}{3}c$ in (5),

$$M = \frac{w}{18}(2c - 3x)x, \qquad (7)$$

and for the maximum negative moment at any point of the loaded left half span, by putting $z = \frac{2}{3}c$ in (6),

$$M = -\frac{w}{18}(2c - 3x)x. \qquad (8)$$

From equations (7) and (8) we have the maximum moments for

$$\begin{array}{cccc} x = 0 & \frac{1}{6}c & \frac{1}{8}c & \frac{1}{2}c \\ M = 0 & \pm\frac{wc^2}{72} & \pm\frac{wc^2}{54} & \pm\frac{wc^2}{72} \end{array}$$

That is, the maximum moment beyond $\frac{1}{6}c$ is practically constant and varies but little from $\frac{wc^2}{54}$.

It is therefore customary by the old method to *design every chord panel for the maximum moments due to live load*

$$M = \pm\frac{wc^2}{54}. \qquad (II)$$

Temperature Load for Truss.—From (I) and (II) we can then easily find the area a of each truss member due to live load. Thus for straight truss of height h, if σ is the working stress, we have for the area a of the chords

$$a = \frac{wc^2}{54h\sigma}, \qquad (9)$$

and for the area of a brace which makes the angle θ with the vertical

$$a = \frac{wc}{8\sigma}\sec\theta. \qquad (10)$$

We have then from (8), page 613, the temperature load per unit of length,

$$w_t = \frac{En\epsilon t\Sigma s_c^2}{8r\Sigma\frac{pu_0 s}{a}}. \qquad (III)$$

Stress in Truss.—This temperature load should be taken into account together with the live load in finding the maximum truss stresses. We have then to add to the shear and moment given by (I) and (III) the shear and moment due to the temperature load w_t. The actual stresses in the truss members, then, are greater than those due to the live load only, and hence the areas assumed in (9) and (10) are too small and the corresponding value of w_t

given by (III) is too small. We should therefore assume w_t somewhat larger than given by (III), and then find the stresses for this assumed w_t and the live load. The corresponding areas should, when inserted in (III), give us pretty closely the value for w_t we assumed. If not, we can make another approximation.

Cable Stress and Area.—We can now find the stress and area in any cable segment. Thus let the dead load per unit of length be w_0, the live load w, and the temperature load w_t, assumed as above. Then the total unit load for the cable is $(w + w_0 + w_t)$, and from (5) we have for the stress in any segment whose length is s_c

$$\text{Stress in segment} = \frac{n(w + w_0 + w_t)cs_c}{8r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IV})$$

If the working stress for cable is σ_c, we have only to divide by σ_c to have the area of cross-section.

If the cable is made of links, (IV) will give the stress for any link according to the value of s_c, and dividing by σ_c we have the area of cross-section of the link. If the links are of constant area of cross-section, or if we have a wire cable, we must take for s_c the length of the end segment for which s_c is greatest.

Suspender Stress and Area.—Let n be the number of segments of the cable; then $\frac{c}{n}$ is the distance between suspenders. The unit load is $(w + w_0 + w_t)$; hence

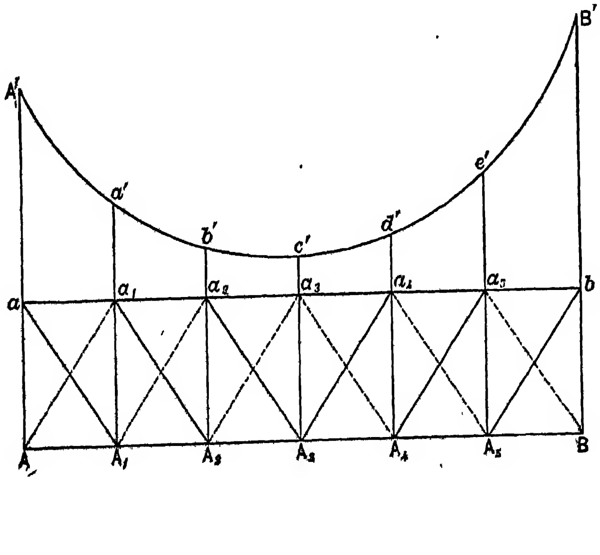
$$\text{Stress on suspender} = (w + w_0 + w_t)\frac{c}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{V})$$

If σ_s is the working stress, we have only to divide by σ_s to have the area of cross-section of the suspender.

Example.—OLD METHOD.—In order to abridge the work of computation we take a short span for illustration.

Data.—Let the span $c = 54$ feet; the versine of cable $r = 9$ feet; the depth of truss $h = 12$ feet; the panel length 9 feet, so that the number of segments of the cable is $n = 6$; the truss of steel and coefficient of elasticity for truss members $E = 30\,000\,000$ pounds per square inch; working stress for truss members and steel suspenders $\sigma = \sigma_s = 10000$ pounds per square inch; cable of steel wire working stress $= \sigma_c = 30000$ pounds per square inch; live load $w = 2000$ pounds per foot; dead load $w_0 = 1000$ pounds per foot; coefficient of expansion $\epsilon = 0.00000686$; range of temperature $t = 80°$.

Let the members be notated as in the following figure. The angle θ of the braces with the vertical is then such that $\sec\theta = \frac{5}{4}$.



Calculation.—We have, from (I),

$$\text{Shear} = \pm \frac{wc}{8} = \frac{2000 \times 54}{8} = 13500 \text{ pounds.}$$

This, then, is the stress for every post. For every brace the stress is

$$\text{Shear} \times \sec\theta = 19089 \text{ pounds.}$$

From (II), the moment for any chord panel is

$$\frac{wc^2}{54} = \frac{2000 \times 54 \times 54}{54} = 108000.$$

Hence the stress for each chord panel is

$$\frac{108000}{12} = 9000.$$

If the working stress $\sigma = 10000$, we have then for each post

$$\text{post area} = 1.35 \text{ sq. in.,}$$

for every brace

$$\text{brace area} = 1.9 \text{ sq. in.,}$$

for every chord panel

$$\text{panel area} = 0.9 \text{ sq. in.}$$

Since we have disregarded the stress due to temperature, let us take post area = 1.5 sq. in., brace area = 2 sq. in., panel area = 1 sq. in.

The length of a cable segment is, from (4), page 611,

$$s_c = \sqrt{81 + \frac{1}{81}(2x + 9)^2},$$

where x is the ordinate from the centre to nearest end of segment.

We have then for the length of cable segments:

	$A'a'$	$a'b$	$b'c'$	$c'd'$	$d'e'$	$e'B'$
$s_c =$	$\sqrt{106}$	$\sqrt{90}$	$\sqrt{82}$	$\sqrt{82}$	$\sqrt{90}$	$\sqrt{106}$

Hence $\Sigma s_c^2 = 556$, and this is to be inserted in (III) in order to find the temperature load w_t.

We can now draw up the following table.

Member.	Area a in Sq. Inches.	Length s in Feet.	Stress u_0 in Pounds.	Stress p in Pounds.	$\frac{pu_0s}{a}$	pu_0s
aA	1.5	12	− 22.5	− 0.5	+ 90	+ 135
a_1A_1	1.5	12	− 13.5	− 0.5	+ 54	+ 81
a_2A_2	1.5	12	− 4.5	− 0.5	+ 18	+ 24
aA_1	2	15	+ 28.125	+ 0.625	+ 131.836	+ 263.672
a_1A_2	2	15	+ 16.875	+ 0.625	+ 79.101	+ 158.202
a_2A_3	2	15	+ 5.625	+ 0.625	+ 26.367	+ 52.734
aa_1	1	9	− 16.875	− 0.375	+ 56.953	+ 56.953
a_1a_2	1	9	− 27	− 0.75	+ 182.25	+ 182.25
a_2a_3	1	9	− 30.375	− 1.125	+ 307.547	+ 307.547
AA_1	1	9	0	0	0	0
A_1A_2	1	9	+ 16.875	+ 0.375	+ 56.953	+ 56.953
A_2A_3	1	9	+ 27	+ 0.75	+ 182.25	+ 182.25
					+ 1185.257	

For each member in the half truss we give in the table the area a in square inches already found, the length s in feet, the stress u_0 in pounds due to a uniform load of one pound per foot of length or 9 pounds at each lower apex, and the stress p in pounds due to one pound at the centre-line apex. In the last column we have then the quantities pu_0s and $\frac{pu_0s}{a}$ for each member of the half span.

The table gives us for the half span $\Sigma \frac{pu_0 s}{a} = +\ 1185.257$, and hence for the whole span

$$\Sigma \frac{pu_0 s}{a} = +\ 2370.514.$$

Inserting this and the value of $\Sigma s_c^2 = 556$ in (III), we have

$$w_t = \frac{30000000 \times 6 \times 0.00000686 \times 80 \times 556}{8 \times 9 \times 2370.514} = 322.$$

This, as we have seen, must be too small. Let us then assume

$$w_t = 600 \text{ pounds per foot of length.}$$

Stresses in Truss.—If we take for the braces the live-load shear $\pm \frac{wc}{8} = \pm\ 13500$ pounds, as given by (I), for the chords the moment $\pm \frac{wc^2}{54} = \pm\ 108000$ as given by (II), and in addition the shear and moment due to $w_t = 600$ pounds per foot of length, we obtain the following stresses in the truss members in pounds:

For the posts

$aA = -\ 27000$ $\quad a_1A_1 = -\ 21600$ $\quad a_2A_2 = -\ 16200$ $\quad a_3A_3 = -\ 13500$

For the braces

$aA_1 = +\ 33750$ $\quad a_1A_2 = +\ 27000$ $\quad a_2A_3 = +\ 20250$

$Aa_1 = +\ 33750$ $\quad A_1a_2 = +\ 27000$ $\quad A_2a_3 = +\ 20250$

For the chords

$aa_1 = -\ 19125$ $\quad a_1a_2 = \pm\ 25200$ $\quad a_2a_3 = \pm\ 27225$

$AA_1 = -\ 19125$ $\quad A_1A_2 = \pm\ 19125$ $\quad A_2A_3 = \pm\ 25200$

Taking the working stress $\sigma = 10000$ pounds per square inch, these stresses give us the following areas of cross-section in square inches:

$aA = 2.7$ $\quad a_1A_1 = 2.16$ $\quad a_2A_2 = 1.62$ $\quad a_3A_3 = 1.35$ $\quad aA_1 = 3.37$ $\quad a_1A_2 = 2.7$

$a_2A_3 = 2.02$ $\quad aa_1 = 1.91$ $\quad a_1a_2 = 2.52$ $\quad a_2a_3 = 2.72$ $\quad AA_1 = 1.91$

$A_1A_2 = 1.91$ $\quad A_2A_3 = 2.52$

If we use these values of a in place of the values of a in the table page 618, we obtain

$$\Sigma \frac{pu_0 s}{a} = 1165.06,$$

and inserting this in (III) we obtain

$$w_t = 655,$$

whereas we assumed w_t at only 600.

Let us then again assume $w_t = 700$. We have then the following stresses:

For the posts

$aA = -\ 29250$ $\quad a_1A_1 = -\ 22950$ $\quad a_2A_2 = -\ 16650$ $\quad a_3A_3 = -\ 13500$

For the braces

$aA_1 = +\ 36562$ $\quad a_1A_2 = +\ 28687$ $\quad a_2A_3 = +\ 20812$

$Aa_1 = +\ 36562$ $\quad A_1a_2 = +\ 27000$ $\quad A_2a_3 = +\ 20812$

For the chords

$aa_1 = -\ 20812$ $\quad a_1a_2 = \pm\ 27900$ $\quad a_2a_3 = \pm\ 30262$

$AA_1 = -\ 20812$ $\quad A_1A_2 = \pm\ 20812$ $\quad A_2A_3 = \pm\ 27900$

We have then for $\sigma = 10000$ the areas of cross-section $aA = 2.92$, $a_1A_1 = 2.29$, $a_2A_2 = 1.66$, $a_3A_3 = 1.35$, $aA_1 = 3.66$, $a_1A_2 = 2.87$, $a_2A_3 = 2.08$, $aa_1 = 2.08$, $a_1a_2 = 2.79$, $a_2a_3 = 3.02$, $AA_1 = 2.08$, $A_1A_2 = 2.08$, $A_2A_3 = 2.79$.

Taking these values of a in place of the values of a in the table page 618, we have

$$\sum \frac{pu_0 s}{a} = 1071.58,$$

and inserting this in (III) we obtain $w_t = 711$. Since we assumed $w_t = 700$, we obtain practically the same value we assumed.

We have then $w_t = 700$, and the stresses last found are the truss stresses.

Since the live-load post stresses are 13500, the live-load brace stresses 16875, and the live-load chord stresses 9000 pounds, we see that the temperature-load stresses are in this case much greater and of greater importance than the live-load stresses.

The truss stresses just found should be divided by 2 for two trusses. The calculation supposes all the load carried by one truss and one cable.

CABLE STRESS AND AREA.—We have found already for the lengths of the cable segments

$$A'a' = \sqrt{106}, \qquad a'b' = \sqrt{90}, \qquad b'c' = \sqrt{82} \text{ feet.}$$

Hence, from (IV), we have for the stresses

$$A'a' = 171495, \qquad a'b' = 158175, \qquad b'c' = 149850 \text{ pounds.}$$

If the working stress is 30000 pounds per square inch, the areas of cross-section should be

$$A'a' = 5.72, \qquad a'b' = 5.27, \qquad b'c' = 4.99 \text{ square inches.}$$

If the cable is to have a constant area of cross-section or is a wire cable, the area should be then 5.72 square inches. Stresses and areas should be divided by 2 for two cables, etc.

Suspender Stress and Area.—For the suspender stress we have, from (V),

$$\text{Suspender stress} = (2000 + 1000 + 700)\tfrac{49}{6} = 30300 \text{ pounds.}$$

If the working stress is 10000 pounds per square inch, the area of cross-section should be 3 square inches. For two cables we have then 1.5 square inches cross-section for each suspender.

If the suspenders are also of steel wire, so that the working stress is 30000 pounds per square inch, the area of cross-section would be 1 square inch, or for two cables 0.5 square inch for each suspender.

New Theory of Suspension System.—The old method which we have just illustrated is based upon the assumption that the cable carries the entire load, dead and live, so that the office of the truss is simply to prevent change of shape. Unless, however, the truss swings clear of the supports even when fully loaded, this assumption is not correct. If we suppose that the truss just bears on the supports when unloaded, then when the live load comes on, the truss must bear a portion of this load as well as prevent change of shape, and the cable carries then the dead load and only a portion of the live load. The portion carried by the cable and by the truss must then be determined by the principle of least work.

Work on Suspenders.—Let the truss just bear on the supports when unloaded, and suppose a load P to be placed at any apex. Let a certain fraction of P represented by ϕ be carried by the cable. Then if there is no change of shape, the load ϕP must act on the cable as a uniformly distributed load, and the load on a suspender is $\frac{\phi P}{n}$. If l is the length of a suspender, the work on the suspender is (page 515) $\frac{\phi^2 P^2 l}{2n^2 E_s a_s}$. The work on all the suspenders is then

$$\text{Work on suspenders} = \frac{\phi^2 P^2 \Sigma l}{2n^2 E_s a_s}, \qquad \qquad (1)$$

where a_s is the area of cross-section and E_s is the coefficient of elasticity for the suspenders

Work on Cable.—The load per foot of length on the cable for a load P placed on the truss is $\frac{\phi P}{c}$. We have then, by putting $\frac{\phi P}{c}$ in place of w in equation (5), page 611, for the stress on a segment of the cable of length s_c

$$\text{Stress} = \frac{n\phi P s_c}{8r}. \qquad \qquad (2)$$

Hence (page 515) the work on a segment is

$$\frac{n^2\phi^2P^2s_c^{\,3}}{128r^2E_ca_c},$$

and the total work on cable is

$$\text{Work on cable} = \frac{n^2\phi^2P^2}{128r^2E_c}\sum\frac{s_c^{\,3}}{a_c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where a_c is the area of cross-section of cable segment, and E_c is the coefficient of elasticity for the cable.

Work on Truss.—The truss is subjected to a uniformly distributed upward load due to the cable of $\frac{\phi P}{c}$ pounds per foot of length, and also supports a load P at any distance z from the left end.

Let u_0 be the stress in pounds in any member for a uniformly distributed load of *one pound per foot* of length. Then the stress for $\frac{\phi P}{c}$ pounds per foot will be

$$\frac{\phi P u_0}{c}.$$

Let p be the stress in pounds in any member due to one pound at the distance z from the left end. Then the stress due to P will be

$$Pp.$$

The stress, then, in any member can be written

$$\text{Stress} = Pp - \frac{\phi P u_0}{c} = P\left(p - \frac{\phi u_0}{c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the stresses u_0 and p are to be inserted with their proper signs (+) for tension and (−) for compression.

Now the work of the member is, from page 515,

$$\text{Work} = \frac{(\text{Stress})^2 s}{2Ea},$$

where s is the length, a the area of cross-section and E the coefficient of elasticity. Inserting the value of the stress just found, we have for the work of all the members, or

$$\text{Work of truss} = P^2\sum\left(p - \frac{\phi u_0}{c}\right)^2\frac{s}{2Ea}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Value of ϕ.—We can now find the value of ϕ or that fraction of the load P carried by the cable.

Thus, adding the works given by (1), (3) and (5), we have

$$\text{Total work} = \frac{\phi^2P^2\Sigma l}{2n^2E_sa_s} + \frac{n^2\phi^2P^2}{128r^2E_c}\sum\frac{s_c^{\,3}}{a_c} + P^2\sum\left(p - \frac{\phi u_0}{c}\right)^2\frac{s}{2Ea}.$$

The value of ϕ is that which makes this work a minimum.

If then we differentiate with reference to ϕ and put $\frac{d(\text{work})}{d\phi} = 0$, we have

$$\frac{\phi\Sigma l}{n^2E_sa_s} + \frac{n^2\phi}{64r^2E_c}\sum\frac{s_c^{\,3}}{a_c} + \frac{\phi}{c^2}\sum\frac{u_0^{\,2}s}{Ea} - \frac{1}{c}\sum\frac{pu_0s}{Ea} = 0.$$

Hence we have

$$\phi = \frac{\frac{1}{c}\sum\frac{pu_0 s}{Ea}}{\frac{\Sigma l}{n^2 E_s a_s} + \frac{n^2}{64 r^2 E_c}\sum\frac{s_c^3}{a_c} + \frac{1}{c^2}\sum\frac{u_0^2 s}{Ea}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (VI)$$

This value of ϕ requires that the values of the cross-sections, a, a_c, and a_s of the truss members, cable, and suspenders shall be known.

New Method.—We therefore assume for a first approximation the values of a, a_c and a_s as already found by the old method. Then, from (VI), we can find the value of ϕ for each apex live load P, and we thus know for each apex live load P the amount ϕP carried by the cable which acts as a uniformly distributed upward load on the truss. This gives an upward apex load at every apex of $\frac{\phi P}{n}$.

We can now find and tabulate the stress in every truss member due to each apex live load P and the upward apex load $\frac{\phi P}{n}$ at every apex, and thus obtain the maximum live-load stresses. To these must be added the stresses due to temperature load w_t as given by equation (III).

The cable stress is then given from (IV) by substituting $\frac{\Sigma\phi P}{c}$ for w:

$$\text{Stress in cable segment} = \frac{n\left[\frac{\Sigma\phi P}{c} + w_0 + w_t\right]cs_c}{8r}. \quad . \quad . \quad . \quad . \quad (VII)$$

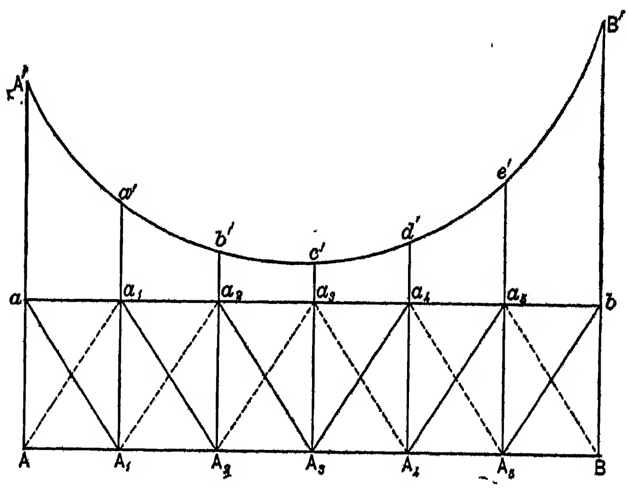
The suspender stress is in the same way, from (V),

$$\text{Stress on suspender} = \left[\frac{\Sigma\phi P}{c} + w_0 + w_t\right]\frac{c}{n}. \quad . \quad . \quad . \quad . \quad . \quad (VIII)$$

From these stresses the corresponding areas a, a_c and a_s can be determined, and if not sufficiently close to those assumed, another approximation can be made.

Example.—NEW METHOD.—We take the same example as before.

Data.—$c = 54$ ft., $r = 9$ ft., $h = 12$ ft., $n = 6$, $E = E_c = E_s = 30\,000\,000$ lbs. per sq. in., $\sigma_c = 30000$, and $\sigma_s = \sigma = 10000$ lbs. per sq. in., apex live load $P = 18000$ lbs., dead load $w_0 = 1000$ lbs. per ft., $e = 0.00000686$, $t = 80°$, length of centre suspender $l_0 = 14$ ft.



Members notated as in the figure. The angle θ of braces with vertical such that $\sec\theta = \frac{5}{4}$.

Calculation.—We proceed first by the old method and find, as already shown, the areas of cross-section for suspenders, cable and truss, and the temperature load w_t.

We have already found by the old method

$$w_t = 700 \quad \text{and} \quad a_s = 3,$$

and for cable of uniform cross-section

$$a_c = 5.72, \qquad s_c = \sqrt{106}.$$

If the cable is of links, we have

	Aa'	$a'b'$	$b'c'$
$a_c =$	5.72	5.27	5
$s_c =$	$\sqrt{106}$	$\sqrt{90}$	$\sqrt{82}$

We have then for uniform cross-section of cable

$$\sum\frac{s_c^2}{a_c} = 1145,$$

and for cable of links

$$\sum\frac{s_c^2}{a_c} = 1000.$$

We have from equation (3), page 612, the length of a suspender,

$$l = 14 + \frac{x^2}{81}.$$

Hence we obtain

$$\Sigma l = 14 + 2(15 + 18 + 23) = 126.$$

Inserting these values in (VI), we have, since $E_c = E_s = E$, for cable of uniform cross section

$$\phi = \frac{\sum\frac{pu_0s}{a}}{492.375 + \frac{1}{54}\sum\frac{u_0^2s}{a}}. \qquad (6)$$

For cable of links we have 438 in place of 492.375.

We can now form the table on page 624.

For each member in the truss (omitting counters) we give in the table the area a in square inches as found already by the old method, the length s in feet, the stress u_0 in pounds due to a uniform load of one pound per foot of length or 9 pounds at each lower apex.

We can determine for each member the quantity $\frac{u_0^2s}{a}$, and we have

$$\sum\frac{u_0^2s}{a} = 36210.$$

Equation (6) thus becomes, for cable of uniform section,

$$\phi = \frac{\sum\frac{pu_0s}{a}}{492.375 + 670.55}. \qquad (7)$$

	s	a	u_0	$\frac{u_0^2 s}{a}$	p_1	p_2	p_3	p_4	p_5
aA	12	2.92	− 22.5	+ 2080	− 0.833	− 0.666	− 0.5	− 0.333	− 0.166
a_1A_1	12	2.29	− 13.5	+ 955	+ 0.166	− 0.666	− 0.5	− 0.333	− 0.166
a_2A_2	12	1.66	− 4.5	+ 146	+ 0.166	+ 0.333	− 0.5	− 0.333	− 0.166
a_3A_3	12	1.35	0	0	0	0	0	0	0
a_4A_4	12	1.66	− 4.5	+ 146	− 0.166	− 0.333	− 0.5	+ 0.333	+ 0.166
a_5A_5	12	2.29	− 13.5	+ 955	− 0.166	− 0.333	− 0.5	− 0.666	+ 0.166
bB	12	2.92	− 22.5	+ 2080	− 0.166	− 0.333	− 0.5	− 0.666	− 0.833
aA_1	15	3.66	+ 28.125	+ 3241	+ 1.041	+ 0.833	+ 0.625	+ 0.416	+ 0.208
a_1A_2	15	2.87	+ 16.875	+ 1488	− 0.208	+ 0.833	+ 0.625	+ 0.416	+ 0.208
a_2A_3	15	2.08	+ 5.625	+ 228	− 0.208	− 0.416	+ 0.625	+ 0.416	+ 0.208
a_4A_3	15	2.08	+ 5.625	+ 228	+ 0.208	+ 0.416	+ 0.625	− 0.416	− 0.208
a_5A_4	15	2.87	+ 16.875	+ 1488	+ 0.208	+ 0.416	+ 0.625	+ 0.833	− 0.208
bA_5	15	3.66	+ 28.125	+ 3241	+ 0.208	+ 0.416	+ 0.625	+ 0.833	+ 1.041
aa_1	9	2.08	− 16.875	+ 1232	− 0.625	− 0.5	− 0.375	− 0.25	− 0.125
a_1a_2	9	2.79	− 27	+ 2352	− 0.5	− 1.0	− 0.75	− 0.5	− 0.25
a_2a_3	9	3.02	− 30.375	+ 2749	− 0.375	− 0.75	− 1.125	− 0.75	− 0.375
a_3a_4	9	3.02	− 30.375	+ 2749	− 0.375	− 0.75	− 1.125	0.75	− 0.375
a_4a_5	9	2.79	− 27	+ 2352	− 0.25	− 0.5	− 0.75	− 1.0	− 0.5
a_5b	9	2.08	− 16.875	+ 1232	− 0.125	− 0.25	− 0.375	− 0.5	− 0.625
AA_1	9	2.08	0	0	0	0	0	0	0
A_1A_2	9	2.08	+ 16.875	+ 1232	+ 0.625	+ 0.5	+ 0.375	+ 0.25	+ 0.125
A_2A_3	9	2.79	+ 27	+ 2352	+ 0.5	+ 1.0	+ 0.75	+ 0.5	+ 0.25
A_3A_4	9	2.79	+ 27	+ 2352	+ 0.25	+ 0.5	+ 0.75	+ 1.0	+ 0.5
A_4A_5	9	2.08	+ 16.875	+ 1232	+ 0.125	+ 0.25	+ 0.375	+ 0.5	+ 0.625
A_5B	9	2.08	0	0	0	0	0	0	0
				36210					

For cable of links we have 438 in place of 492.375.

We also give in the table the stress in every member due to loads p_1, p_2, p_3, p_4, p_5 of one pound placed at each lower apex.

We can now form the following table giving the quantity $\frac{pu_0s}{a}$ for each member, and hence $\sum\frac{pu_0s}{a}$ for one pound at A_1, A_2, A_3, A_4, and A_5.

	$\frac{p_1u_0s}{a}$	$\frac{p_2u_0s}{a}$	$\frac{p_3u_0s}{a}$	$\frac{p_4u_0s}{a}$	$\frac{p_5u_0s}{a}$
aA	+ 77.05	+ 61.64	+ 46.23	+ 30.82	+ 15.41
a_1A_1	− 11.79	+ 47.16	+ 35.37	+ 23.58	+ 11.79
a_2A_2	− 5.4	− 10.8	+ 16.2	+ 10.8	+ 5.4
a_3A_3	0	0	0	0	0
a_4A_4	+ 5.4	+ 10.8	+ 16.2	− 10.8	− 5.4
a_5A_5	+ 11.79	+ 23.58	+ 35.37	+ 47.16	− 11.79
bB	+ 15.41	+ 30.82	+ 46.23	+ 61.64	+ 77.05
aA_1	+ 119.87	+ 95.90	+ 71.92	+ 47.95	+ 23.97
a_1A_2	− 18.34	+ 73.36	+ 55.02	+ 36.68	+ 18.34
a_2A_3	− 8.44	+ 16.87	+ 25.31	+ 16.87	+ 8.44
a_4A_3	+ 8.44	− 16.87	+ 25.31	− 16.87	− 8.44
a_5A_4	+ 18.34	+ 36.68	+ 55.02	+ 73.36	− 18.34
bA_5	+ 23.97	+ 47.95	+ 71.92	+ 95.90	+ 119.87
aa_1	+ 45.65	+ 36.52	+ 27.39	+ 18.26	+ 9.13
a_1a_2	+ 43.54	+ 86.08	+ 65.31	+ 43.54	+ 21.77
a_2a_3	+ 33.94	+ 67.88	+ 101.82	+ 67.88	+ 33.94
a_3a_4	+ 33.94	+ 67.88	+ 101.82	+ 67.88	+ 33.94
a_4a_5	+ 21.77	+ 43.54	+ 65.31	+ 86.08	+ 43.54
a_5b	+ 9.13	+ 18.26	+ 27.39	+ 36.52	+ 45.65
AA_1	0	0	0	0	0
A_1A_2	+ 45.65	+ 36.52	+ 27.39	+ 18.26	+ 9.13
A_2A_3	+ 43.54	+ 86.08	+ 65.31	+ 43.54	+ 21.77
A_3A_4	+ 21.77	+ 43.54	+ 65.31	+ 86.08	+ 43.54
A_4A_5	+ 9.13	+ 18.26	+ 27.39	+ 36.52	+ 45.65
A_5B	0	0	0	0	0
	+ 544.36	+ 921.65	+ 1074.54	+ 921.65	+ 544.36

We have then from (7) for the apex loads P_1, P_2, P_3, P_4, P_5, for cable of links,

	P_1	P_2	P_3	P_4	P_5
$\phi =$	0.49	0.83	0.96	0.83	0.49

We see that the fraction ϕ of a load P_1 carried by the cable varies with the position of the load from about 0.5 at the end to 0.96 at centre.

We have $P = 18000$ pounds in the present case and the *upward* load on truss $\frac{\phi P}{n} = 3000\phi$ pounds at every apex of the loaded chord. We also have the temperature load $w_t = 700$ pounds per foot.

We can then draw up the following table of stresses for the truss.

	aA	a_1A_1	a_2A_2	a_3A_3	aA_1	a_1A_2	a_2A_3
P_1	− 11325	0	− 5205	− 3735	+ 14156	− 6506	− 4668
P_2	− 5775	− 8265	0	− 7245	+ 7219	+ 10331	− 9056
P_3	− 1800	− 4680	− 7560	0	+ 2250	+ 5850	+ 9450
P_4	0	− 2265	− 4755	− 7245	− 281	+ 2831	+ 5944
P_5	0	− 795	− 2265	− 3735	− 844	+ 994	+ 2831
w_t	.... − 15750	.... − 15750	.... − 9450	.... − 3150	+ 19687 − 19687	+ 11812 − 11812	+ 3937 − 3937
Max. Stresses...	.... − 34650	.... − 31755	.... − 29235	.... − 25110	+ 43312 − 20812	+ 31818 − 18318	+ 22162 − 17661

	aa_1	a_1a_2	a_2a_3	AA_1	A_1A_2	A_2A_3
P_1	− 8494	− 8494	− 4590	0	+ 4590	+ 1789
P_2	− 4331	− 10530	− 10530	0	+ 4331	+ 5096
P_3	− 1350	− 4860	− 10530	0	+ 1350	+ 4860
P_4	0	− 1530	− 5096	− 168	− 168	+ 1530
P_5	0	− 90	− 1789	− 506	− 506	+ 90
w_t	.... − 11812	+ 11812 − 18900	+ 18900 − 21262	.... − 11812	+ 11812 − 18900	+ 18900 − 21262
Max. Stresses...	.... − 25987	+ 11812 − 44404	+ 18900 − 53797	.... − 12486	+ 22083 − 19574	+ 32265 − 21262

From this table we find the maximum stresses in the truss members.

We have now from (VII), since $w_t = 700$, $w_0 = 1000$, $\Sigma\phi = 3.6$, for the stresses in the cable segments,

$A'a'$.	$a'b'$	$b'c'$
134415	123714	118102

The suspender stress is, from (VIII), 26100 pounds.

We can now find the corresponding areas of cross-section of the members, and if these areas differ too much from those assumed, should make a new calculation with the new areas.

COMPARISON OF THE TWO METHODS.—In the present case we have then the following results:

	Old Method.	New Method.		Old Method.	New Method.
aA	$-$ 29250	$-$ 34650	a_2a_3	$\pm$ 30262	$+$ 18000 $-$ 53797
a_1A_1	$-$ 22950	$-$ 31755	AA_1	$-$ 20812	$-$ 12486
a_2A_2	$-$ 16650	$-$ 29230	A_1A_2	$\pm$ 20812	$+$ 22083 $-$ 19574
a_3A_3	$-$ 13500	$-$ 25115	A_2A_3	$\pm$ 27900	$+$ 32265 $-$ 21262
aA_1	$\pm$ 36562	$+$ 43312 $-$ 20812	$A'a'$	$+$ 171495	$+$ 134415
a_1A_2	$+$ 28687 $-$ 27000	$+$ 31818 $-$ 18318	$a'b'$	$+$ 158175	$+$ 123714
a_2A_3	$\pm$ 20812	$+$ 22162 $-$ 17661	$b'c'$	$+$ 149850	$+$ 118102
aa_1	$-$ 20812	$-$ 25987	Susp.	$+$ 30300	$+$ 26100
a_1a_2	$\pm$ 27900	$+$ 11812 $-$ 44404			

We see that the cable and suspender stresses are much less by the new method. For the truss members by the new method the direct stresses are greater and the counter-stresses less.

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